GAUGE TRANSFORMATIONS AS LORENTZ-BOOSTED ROTATIONS

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The E(2)-like little group for massless particles is obtained from the O(3)-like little group for massive particles in the infinite-momentum/zero-mass limit. It is shown therefore that rotations around the axes perpendicular to the direction of the momentum become gauge transformations in this limit. The limiting procedure is shown to be identical to that of group contraction through which O(3) becomes E(2).

One of the beauties of Einsteins's special relativity is the unified description of the energy–momentum relation for massive and massless particles through $E = (P^2 + M^2)^{1/2}$, where E, P and M are respectively the energy, momentum, and mass of a given free particle. Relativistic particles are known to have internal space–time symmetries. Is there then a unified way to describe internal space–time symmetries for both massive and massless particles?

The purpose of the present paper is to discuss this problem [‡1]. This is an interesting problem in view of the recent success of the gauge theory in which the photon and massive vector bosons form a gauge multiplet [2,3]. In 1939, Wigner formulated a method of studying the internal space–time symmetries of massive and massless particles based on the little groups [4].

[‡1] According to O.W. Greenberg, this problem was discussed as one of the unsolved problems at one of the lectures E.P. Wigner gave in Trieste and Istanbul in 1962 [1], although the content of this discussion was not included in the published lecture notes (ref. [2]). We thank O.W. Greenberg for providing this valuable information.

The little group is a subgroup of the Poincaré group which leaves the four-momentum of a given particle invariant. The little groups for massive and massless particles are locally isomorphic to the three-dimensional rotation group and the two-dimensional euclidean group respectively.

For this reason, in order to obtain a unified description of the little groups for massive and massless particles, we are let to consider the possibility of obtaining E(2) as a limiting case of O(3). This idea is not new. Inōnu and Wigner in 1953 introduced the method of group contraction to the physics world and worked out in detail how the infinite-dimensional unitary representation of the E(2) group can be obtained as a limiting case of the spherical harmonics for large values of angular momentum [5]. However, it was not known until recently that the four-vector representation for photons corresponds to a *finite-dimensional non-unitary representation* of the E(2) group [6].

The E(2) group consists of rotations and translations on a two-dimensional xy plane. The

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coordinate transformation in which a rotation is followed by a translation takes the form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & u \\ \sin\phi & \cos\phi & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{1}$$

This is the simplest non-trivial finite-dimensional non-unitary representation of the E(2) group. The way in which the above three-parameter matrix describes the helicity and gauge degrees of freedom in the four-vector representation of the electromagnetic field has been discussed in detail in refs. [6,7].

The three-by-three matrix in eq. (1) can be exponentiated as

$$D(\phi, u, v) = \exp[-\mathrm{i}(uP_1 + vP_2)] \exp(-\mathrm{i}\phi L_3). \tag{2}$$

The generators in this case are

$$L_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

$$P_1 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \end{pmatrix}.$$

These generators satisfy the following commutation relations:

$$[P_1, P_2] = 0, \quad [L_3, P_1] = \mathrm{i}P_2, \quad [L_3, P_2] = -\mathrm{i}P_1. \tag{4}$$

On the other hand, in O(3), three-by-three rotation matrices applicable to coordinate variables (x, y, z) are generated by L_3 of eq. (3) and

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}. \tag{5}$$

Both E(2) and O(3) share the same L_3. The question is how P_1 and P_2 can be obtained from L_1 and L_2. For this purpose, let us consider the surface of a sphere with a large radius, and a small area near the north pole [8]. Then z is very large and is approximately equal to the radius of the sphere R. We can then write

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{6}$$

The column vectors on the left- and right-hand sides are respectively the coordinate vectors on which O(3) and E(2) transformations are applicable. We shall use A for the three-by-three matrix on the right hand side. Then, in the limit of large R,

$$L_3 = AL_3A^{-1}, \quad P_1 = (1/R)AL_2A^{-1}, \\ P_2 = -(1/R)AL_1A^{-1}, \tag{7}$$

where L_3, P_1, and P_2 are given in eq. (5). This kind of limiting procedure is called the contraction of O(3) to E(2).

