Statistical Mechanics Approach to Econophysics

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Glossary

Probability density $P(x)$ is defined so that the probability of finding a random variable x in the interval from x to $x + dx$ is equal to $P(x)\,dx$.

Cumulative probability $C(x)$ is defined as the integral $C(x) = \int_x^\infty P(x)dx$. It gives the probability that the random variable exceeds a given value x.

The Boltzmann-Gibbs distribution gives the probability of finding a physical system in a state with the energy ε. Its probability density is given by the exponential function (1).

The Gamma distribution has the probability density given by a product of an exponential function and a power-law function, as in (9).

The Pareto distribution has the probability density $P(x) \propto 1/x^{1+\alpha}$ and the cumulative probability $C(x) \propto 1/x^\alpha$ given by a power law. These expressions apply only for high enough values of x and do not apply for $x \to 0$.

The Lorenz curve was introduced by American economist Max Lorenz to describe income and wealth inequality. It is defined in terms of two coordinates $x(r)$ and $y(r)$ given by (19). The horizontal coordinate $x(r)$ is the fraction of the population with income below r, and the vertical coordinate $y(r)$ is the fraction of income this population accounts for. As r changes from 0 to ∞, x and y change from 0 to 1, parametrically defining a curve in the (x, y)-plane.

The Gini coefficient G was introduced by the Italian statistician Corrado Gini as a measure of inequality in a society. It is defined as the area between the Lorenz curve and the straight diagonal line, divided by the area of the triangle beneath the diagonal line. For perfect equality (everybody has the same income or wealth) $G = 0$, and for total inequality (one person has all income or wealth, and the rest have nothing) $G = 1$.

The Fokker-Planck equation is the partial differential Eq. (22) that describes evolution in time t of the probability density $P(r, t)$ of a random variable r experiencing small random changes Δr during short time intervals Δt. It is also known in mathematical literature as the Kolmogorov forward equation. The diffusion equation is an example of the Fokker-Planck equation.

Definition of the Subject

Econophysics is an interdisciplinary research field applying methods of statistical physics to problems in economics and finance. The term "econophysics" was first introduced by the prominent theoretical physicist Eugene Stanley in 1995 at the conference *Dynamics of Complex Systems*, which was held in Calcutta (now known as Kolkata) as a satellite meeting to the

STATPHYS-19 conference in China (Chakrabarti 2005; Carbone et al. 2007). The term appeared in print for the first time in the paper by Stanley et al. (1996) in the proceedings of the Calcutta conference. The paper presented a manifesto of the new field, arguing that "behavior of large numbers of humans (as measured, e. g., by economic indices) might conform to analogs of the scaling laws that have proved useful in describing systems composed of large numbers of inanimate objects" (Stanley et al. 1996). Soon the first econophysics conferences were organized: *International Workshop on Econophysics*, Budapest, 1997, and *International Workshop on Econophysics and Statistical Finance*, Palermo, 1998 (Carbone et al. 2007), and the book *An Introduction to Econophysics* (Mantegna and Stanley 1999) was published.

The term "econophysics" was introduced by analogy with similar terms, such as "astrophysics," "geophysics," and "biophysics," which describe applications of physics to different fields. Particularly important is the parallel with biophysics, which studies living creatures, which still obey the laws of physics. It should be emphasized that econophysics does not literally apply the laws of physics, such as Newton's laws or quantum mechanics, to humans, but rather uses mathematical methods developed in statistical physics to study statistical properties of complex economic systems consisting of a large number of humans. So, it may be considered as a branch of applied theory of probabilities. However, statistical physics is distinctly different from mathematical statistics in its focus, methods, and results.

Originating from physics as a quantitative science, econophysics emphasizes quantitative analysis of large amounts of economic and financial data, which became increasingly available with the massive introduction of computers and the Internet. Econophysics distances itself from the verbose, narrative, and ideological style of political economy and is closer to **econometrics** in its focus. Studying mathematical models of a large number of interacting economic agents, econophysics has much common ground with the **agent-based modeling and simulation**. Correspondingly, it distances itself from the representative-agent approach of traditional economics, which, by definition, ignores statistical and heterogeneous aspects of the economy.

Two major directions in econophysics are applications to finance and economics. Observational aspects are covered in the entry "Econophysics, Observational." This entry, "Econophysics, Statistical Mechanics Approach to," concentrates primarily on statistical distributions of money, wealth, and income among interacting economic agents.

Another direction related to econophysics has been advocated by the theoretical physicist Serge Galam since the early 1980s under the name "**sociophysics**" (Galam 2004), with the first appearance of the term in print in Galan et al. (1982). It echoes the term *physique sociale* proposed in the nineteenth century by Auguste Comte, the founder of sociology. Unlike econophysics, the term "sociophysics" did not catch on when first introduced, but it is coming back with the popularity of econophysics and active promotion by some physicists (Stauffer 2004; Schweitzer 2003; Weidlich 2000). While the principles of both fields have a lot in common, econophysics focuses on the narrower subject of economic behavior of humans, where more quantitative data are available, whereas sociophysics studies a broader range of social issues. The boundary between econophysics and sociophysics is not sharp, and the two fields enjoy a good rapport (Chakrabarti et al. 2006). A more detailed description of the historical development is presented in section "Historical Introduction."

Historical Introduction

Statistical mechanics was developed in the second half of the nineteenth century by James Clerk Maxwell, Ludwig Boltzmann, and Josiah Willard Gibbs. These physicists believed in the existence of atoms and developed mathematical methods for describing their statistical properties, such as the probability distribution of velocities of molecules in a gas (the Maxwell-Boltzmann distribution) and the general probability distribution of states with different energies (the Boltzmann-Gibbs distribution). There are interesting

connections between the development of statistical physics and statistics of social phenomena, which were recently brought up by the science journalist Philip Ball (Ball 2002, 2004).

Collection and study of "social numbers," such as the rates of death, birth, and marriage, has been growing progressively since the seventeenth century (see Chap. 3 in Ball 2004). The term "statistics" was introduced in the eighteenth century to denote these studies dealing with the civil "states," and its practitioners were called "statists." Popularization of social statistics in the nineteenth century is particularly accredited to the Belgian astronomer Adolphe Quetelet. Before the 1850s, statistics was considered an empirical arm of political economy, but then it started to transform into a general method of quantitative analysis suitable for all disciplines. It stimulated physicists to develop statistical mechanics in the second half of the nineteenth century.

Rudolf Clausius started development of the kinetic theory of gases, but it was James Clerk Maxwell who made a decisive step of deriving the probability distribution of velocities of molecules in a gas. Historical studies show (see Chap. 3 in Ball 2004) that, in developing statistical mechanics, Maxwell was strongly influenced and encouraged by the widespread popularity of social statistics at the time. This approach was further developed by Ludwig Boltzmann, who was very explicit about its origins (see p. 69 in Ball 2004):

> The molecules are like individuals, and the properties of gases only remain unaltered, because the number of these molecules, which on the average have a given state, is constant.

In his book *Populäre Schriften* from 1905 (Boltzmann 1905), Boltzmann praises Josiah Willard Gibbs for systematic development of statistical mechanics. Then, Boltzmann says (cited from Austrian Central Library for Physics (2006)):

> This opens a broad perspective if we do not only think of mechanical objects. Let's consider to apply this method to the statistics of living beings, society, sociology and so forth.

(The author is grateful to Michael E. Fisher for bringing this quote to his attention.)

It is worth noting that many now-famous economists were originally educated in physics and engineering. Vilfredo Pareto earned a degree in mathematical sciences and a doctorate in engineering. Working as a civil engineer, he collected statistics demonstrating that distributions of income and wealth in a society follow a power law (Pareto 1897). He later became a professor of economics at Lausanne, where he replaced Léon Walras, also an engineer by education. The influential American economist Irving Fisher was a student of Gibbs. However, most of the mathematical apparatus transferred to economics from physics was that of Newtonian mechanics and classical thermodynamics (Mirowski 1989). It culminated in the neoclassical concept of mechanistic equilibrium where the "forces" of supply and demand balance each other. The more general concept of statistical equilibrium largely eluded mainstream economics.

With time, both physics and economics became more formal and rigid in their specializations, and the social origin of statistical physics was forgotten. The situation is well summarized by Philip Ball (see p. 69 in Ball (2004)):

> Today physicists regard the application of statistical mechanics to social phenomena as a new and risky venture. Few, it seems, recall how the process originated the other way around, in the days when physical science and social science were the twin siblings of a mechanistic philosophy and when it was not in the least disreputable to invoke the habits of people to explain the habits of inanimate particles.

Some physicists and economists attempted to connect the two disciplines during the twentieth century. The theoretical physicist Ettore Majorana argued in favor of applying the laws of statistical physics to social phenomena in a paper published after his mysterious disappearance (Majorana 1942). The statistical physicist Elliott Montroll co-authored the book *Introduction to Quantitative Aspects of Social Phenomena* (Montroll and Badger 1974). Several economists applied statistical physics to economic problems (Föllmer 1974; Blume 1993; Foley 1994; Durlauf 1997). An early attempt to bring together the leading theoretical physicists and economists at the Santa Fe Institute was not entirely successful

(Anderson et al. 1988). However, by the late 1990s, the attempts to apply statistical physics to social phenomena finally coalesced into the robust movements of econophysics and sociophysics, as described in section "Definition of the Subject."

The current standing of econophysics within the physics and economics communities is mixed. Although an entry on econophysics has appeared in the *New Palgrave Dictionary of Economics* (Rosser 2008), it is fair to say that econophysics is not accepted yet by mainstream economics. Nevertheless, a number of open-minded, non-traditional economists have joined this movement, and the number is growing. Under these circumstances, econophysicists have most of their papers published in physics journals. The journal *Physica A: Statistical Mechanics and Its Applications* emerged as the leader in econophysics publications and has even attracted submissions from some bona fide economists. The mainstream physics community is generally sympathetic to econophysics, although it is not uncommon for econophysics papers to be rejected by *Physical Review Letters* on the grounds that "it is not physics." There are regular conferences on econophysics, such as *Applications of Physics in Financial Analysis* (sponsored by the European Physical Society), *Nikkei Econophysics Symposium*, and *Econophysics Colloquium*. Econophysics sessions are included in the annual meetings of physical societies and statistical physics conferences. The overlap with economics is the strongest in the field of agent-based simulation. Not surprisingly, the conference series WEHIA/ESHIA, which deals with heterogeneous interacting agents, regularly includes sessions on econophysics.

Statistical Mechanics of Money Distribution

When modern econophysics started in the middle of the 1990s, its attention was primarily focused on analysis of financial markets. However, three influential papers (Drăgulescu and Yakovenko 2000; Chakraborti and Chakrabarti 2000; Bouchaud and Mézard 2000), dealing with the subject of money and wealth distributions, were

published in 2000. They started a new direction that is closer to economics than finance and created an expanding wave of follow-up publications. We start reviewing this subject with Drăgulescu and Yakovenko (2000), whose results are the most closely related to the traditional statistical mechanics in physics.

The Boltzmann-Gibbs Distribution of Energy

The fundamental law of equilibrium statistical mechanics is the Boltzmann-Gibbs distribution. It states that the probability $P(\varepsilon)$ of finding a physical system or subsystem in a state with the energy ε is given by the exponential function

$$P(\varepsilon) = c e^{\frac{-\varepsilon}{T}}, \tag{1}$$

where T is the temperature, and c is a normalizing constant (Wannier 1987). Here we set the Boltzmann constant k_{B} to unity by choosing the energy units for measuring the physical temperature T. Then, the expectation value of any physical variable x can be obtained as

$$\langle x \rangle = \frac{\sum\limits_{k} x_k e^{\frac{-\varepsilon_k}{T}}}{\sum\limits_{k} e^{\frac{-\varepsilon_k}{T}}}, \tag{2}$$

where the sum is taken over all states of the system. Temperature is equal to the average energy per particle: $T \sim \langle \varepsilon \rangle$, up to a numerical coefficient of the order of 1.

Equation (1) can be derived in different ways (Wannier 1987). All derivations involve the two main ingredients: statistical character of the system and conservation of energy ε. One of the shortest derivations can be summarized as follows. Let us divide the system into two (generally unequal) parts. Then, the total energy is the sum of the parts, $\varepsilon = \varepsilon_1 + \varepsilon_2$, whereas the probability is the product of probabilities, $P(\varepsilon) = P(\varepsilon_1)P(\varepsilon_2)$. The only solution of these two equations is the exponential function (1).

A more sophisticated derivation, proposed by Boltzmann himself, uses the concept of entropy. Let us consider N particles with total energy E. Let us divide the energy axis into small intervals

(bins) of width $\Delta\varepsilon$ and count the number of particles N_k having energies from ε_k to $\varepsilon_k + \Delta\varepsilon$. The ratio $N_k/N = P_k$ gives the probability for a particle having the energy ε_k. Let us now calculate the multiplicity W, which is the number of permutations of the particles between different energy bins such that the occupation numbers of the bins do not change. This quantity is given by the combinatorial formula in terms of the factorials

$$W = \frac{N!}{N_1! N_2! N_3! \dots}. \quad (3)$$

The logarithm of multiplicity of called the entropy $S = \ln W$. In the limit of large numbers, the entropy per particle can be written in the following form using the Stirling approximation for the factorials:

$$\frac{S}{N} = -\sum_k \frac{N_k}{N} \ln\left(\frac{N_k}{N}\right) = -\sum_k P_k \ln P_k. \quad (4)$$

Now we would like to find what distribution of particles between different energy states has the highest entropy, that is, the highest multiplicity, provided that the total energy of the system, $E = \sum_k N_k \varepsilon_k$, has a fixed value. Solution of this problem can be easily obtained using the method of Lagrange multipliers (Wannier 1987), and the answer gives the exponential distribution (1).

The same result can be derived from the **ergodic theory**, which says that the many-body system occupies all possible states of a given total energy with equal probabilities. Then it is straightforward to show (Lopez-Ruiz et al. 2007a, b) that the probability distribution of the energy of an individual particle is given by (1).

Conservation of Money

The derivations outlined in section "The Boltzmann-Gibbs Distribution of Energy" are very general and use only the statistical character of the system and the conservation of energy. So, one may expect that the exponential Boltzmann-Gibbs distribution (1) may apply to other statistical systems with a conserved quantity.

The economy is a big statistical system with millions of participating agents, so it is a promising target for applications of statistical mechanics. Is there a conserved quantity in economy? Drăgulescu and Yakovenko (2000) argue that such a conserved quantity is money m. Indeed, the ordinary economic agents can only receive money from and give money to other agents. They are not permitted to "manufacture" money, for example, to print dollar bills. When one agent i pays money Δm to another agent j for some goods or services, the money balances of the agents change as follows:

$$\begin{aligned} m_i &\rightarrow m_i' = m_i - \Delta m, \\ m_j &\rightarrow m_j' = m_j + \Delta m. \end{aligned} \quad (5)$$

The total amount of money of the two agents before and after the transaction remains the same,

$$m_i + m_j = m_i' + m_j', \quad (6)$$

that is, there is a local conservation law for money. The rule (5) for the transfer of money is analogous to the transfer of energy from one molecule to another in molecular collisions in a gas, and (6) is analogous to conservation of energy in such collisions.

Addressing some misunderstandings developed in economic literature (Anglin 2005; Lux 2005; Gallegati et al. 2006; Lux 2008), we should emphasize that, in the model of Drăgulescu and Yakovenko (2000), the transfer of money from one agent to another happens voluntarily, as a payment for goods and services in a market economy. However, the model only keeps track of money flow, and does not keep track of what kinds of goods and service are delivered. One reason for this is that many goods, for example, food and other supplies, and most services, for example, getting a haircut or going to a movie, are not tangible and disappear after consumption. Because they are not conserved and also because they are measured in different physical units, it is not very practical to keep track of them. In contrast, money is measured in the same unit (within a given country with a single currency) and is

conserved in transactions, so it is straightforward to keep track of money flow.

Unlike ordinary economic agents, a central bank or a central government can inject money into the economy. This process is analogous to an influx of energy into a system from external sources, for example, the Earth receives energy from the Sun. Dealing with these situations, physicists start with an idealization of a closed system in thermal equilibrium and then generalize to an open system subject to an energy flux. As long as the rate of money influx from central sources is slow compared with relaxation processes in the economy and does not cause hyperinflation, the system is in quasi-stationary statistical equilibrium with slowly changing parameters. This situation is analogous to heating a kettle on a gas stove slowly, where the kettle has a well-defined, but slowly increasing temperature at any moment of time.