Let us return to physics. As was noted before, the internal spacetime symmetries of free particles are governed by the little groups. If a massive particle is at rest, the symmetry group is generated by the angular momentum operators J_1, J_2 and J_3. If this particle moves along the z direction, J_3 remains invariant, and its eigenvalue is the helicity. However, we have been avoiding in the past the question of what happens to J_1 and J_2, particularly in the infinite-momentum limit where the particle appears massless. Do they transform themselves to accommodate the E(2)-like symmetry for massless particles?

The purpose of this paper is to show that J_1 and J_2 become proportional to the generators of gauge transformations in the infinite-momentum/zero-mass limit. Let us start with a massive particle at rest. If we use the four-vector convention:

$$x^\mu = (x, y, z, t), \quad x_\mu = (x, y, z, -t), \tag{8}$$

the generators of the O(3)-like little group applicable to this vector space are the four-by-four matrices of the form

$$J_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

and similar expressions for J_1 and J_2.

If we boost this massive particle along the z direction, its momentum and energy will become P and $E = (P^2 + M^2)^{1/2}$ respectively. The boost matrix is

$$B(P)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & E/M & P/M\\ 0 & 0 & P/M & E/M\end{pmatrix}. \tag{10}$$

Under this boost operation, J_3 given in eq. (9) remains invariant:

$$BJ_3B^{-1}=J_3. \tag{11}$$

However, for J_2 and J_1, we can consider

$$G_1=-(M/E)BJ_2B^{-1}=-J_2+(P/E)K_1,$$
$$G_2=(M/E)BJ_1B^{-1}=J_1+(P/E)K_2, \tag{12}$$

where K_1 and K_2 in eq. (12) are the boost generators along the x and y directions respectively, and take the form

$$K_1=\begin{pmatrix}0 & 0 & 0 & \mathrm{i}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \mathrm{i} & 0 & 0 & 0\end{pmatrix},$$
$$K_2=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathrm{i}\\ 0 & 0 & 0 & 0\\ 0 & \mathrm{i} & 0 & 0\end{pmatrix}. \tag{13}$$

Then the forms given in eq. (12) are not unlike those in eq. (7). The B matrix in eq. (12) is like the A matrix in eq. (7), and the ratio M/E is like $1/R$ in eq. (7) measured in a suitable unit.

Now, in terms of the operators J_3, G_1 and G_2, the O(3) commutation relations for J_i can be written as

$$[J_3,G_1]=\mathrm{i}G_2,\quad [J_3,G_2]=-\mathrm{i}G_1,$$
$$[G_1,G_2]=(M/E)^2J_3. \tag{14}$$

In the infinite-momentum/zero-mass limit, the quantity $(M/E)^2$ vanishes, and the G operators become

$$G_1\to N_1\quad\text{and}\quad G_2\to N_2, \tag{15}$$

where

$$N_1=K_1-J_2,\quad N_2=K_2+J_1, \tag{16}$$

and the O(3) commutation relations of eq. (14) become

$$[N_1,N_2]=0,\quad [J_3,N_1]=\mathrm{i}N_2,\quad [J_3,N_2]=-\mathrm{i}N_1. \tag{17}$$

The matrices N_1 and N_2 defined in eq. (16) together with J_3 form the generators of the E(2)-like little group for massless particles [4]. They satisfy the above commutation relations which are identical to those for the generators of E(2) given in eq. (4). J_3 is like the generator of rotation while N_1 and N_2 are like the generators of translations. These N operators are known to generate gauge transformations [6,7,9].

We have thus shown that rotations around the axes perpendicular to the momentum become gauge transformations in the infinite-momentum/zero-mass limit. While the above calculations are carried out in the O(3, 1) regime where both the O(3)-like little group and the E(2)-like little group are subgroups of O(3, 1), it is straightforward to show that the E(2)-like subgroup of SL(2, **c**) applicable to neutrinos [6] is the same limiting case of SU(2) within the framework of the SL(2, **c**) formalism of Lorentz transformations.

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