Another potential problem with conservation of money is debt. This issue is discussed in more detail in section "Models with Debt." As a starting point, Drăgulescu and Yakovenko (2000) first considered simple models, where debt is not permitted. This means that money balances of agents cannot go below zero: $m_i \geq 0$ for all i. Transaction (5) takes place only when an agent has enough money to pay the price, $m_i \geq \Delta m$, otherwise the transaction does not take place. If an agent spends all the money, the balance drops to zero $m_i = 0$, so the agent cannot buy any goods from other agents. However, this agent can still produce goods or services, sell them to other agents, and receive money for them. In real life, cash balance dropping to zero is not at all unusual for people who live from paycheck to paycheck.

The conservation law is the key feature for the successful functioning of money. If the agents were permitted to "manufacture" money, they would be printing money and buying all goods for nothing, which would be a disaster. The physical medium of money is not essential, as long as the conservation law is enforced. Money may be in the form of paper cash, but today it is more often represented by digits in computerized bank accounts. The conservation law is the fundamental principle of accounting, whether in the single-

entry or in the double-entry form. More discussion of banks and debt is given in section "Models with Debt."

The Boltzmann-Gibbs Distribution of Money

Having recognized the principle of money conservation, Drăgulescu and Yakovenko (2000) argued that the stationary distribution of money should be given by the exponential Boltzmann-Gibbs function analogous to (1):

$$P(m) = c e^{\frac{-m}{T_m}}. \tag{7}$$

Here c is a normalizing constant, and T_m is the "money temperature," which is equal to the average amount of money per agent: $T = \langle m \rangle = M/N$, where M is the total money, and N is the number of agents.

To verify this conjecture, Drăgulescu and Yakovenko (2000) performed agent-based computer simulations of money transfers between agents. Initially all agents were given the same amount of money, say, $1000. Then, a pair of agents (i, j) were randomly selected, the amount Δm was transferred from one agent to another, and the process was repeated many times. Time evolution of the probability distribution of money $P(m)$ can be seen in computer animation videos at the Web pages (Computer animation videos of money-transfer models 2008; Wright 2007). After a transitory period, money distribution converges to the stationary form shown in Fig. 1. As expected, the distribution is very well fitted by the exponential function (7).

Several different rules for Δm were considered in Drăgulescu and Yakovenko (2000). In one model, the amount transferred was fixed to a constant $\Delta m = 1$\$. Economically, it means that all agents were selling their products for the same price $\Delta m = 1$\$. Computer animation (Computer animation videos of money-transfer models 2008) shows that the initial distribution of money first broadens to a symmetric, Gaussian curve, characteristic for a diffusion process. Then, the distribution starts to pile up around the $m = 0$ state, which acts as the impenetrable boundary, because of the imposed condition $m \geq 0$. As a result, $P(m)$

becomes skewed (asymmetric) and eventually reaches the stationary exponential shape, as shown in Fig. 1. The boundary at $m = 0$ is analogous to the ground-state energy in statistical physics. Without this boundary condition, the probability distribution of money would not reach a stationary state. Computer animation (Computer animation videos of money-transfer models 2008; Wright 2007) also shows how the entropy of money distribution, defined as $S/N = -\sum_{k} P(m_k) \ln P(m_k)$, grows from the initial value $S = 0$, when all agents have the same money, to the maximal value at the statistical equilibrium.

While the model with $\Delta m = 1$ is very simple and instructive, it is not very realistic, because all prices are taken to be the same. In another model considered in Drăgulescu and Yakovenko (2000), Δm in each transaction is taken to be a random fraction of the average amount of money per agent, that is, $\Delta m = v(M/N)$, where v is a uniformly distributed random number between 0 and 1. The random distribution of Δm is supposed to represent the wide variety of prices for different products in the real economy. It reflects the fact that agents buy and consume many different types of products, some of them simple and cheap, some sophisticated and expensive. Moreover, different agents like to consume these products in different quantities, so there is variation in the amounts Δm paid, even though the unit price of the same product is constant. Computer simulation of this model produces exactly the same stationary distribution (7) as in the first model. Computer animation for this model is also available on the Web page (Computer animation videos of money-transfer models 2008).

The final distribution is universal despite different rules for Δm. To amplify this point further, Drăgulescu and Yakovenko (2000) also considered a toy model, where Δm was taken to be a random fraction of the average amount of money of the two agents: $\Delta m = v(m_i + m_j)/2$. This model produced the same stationary distribution (7) as the other two models.

The pairwise models of money transfer are attractive in their simplicity, but they represent a rather primitive market. The modern economy is dominated by big firms, which consist of many agents, so Drăgulescu and Yakovenko (2000) also studied a model with firms. One agent at a time is appointed to become a "firm." The firm borrows capital K from another agent and returns it with interest hK, hires L agents and pays them wages W, manufactures Q items of a product, sells them to Q agents at price R, and receives profit $F = RQ - LW - hK$. All of these agents are randomly selected. The parameters of the model are optimized following a procedure from economics textbooks (McConnell and Brue 1996). The aggregate demand-supply curve for the product is used in the form $R(Q) = v/Q^{\eta}$, where Q is the quantity consumers would buy at price R, and η and v are some parameters. The production function of the firm has the traditional Cobb-Douglas form: $Q(L, K) = L^{\chi} K^{1-\chi}$, where χ is a parameter. Then the profit of the firm F is maximized with respect to K and L. The net result of the firm activity is a many-body transfer of money, which still satisfies the conservation law. Computer simulation of this model generates the same exponential distribution (7), independently of the model parameters. The reasons for the universality of the Boltzmann-Gibbs distribution and its limitations are discussed from a different perspective in section "Additive Versus Multiplicative Models."

Well after paper (Drăgulescu and Yakovenko 2000) appeared, Italian econophysicists (Patriarca et al. 2005) found that similar ideas had been published earlier in obscure journals in Italian by Eleonora Bennati (Bennati 1988, 1993). They proposed calling these models the Bennati-Drăgulescu-Yakovenko game (Scalas et al. 2006). The Boltzmann distribution was independently applied to social sciences by Jürgen Mimkes using the Lagrange principle of maximization with constraints (Mimkes 2000, 2005). The exponential distribution of money was also found in Shubik (1999) using a Markov chain approach to strategic market games. A long time ago, Benoit Mandelbrot observed (see p. 83 in Mandelbrot (1960)):

> There is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between gas molecules.

Statistical Mechanics Approach to Econophysics, Fig. 1 Stationary probability distribution of money $P(m)$ obtained in agent-based computer simulations. *Solid curves*: fits to the Boltzmann-Gibbs law (7). *Vertical lines*: the initial distribution of money. (Reproduced from Drăgulescu and Yakovenko (2000))
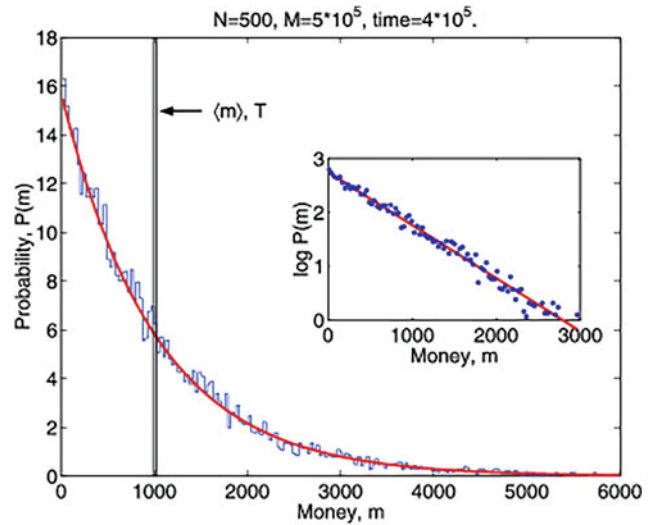
He realized that this process should result in the exponential distribution, by analogy with the barometric distribution of density in the atmosphere. However, he discarded this idea, because it does not produce the Pareto power law, and proceeded to study the stable Lévy distributions. Ironically, the actual economic data, discussed in section "Empirical Data on Money and Wealth Distributions" and "Empirical Data on Income Distribution," do show the exponential distribution for the majority of the population. Moreover, the data have finite variance, so the stable Lévy distributions are not applicable because of their infinite variance.

Models with Debt

Now let us discuss how the results change when debt is permitted. Debt may be considered as negative money. When an agent borrows money from a bank (considered here as a big reservoir of money), the cash balance of the agent (positive money) increases, but the agent also acquires a debt obligation (negative money), so the total balance (net worth) of the agent remains the same, and the conservation law of total money (positive and negative) is satisfied. After spending some cash, the agent still has the debt obligation, so the money balance of the agent becomes negative. Any stable economic system must have a

mechanism preventing unlimited borrowing and unlimited debt. Otherwise, agents can buy any products without producing anything in exchange by simply going into unlimited debt. The exact mechanisms of limiting debt in the real economy are complicated and obscured. Drăgulescu and Yakovenko (2000) considered a simple model where the maximal debt of any agent is limited by a certain amount m_d. This means that the boundary condition $m_i \geq 0$ is now replaced by the condition $m_i \geq -m_d$ for all agents i. Setting interest rates on borrowed money to be zero for simplicity, Drăgulescu and Yakovenko (2000) performed computer simulations of the models described in section "The Boltzmann-Gibbs Distribution of Money" with the new boundary condition. The results are shown in Fig. 2. Not surprisingly, the stationary money distribution again has an exponential shape, but now with the new boundary condition at $m = -m_d$ and the higher money temperature $T_d = m_d + M/N$. By allowing agents to go into debt up to m_d, we effectively increase the amount of money available to each agent by m_d. So, the money temperature, which is equal to the average amount of effectively available money per agent, increases. A model with nonzero interest rates was also studied in Drăgulescu and Yakovenko (2000).

Statistical Mechanics Approach to Econophysics, Fig. 2 Stationary distributions of money with and without debt. The debt is limited to $m_d = 800$. *Solid curves*: fits to the Boltzmann-Gibbs laws with the "temperatures" $T = 1800$ and $T = 1000$. (Reproduced from Drăgulescu and Yakovenko (2000))
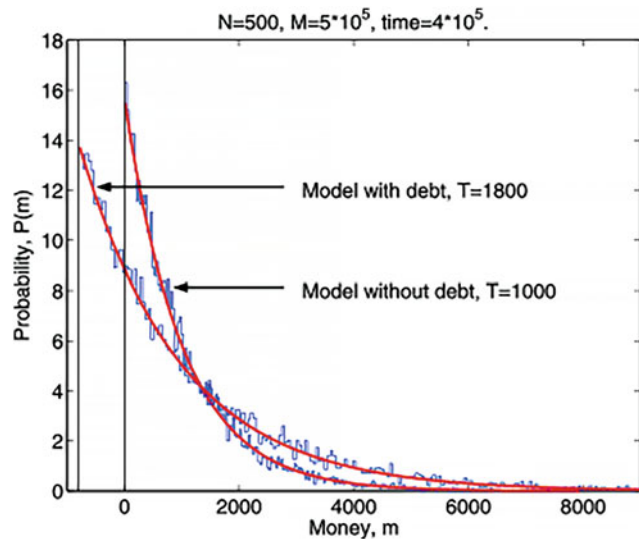
We see that debt does not violate the conservation law of money, but rather modifies boundary conditions for $P(m)$. When economics textbooks describe how "banks create money" or "debt creates money" (McConnell and Brue 1996), they count only positive money (cash) as money, but do not count liabilities (debt obligations) as negative money. With such a definition, money is not conserved. However, if we include debt obligations in the definition of money, then the conservation law is restored. This approach is in agreement with the principles of double-entry accounting, which records both assets and debts. Debt obligations are as real as positive cash, as many borrowers painfully realized in their experience. A more detailed study of positive and negative money and bookkeeping from the point of view of econophysics is presented in a series of papers by the physicist Dieter Braun (2001; Fischer and Braun 2003a, b).

One way of limiting the total debt in the economy is the so-called required reserve ratio r (McConnell and Brue 1996). Every bank is required by law to set aside a fraction r of the money deposited with the bank, and this reserved money cannot be loaned further. If the initial amount of money in the system (the money base) is M_0, then with loans and borrowing the total amount of positive money available to the agents increases to $M = M_0/r$, where the factor $1/r$

is called the money multiplier (McConnell and Brue 1996). This is how "banks create money." Where does this extra money come from? It comes from the increase of the total debt in the system. The maximal total debt is equal to $D = M_0/r - M_0$ and is limited by the factor r. When the debt is maximal, the total amounts of positive, M_0/r, and negative, $M_0(1 - r)/r$, money circulate between the agents in the system, so there are effectively two conservation laws for each of them (Xi et al. 2005). Thus, we expect to see the exponential distributions of positive and negative money characterized by two different temperatures: $T_+ = M_0/rN$ and $T_- = M_0(1 - r)/rN$. This is exactly what was found in computer simulations in Xi et al. (2005), shown in Fig. 3. Similar two-sided distributions were also found in Fischer and Braun (2003a).

Proportional Money Transfers and Saving Propensity

In the models of money transfer considered thus far, the transferred amount Δm is typically independent of the money balances of agents. A different model was introduced in the physics literature earlier (Ispolatov et al. 1998) under the name multiplicative asset exchange model. This model also satisfies the conservation law, but the amount of money transferred is a fixed fraction γ of the payer's money in (5):

$$\Delta m = \gamma m_i. \tag{8}$$

The stationary distribution of money in this model, shown in Fig. 4 with an exponential function, is close, but not exactly equal, to the Gamma distribution:

$$P(m) = cm^{\beta}e^{\frac{-m}{T}}. \tag{9}$$

Equation (9) differs from (7) by the power-law prefactor m^{β}. From the Boltzmann kinetic equation (discussed in more detail in section "Additive Versus Multiplicative Models"), Ispolatov et al.

(1998) derived a formula relating the parameters γ and β in (8) and (9):

$$\beta = \frac{-1 - \ln 2}{\ln(1 - \gamma)}. \tag{10}$$

When payers spend a relatively small fraction of their money $\gamma < 1/2$, (10) gives $\beta > 0$, so the low-money population is reduced and $P(m \to 0) = 0$, as shown in Fig. 4.

Later, Thomas Lux brought to the attention of physicists (Lux 2005) that essentially the same model, called the inequality process, had been introduced and studied much earlier by the

Statistical Mechanics Approach to Econophysics, Fig. 3 The stationary distribution of money for the required reserve ratio $r = 0.8$. The distribution is exponential for positive and negative money with different "temperatures" T_+ and T_-, as illustrated by the *inset* on log-linear scale. (Reproduced from Xi et al. (2005))
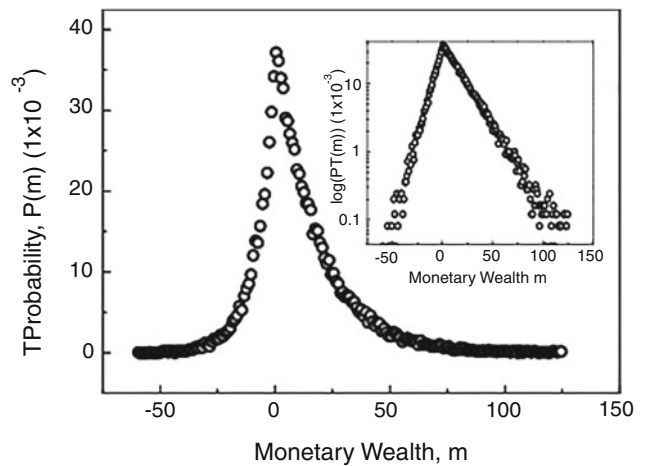
Statistical Mechanics Approach to Econophysics, Fig. 4 Stationary probability distribution of money in the multiplicative random exchange model (8) for $\gamma = 1/3$. *Solid curve*: the exponential Boltzmann-Gibbs law. (Reproduced from Drăgulescu and Yakovenko (2000))
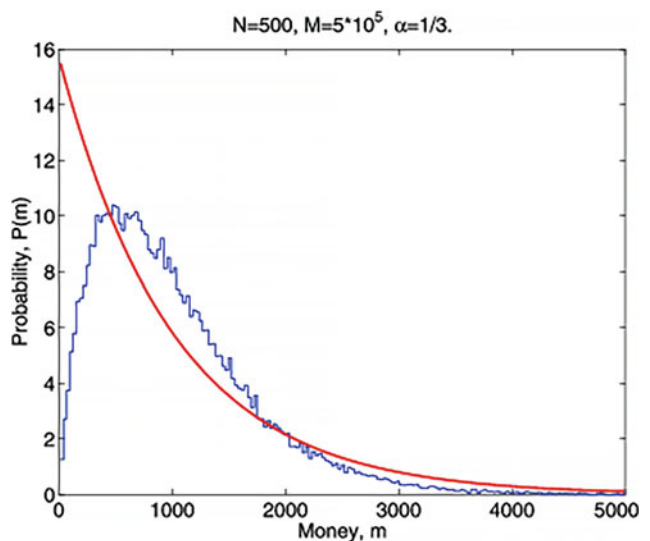
sociologist John Angle (1986, 1992a, b, 1996, 2002), see also the review Angle (2006) for additional references. While Ispolatov et al. (Ispolatov et al. 1998) did not give much justification for the proportionality law (8), Angle (1986) connected this rule with the surplus theory of social stratification (Engels 1972), which argues that inequality in human society develops when people can produce more than necessary for minimal subsistence. This additional wealth (surplus) can be transferred from original producers to other people, thus generating inequality. In the first paper by Angle (1986), the parameter γ was randomly distributed, and another parameter, δ, gave a higher probability of winning to the agent with a higher money balance in (5). However, in the following papers, he simplified the model to a fixed γ (denoted as ω by Angle) and equal probabilities of winning for higher- and lower-balance agents, which makes it completely equivalent to the model of Ispolatov et al. (1998). Angle also considered a model (Angle 2002, 2006) where groups of agents have different values of γ, simulating the effect of education and other "human capital." All of these models generate a Gamma-like distribution, well approximated by (9).

Another model with an element of proportionality was proposed in Chakraborti and Chakrabarti (2000). (This paper originally appeared as a follow-up preprint cond-mat/0004256 to the preprint cond-mat/0001432 of Drăgulescu and Yakovenko (2000).) In this model, the agents set aside (save) some fraction of their money λm_i, whereas the rest of their money balance $(1 - \lambda)m_i$ becomes available for random exchanges. Thus, the rule of exchange (5) becomes

$$\begin{aligned} m'_i &= \lambda m_i + \xi(1 - \lambda)\big(m_i + m_j\big), \\ m'_j &= \lambda m_j + (1 - \xi)(1 - \lambda)\big(m_i + m_j\big). \end{aligned} \quad (11)$$

Here the coefficient λ is called the saving propensity, and the random variable ξ is uniformly distributed between 0 and 1. It was pointed out in Angle (2006) that, by the change of notation $\lambda \to (1 - \gamma)$, (11) can be transformed to the

same form as (8), if the random variable ξ takes only discrete values 0 and 1. Computer simulations (Chakraborti and Chakrabarti 2000) of the model (11) found a stationary distribution close to the Gamma distribution (9). It was shown that the parameter β is related to the saving propensity λ by the formula $\beta = 3\lambda/(1 - \lambda)$ (Patriarca et al. 2004a, b, 2005, Repetowicz et al. 2005). For $\lambda \neq 0$, agents always keep some money, so their balances never go to zero and $P(m \to 0) = 0$, whereas for $\lambda = 0$ the distribution becomes exponential.

In the subsequent papers by the Kolkata school (Chakrabarti 2005) and related papers, the case of random saving propensity was studied. In these models, the agents are assigned random parameters λ drawn from a uniform distribution between 0 and 1 (Chatterjee et al. 2004). It was found that this model produces a power-law tail $P(m) \propto 1/m^2$ at high m. The reasons for stability of this law were understood using the Boltzmann kinetic equation (Repetowicz et al. 2005; Das and Yarlagadda 2005; Chatterjee et al. 2005), but most elegantly in the mean-field theory (Mohanty 2006). The fat tail originates from the agents whose saving propensity is close to 1, who hoard money and do not give it back (Patriarca et al. 2005, 2006). An interesting matrix formulation of the problem was presented in Gupta (2006). Patriarca et al. (2007) studied the relaxation rate in the money transfer models. Drăgulescu and Yakovenko (2000) studied a model with taxation, which also has an element of proportionality. The Gamma distribution was also studied for conservative models within a simple Boltzmann approach in Ferrero (2004) and using much more complicated rules of exchange in Scafetta et al. (2004a, b).

Additive Versus Multiplicative Models

The stationary distribution of money (9) for the models of section "Proportional Money Transfers and Saving Propensity" is different from the simple exponential formula (7) found for the models of section "The Boltzmann-Gibbs Distribution of Money." The origin of this difference can be

understood from the Boltzmann kinetic equation (Wannier 1987; Lifshitz and Pitaevskii 1981). This equation describes time evolution of the distribution function $P(m)$ due to pairwise interactions:

$$\frac{dP(m)}{dt} = \int\int \Big\{ f_{[mm']\rightarrow[mm'+]}P(m)P(m') \\ + f_{[mm'+]\rightarrow[mm']}P(m) \\ P(m'+\ ) \Big\}dm'd.$$ (12)

Here $f_{[m,m']\rightarrow[m-\Delta,m'+\Delta]}$ is the probability of transferring money Δ from an agent with money m to an agent with money m' per unit time. This probability, multiplied by the occupation numbers $P(m)$ and $P(m')$, gives the rate of transitions from the state $[m, m']$ to the state $[m - \Delta, m' + \Delta]$. The first term in (12) gives the depopulation rate of the state m. The second term in (12) describes the reverse process, where the occupation number $P(m)$ increases. When the two terms are equal, the direct and reverse transitions cancel each other statistically, and the probability distribution is stationary: $dP(m)/dt = 0$. This is the principle of detailed balance.

In physics, the fundamental microscopic equations of motion of particles obey time-reversal symmetry. This means that the probabilities of the direct and reverse processes are exactly equal:

$$f_{[m,m']\rightarrow[m-\Delta,m'+\Delta]} = f_{[m-\Delta,m'+\Delta]\rightarrow[m,m']}.$$ (13)

When (13) is satisfied, the detailed balance condition for (12) reduces to the equation $P(m)$ $P(m') = P(m - \Delta)P(m' + \Delta)$, because the factors f cancel out. The only solution of this equation is the exponential function $P(m) = c \exp(-m/T_m)$, so the Boltzmann-Gibbs distribution is the stationary solution of the Boltzmann kinetic Eq. (12). Notice that the transition probabilities (13) are determined by the dynamical rules of the model, but the equilibrium Boltzmann-Gibbs distribution does not depend on the dynamical rules at all. This is the origin of the universality of the Boltzmann-Gibbs distribution. It shows that it may be possible to find out the stationary distribution without knowing details of the

dynamical rules (which are rarely known very well), as long as the symmetry condition (13) is satisfied.

The models considered in section "The Boltzmann-Gibbs Distribution of Money" have the time-reversal symmetry. The model with the fixed money transfer Δ has equal probabilities (13) of transferring money from an agent with balance m to an agent with balance m' and vice versa. This is also true when Δ is random, as long as the probability distribution of Δ is independent of m and m'. Thus, the stationary distribution $P(m)$ is always exponential in these models.

However, there is no fundamental reason to expect time-reversal symmetry in economics, so (13) may be not valid. In this case, the system may have a nonexponential stationary distribution or no stationary distribution at all. In model (8), the time-reversal symmetry is broken. Indeed, when an agent i gives a fixed fraction γ of his money m_i to an agent with balance m_j, their balances become $(1 - \gamma)m_i$ and $m_j + \gamma m_i$. If we try to reverse this process and appoint an agent j to be the payer and to give the fraction γ of her money, $\gamma(m_j + \gamma m_i)$, to agent i, the system does not return to the original configuration $[m_i, m_j]$. As emphasized by Angle (2006), the payer pays a deterministic fraction of his money, but the receiver receives a random amount from a random agent, so their roles are not interchangeable. Because the proportional rule typically violates the time-reversal symmetry, the stationary distribution $P(m)$ in multiplicative models is typically not exactly exponential. (However, when Δm is a fraction of the total money $m_i + m_j$ of the two agents, the model is time-reversible and has an exponential distribution, as discussed in section "The Boltzmann-Gibbs Distribution of Money.") Making the transfer dependent on the money balance of the payer effectively introduces a Maxwell's demon into the model. That is why the stationary distribution is not exponential, and, thus, does not maximize entropy (4). Another view on the time-reversal symmetry in economic dynamics is presented in Ao (2007).

These examples show that the Boltzmann-Gibbs distribution does not hold for any

conservative model. However, it is universal in a limited sense. For a broad class of models that have time-reversal symmetry, the stationary distribution is exponential and does not depend on the details of the model. Conversely, when the time-reversal symmetry is broken, the distribution may depend on the details of the model. The difference between these two classes of models may be rather subtle. Deviations from the Boltzmann-Gibbs law may occur only if the transition rates f in (13) explicitly depend on the agent's money m or m' in an asymmetric manner. Drăgulescu and Yakovenko (2000) performed a computer simulation where the direction of payment was randomly selected in advance for every pair of agents (i, j). In this case, money flows along directed links between the agents: $i \rightarrow j \rightarrow k$, and the time-reversal symmetry is strongly violated. This model is closer to the real economy, where one typically receives money from an employer and pays it to a grocery store. Nevertheless, the Boltzmann-Gibbs distribution was found in this model, because the transition rates f do not explicitly depend on m and m' and do not violate (13).

In the absence of detailed knowledge of real microscopic dynamics of economic exchanges, the semiuniversal Boltzmann-Gibbs distribution (7) is a natural starting point. Moreover, the assumption of Drăgulescu and Yakovenko (2000) that agents pay the same prices Δm for the same products, independent of their money balances m, seems very appropriate for the modern anonymous economy, especially for purchases over the Internet. There is no particular empirical evidence for the proportional rules (8) or (11). However, the difference between the additive (7) and multiplicative (9) distributions may be not so crucial after all. From the mathematical point of view, the difference is in the implementation of the boundary condition at $m = 0$. In the additive models of section "The Boltzmann-Gibbs Distribution of Money," there is a sharp cutoff of $P(m)$ at $m = 0$. In the multiplicative models of section "Proportional Money Transfers and Saving Propensity," the balance of an agent never reaches $m = 0$, so $P(m)$ vanishes at $m \rightarrow 0$ in a power-law manner. At the same time, $P(m)$ decreases exponentially for large m for both models.

By further modifying the rules of money transfer and introducing more parameters in the models, one can obtain even more complicated distributions (Scafetta and West 2007). However, one can argue that parsimony is the virtue of a good mathematical model, not the abundance of additional assumptions and parameters, whose correspondence to reality is hard to verify.

Statistical Mechanics of Wealth Distribution

In the econophysics literature on exchange models, the terms "money" and "wealth" are often used interchangeably; however, economists emphasize the difference between these two concepts. In this section, we review the models of wealth distribution, as opposed to money distribution.

Models with a Conserved Commodity

What is the difference between money and wealth? One can argue (Drăgulescu and Yakovenko 2000) that wealth w_i is equal to money m_i plus the other property that an agent i has. The latter may include durable material property, such as houses and cars, and financial instruments, such as stocks, bonds, and options. Money (paper cash, bank accounts) is generally liquid and countable. However, the other property is not immediately liquid and has to be sold first (converted into money) to be used for other purchases. In order to estimate the monetary value of property, one needs to know the price p. In the simplest model, let us consider just one type of property, say, stocks s. Then the wealth of an agent i is given by the formula

$$w_i = m_i + ps_i. \qquad (14)$$

It is assumed that the price p is common for all agents and is established by some kind of market process, such as an auction, and may change in time.

It is reasonable to start with a model where both the total money $M = \sum_i m_i$ and the total stock $S = \sum_i s_i$ are conserved (Chakraborti et al.

2001; Chatterjee and Chakrabarti 2006; Ausloos and Pekalski 2007). The agents pay money to buy stock and sell stock to get money, and so on. Although M and S are conserved, the total wealth $W = \sum_i w_i$ is generally not conserved, because of the price fluctuation (Chatterjee and Chakrabarti 2006) in (14). This is an important difference from the money transfer models of section "Statistical Mechanics of Money Distribution." Here the wealth w_i of an agent i, not participating in any transactions, may change when transactions between other agents establish a new price p. Moreover, the wealth w_i of an agent i does not change after a transaction with an agent j. Indeed, in exchange for paying money Δm, agent i receives the stock $\Delta s = \Delta m/p$, so her total wealth (14) remains the same. In principle, the agent can instantaneously sell the stock back at the same price and recover the money paid. If the price p never changes, then the wealth w_i of each agent remains constant, despite transfers of money and stock between agents.

We see that redistribution of wealth in this model is directly related to price fluctuations. One mathematical model of this process was studied in Silver et al. (2002). In this model, the agents randomly change preferences for the fraction of their wealth invested in stocks. As a result, some agents offer stock for sale and some want to buy it. The price p is determined from the market-clearing auction matching supply and demand. Silver et al. (Silver et al. 2002) demonstrated in computer simulations and proved analytically using the theory of Markov processes that the stationary distribution $P(w)$ of wealth w in this model is given by the Gamma distribution, as in (9). Various modifications of this model (Lux 2005), such as introducing monopolistic coalitions, do not change this result significantly, which shows the robustness of the Gamma distribution. For models with a conserved commodity, Chatterjee and Chakrabarti (2006) found the Gamma distribution for a fixed saving propensity and a power law tail for a distributed saving propensity.

Another model with conserved money and stock was studied in Raberto et al. (2003) for an artificial stock market where traders follow different investment strategies: random, momentum, contrarian, and fundamentalist. Wealth distribution in the model with random traders was found have a power-law tail $P(w) \sim 1/w^2$ for large w. However, unlike in most other simulation, where all agents initially have equal balances, here the initial money and stock balances of the agents were randomly populated according to a power law with the same exponent. This raises the question whether the observed power-law distribution of wealth is an artifact of the initial conditions, because equilibrization of the upper tail may take a very long simulation time.

Models with Stochastic Growth of Wealth
Although the total wealth W is not exactly conserved in the models considered in section "Models with a Conserved Commodity," W nevertheless remains constant on average, because the total money M and stock S are conserved. A different model for wealth distribution was proposed in Bouchaud and Mézard (2000). In this model, time evolution of the wealth w_i of an agent i is given by the stochastic differential equation

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \eta_i(t)w_i + \sum_{j(\neq i)} J_{ij}w_j - \sum_{j(\neq i)} J_{ji}w_i, \quad (15)$$

where $\eta_i(t)$ is a Gaussian random variable with mean $\langle \eta \rangle$ and variance $2\sigma^2$. This variable represents growth or loss of wealth of an agent due to investment in stock market. The last two terms describe transfer of wealth between different agents, which is taken to be proportional to the wealth of the payers with the coefficients J_{ij}. So, the model (15) is multiplicative and invariant under the scale transformation $w_i \rightarrow Zw_i$. For simplicity, the exchange fractions are taken to be the same for all agents: $J_{ij} = J/N$ for all $i \neq j$, where N is the total number of agents. In this case, the last two terms in (15) can be written as $J(\langle w \rangle - w_i)$, where $\langle w \rangle = \sum_i w_i/N$ is the average wealth per agent. This case represents a "mean-field" model, where all agents feel the same environment. It can be easily shown that the average

wealth increases in time as $\langle w \rangle_t = \langle w \rangle_0 e^{(\langle \eta \rangle + \sigma^2)t}$. Then, it makes more sense to consider the relative wealth $\widetilde{w}_i = w_i / \langle w \rangle_t$. Equation (15) for this variable becomes

$$\frac{d\widetilde{w}_i}{dt} = \left(\eta_i(t) - \langle \eta \rangle - \sigma^2\right)\widetilde{w}_i + J(1 - \widetilde{w}_i). \quad (16)$$

The probability distribution $P(\widetilde{w}, t)$ for the stochastic differential Eq. (16) is governed by the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \frac{\partial [J(\widetilde{w} - 1) + \sigma^2 \widetilde{w}]P}{\partial \widetilde{w}}$$
$$+ \sigma^2 \frac{\partial}{\partial \widetilde{w}} \left(\widetilde{w} \frac{\partial(\widetilde{w}P)}{\partial \widetilde{w}} \right). \quad (17)$$

The stationary solution ($\partial P / \partial t = 0$) of this equation is given by the following formula:

$$P(\widetilde{w}) = c \frac{e^{\frac{-J}{\sigma^2 \widetilde{w}}}}{\widetilde{w}^{\frac{2+J}{\sigma^2}}}. \quad (18)$$

The distribution (18) is quite different from the Boltzmann-Gibbs (7) and Gamma (9) distributions. Equation (18) has a power-law tail at large $\widetilde{w}$ and a sharp cutoff at small $\widetilde{w}$. Equation (15) is a version of the generalized Lotka-Volterra model, and the stationary distribution (18) was also obtained in Solomon and Richmond (2001, 2002). The model was generalized to include negative wealth in Huang (2004).

Bouchaud and Mézard (2000) used the mean-field approach. A similar result was found for a model with pairwise interaction between agents in Slanina (2004). In this model, wealth is transferred between the agents using the proportional rule (8). In addition, the wealth of the agents increases by the factor $1 + \zeta$ in each transaction. This factor is supposed to reflect creation of wealth in economic interactions. Because the total wealth in the system increases, it makes sense to consider the distribution of relative wealth $P(\widetilde{w})$. In the limit of continuous trading, Slanina (2004) found the same stationary distribution (18). This result was reproduced using a mathematically more involved treatment of this

model in Cordier et al. (2005). Numerical simulations of the models with stochastic noise η in Scafetta et al. (2004a, b) also found a power-law tail for large w.

Let us contrast the models discussed in section "Models with a Conserved Commodity" and "Models with Stochastic Growth of Wealth." In the former case, where money and commodity are conserved and wealth does not grow, the distribution of wealth is given by the Gamma distribution with an exponential tail for large w. In the latter models, wealth grows in time exponentially, and the distribution of relative wealth has a power-law tail for large $\widetilde{w}$. These results suggest that the presence of a power-law tail is a nonequilibrium effect that requires constant growth or inflation of the economy, but disappears for a closed system with conservation laws.

Reviews of the models discussed were also given in Richmond et al. (2006a, b). Because of lack of space, we omit discussion of models with wealth condensation (Bouchaud and Mézard 2000; Ispolatov et al. 1998; Burda et al. 2002; Pianegonda et al. 2003; Braun 2006), where a few agents accumulate a finite fraction of the total wealth, and studies of wealth distribution on networks (Coelho et al. 2005; Iglesias et al. 2003; Di Matteo et al. 2004; Hu et al. 2007). Here we discuss the models with long-range interaction, where any agent can exchange money and wealth with any other agent. A local model, where agents trade only with the nearest neighbors, was studied in Bak et al. (1999).

Empirical Data on Money and Wealth Distributions

It would be very interesting to compare theoretical results for money and wealth distributions in various models with empirical data. Unfortunately, such empirical data are difficult to find. Unlike income, which is discussed in section "Data and Models for Income Distribution," wealth is not routinely reported by the majority of individuals to the government. However, in many countries, when a person dies, all assets must be reported for the purpose of inheritance tax. So, in principle, there exist good statistics of wealth distribution among dead people, which, of course, is different

from the wealth distribution among living people. Using an adjustment procedure based on the age, gender, and other characteristics of the deceased, the UK tax agency, the Inland Revenue, reconstructed the wealth distribution of the whole population of the UK (Revenue and Customs 2003). Figure 5 shows the UK data for 1996 reproduced from Drăgulescu and Yakovenko (2001a). The figure shows the cumulative probability $C(w) = \int_w^\infty P(w')dw'$ as a function of the personal net wealth w, which is composed of assets (cash, stocks, property, household goods, etc.) and liabilities (mortgages and other debts). Because statistical data are usually reported at nonuniform intervals of w, it is more practical to plot the cumulative probability distribution $C(w)$ rather than its derivative, the probability density $P(w)$. Fortunately, when $P(w)$ is an exponential or a power-law function, then $C(w)$ is also an exponential or a power-law function.

The cumulative probability distribution in Fig. 5 is plotted on a log-log scale, where a straight line represents a power-law dependence. The figure shows that the distribution follows a power law $C(w) \propto 1/w^\alpha$ with exponent $\alpha = 1.9$ for wealth greater than about £100,000. The inset in Fig. 5 shows the data on log-linear scale, where the straight line represents an exponential dependence. We observe that below £100,000 the data are well fitted by the exponential distribution $C(w) \propto \exp(-w/T_w)$ with the effective "wealth temperature" $T_w = £60,000$ (which corresponds to the median wealth of £41,000). So, the distribution of wealth is characterized by the Pareto power law in the upper tail of the distribution and the exponential Boltzmann-Gibbs law in the lower part of the distribution for the great majority (about 90%) of the population. Similar results are found for the distribution of income, as discussed in section "Data and Models for Income Distribution." One may speculate that the wealth distribution in the lower part is dominated by distribution of money, because the corresponding people do not have other significant assets, so the results of section "Statistical Mechanics of Money Distribution" give the Boltzmann-Gibbs law. On the other hand, the upper tail of the wealth distribution is dominated by investment assess, where the results of section "Models with Stochastic Growth of Wealth" give the Pareto law. The power law was studied by many researchers for the upper-tail data, such as the Forbes list of the 400 richest people (Klass et al. 2007; Sinha 2006), but much less attention was paid to the lower part of the wealth distribution. Curiously, Abdul-Magd (2002) found that the wealth distribution in ancient Egyptian society was consistent with (18).

For direct comparison with the results of section "Statistical Mechanics of Money Distribution," it would be very interesting to find data on the distribution of money, as opposed to the distribution of wealth. Making a reasonable

Statistical Mechanics Approach to Econophysics, Fig. 5 Cumulative probability distribution of net wealth in the UK shown on log-log and log-linear (*inset*) scales. *Points* represent the data from the Inland Revenue, and *solid lines* are fits to the exponential (Boltzmann-Gibbs) and power (Pareto) laws. (Reproduced from (Drăgulescu and Yakovenko 2001a))
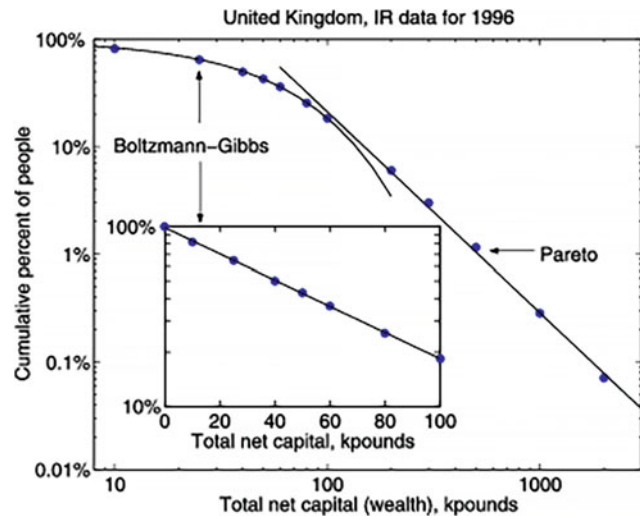
assumption that most people keep most of their money in banks, one can approximate the distribution of money by the distribution of balances on bank accounts. (Balances on all types of bank accounts, such as checking, saving, and money manager, associated with the same person should be added up.) Despite imperfections (people may have accounts with different banks or not keep all their money in banks), the distribution of balances on bank accounts would give valuable information about the distribution of money. The data for a big enough bank would be representative of the distribution in the whole economy. Unfortunately, it has not been possible to obtain such data thus far, even though it would be completely anonymous and not compromise the privacy of bank clients.

Measuring the probability distribution of money would be very useful, because it determines how much people can, in principle, spend on purchases without going into debt. This is different from the distribution of wealth, where the property component, such as house, car, or retirement investment, is effectively locked up and, in most cases, is not easily available for consumer spending. So, although wealth distribution may reflect the distribution of economic power, the distribution of money is more relevant for consumption. Money distribution can be useful for determining prices that maximize revenue of a manufacturer (Drăgulescu and Yakovenko 2000). If a price p is set too high, few people can afford it, and, if a price is too low, the sales revenue is small, so the optimal price must be in-between. The fraction of the population who can afford to pay the price p is given by the cumulative probability $C(p)$, so the total sales revenue is proportional to $pC(p)$. For the exponential distribution $C(p) = \exp(-p/T_m)$, the maximal revenue is achieved at $p = T_m$, that is, at the optimal price equal to the average amount of money per person (Drăgulescu and Yakovenko 2000). Indeed, the prices of mass-market consumer products, such as computers, electronics goods, and appliances, remain stable for many years at a level determined by their affordability to the population, whereas the technical parameters of these products at the same price level

improve dramatically owing to technological progress.

Data and Models for Income Distribution

In contrast to money and wealth distributions, a lot more empirical data are available for the distribution of income r from tax agencies and population surveys. In this section, we first present empirical data on income distribution and then discuss theoretical models.

Empirical Data on Income Distribution

Empirical studies of income distribution have a long history in the economics literature (Kakwani 1980; Champernowne and Cowell 1998; Atkinson and Bourguignon 2000). Following the work by Pareto (1897), much attention was focused on the power-law upper tail of the income distribution and less on the lower part. In contrast to more complicated functions discussed in the literature, Drăgulescu and Yakovenko (2001b) introduced a new idea by demonstrating that the lower part of income distribution can be well fitted with a simple exponential function $P(r) = c\exp(-r/T_r)$ characterized by just one parameter, the "income temperature" T_r. Then it was recognized that the whole income distribution can be fitted by an exponential function in the lower part and a power-law function in the upper part (Drăgulescu and Yakovenko 2001a, 2003), as shown in Fig. 6. The straight line on the log-linear scale in the inset of Fig. 6 demonstrates the exponential Boltzmann-Gibbs law, and the straight line on the log-log scale in the main panel illustrates the Pareto power law. The fact that income distribution consists of two distinct parts reveals the two-class structure of American society (Yakovenko and Silva 2005; Silva and Yakovenko 2005). Coexistence of the exponential and power-law distributions is known in plasma physics and astrophysics, where they are called the "thermal" and "superthermal" parts (Hasegawa et al. 1985; Desai et al. 2003; Collier 2004). The boundary between the lower and upper classes can be defined as the intersection point of the exponential and power-law fits in Fig. 6. For

Statistical Mechanics Approach to Econophysics, Fig. 6 Cumulative probability distribution of tax returns for the USA in 1997 shown on log-log and log-linear (*inset*) scales. *Points* represent the Internal Revenue Service (IRS) data, and *solid lines* are fits to the exponential and power-law functions. (Reproduced from (Drăgulescu and Yakovenko 2003))
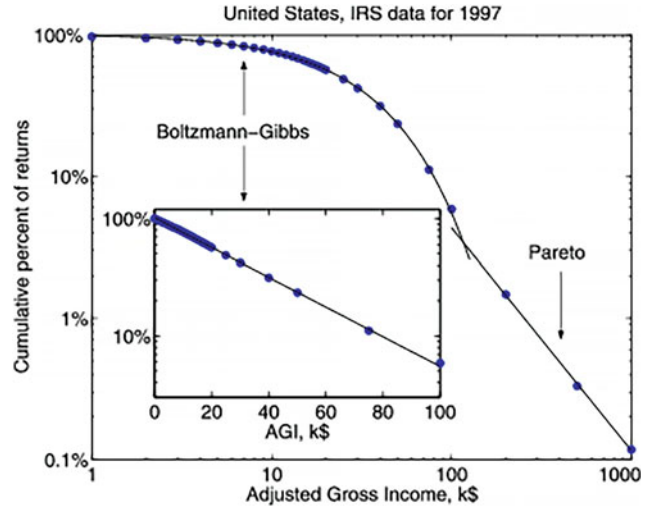

1997, the annual income separating the two classes was about $120,000. About 3% of the population belonged to the upper class, and 97% belonged to the lower class.

Silva and Yakovenko (2005) studied time evolution of income distribution in the USA during 1983–2001 on the basis of data from the Internal Revenue Service (IRS), the government tax agency. The structure of the income distribution was found to be qualitatively the same for all years, as shown in Fig. 7. The average income in nominal dollars approximately doubled during this time interval. So, the horizontal axis in Fig. 7 shows the normalized income r/T_r, where the "income temperature" T_r was obtained by fitting of the exponential part of the distribution for each year. The values of T_r are shown in Fig. 7. The plots for the 1980s and 1990s are shifted vertically for clarity. We observe that the data points in the lower-income part of the distribution collapse on the same exponential curve for all years. This demonstrates that the shape of the income distribution for the lower class is extremely stable and does not change in time, despite the gradual increase of the average income in nominal dollars. This observation suggests that the lower-class distribution is in statistical, "thermal" equilibrium.

On the other hand, Fig. 7 shows that the income distribution in the upper class does not rescale and significantly changes in time. Silva

and Yakovenko (2005) found that the exponent α of the power law $C(r) \propto 1/r^{\alpha}$ decreased from 1.8 in 1983 to 1.4 in 2000. This means that the upper tail became "fatter." Another useful parameter is the total income of the upper class as the fraction f of the total income in the system. The fraction f increased from 4% in 1983 to 20% in 2000 (Silva and Yakovenko 2005). However, in 2001, α increased and f decreased, indicating that the upper tail was reduced after the stock market crash at that time. These results indicate that the upper tail is highly dynamical and not stationary. It tends to swell during the stock market boom and shrink during the bust. Similar results were found for Japan (Souma 2001, 2002; Fujiwara et al. 2003; Aoyama et al. 2003).

Although relative income inequality within the lower class remains stable, the overall income inequality in the USA has increased significantly as a result of the tremendous growth of the income of the upper class. This is illustrated by the Lorenz curve and the Gini coefficient shown in Fig. 8. The Lorenz curve (Kakwani 1980) is a standard way of representing income distribution in the economics literature. It is defined in terms of two coordinates $x(r)$ and $y(r)$ depending on a parameter r:
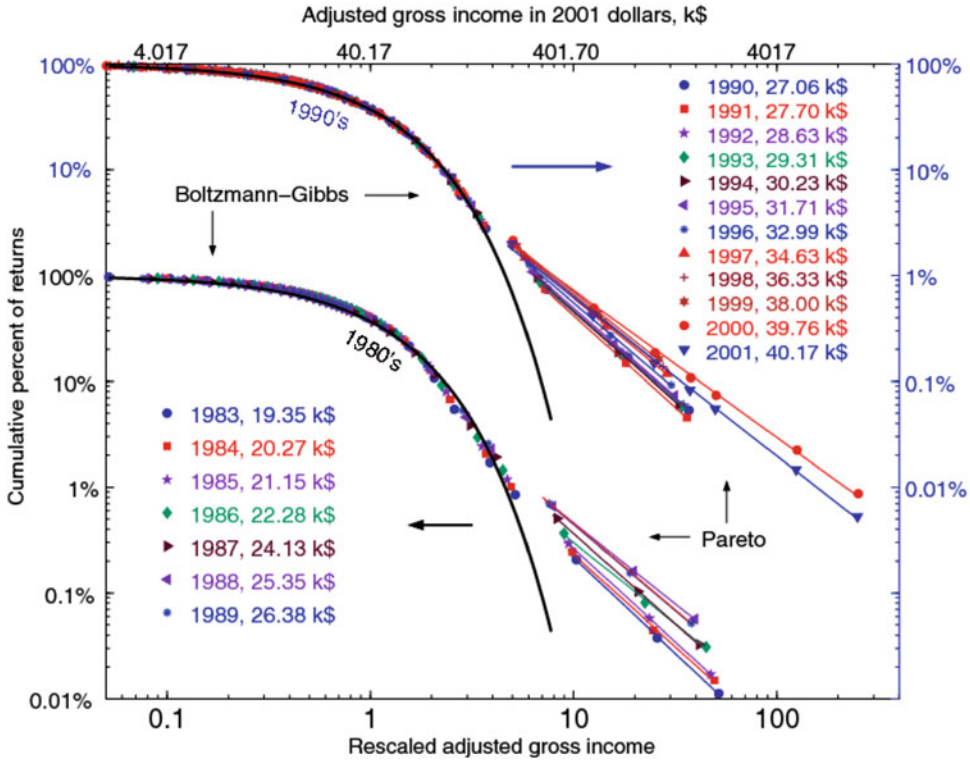
Statistical Mechanics Approach to Econophysics, Fig. 7 Cumulative probability distribution of tax returns plotted on log-log scale versus r/T_r (the annual income r normalized by the average income T_r in the exponential part of the distribution). The IRS data points are for 1983–2001, and the *columns of numbers* give the values of T_r for the corresponding years. (Reproduced from Silva and Yakovenko (2005))

$$x(r) = \int_0^r P(r')\mathrm{d}r',$$

$$y(r) = \frac{\int_0^r r'P(r')\mathrm{d}r'}{\int_0^\infty r'P(r')\mathrm{d}r'}. \tag{19}$$

The horizontal coordinate $x(r)$ is the fraction of the population with income below r, and the vertical coordinate $y(r)$ is the fraction of the income this population accounts for. As r changes from 0 to ∞, x and y change from 0 to 1 and parametrically define a curve in the (x, y)-plane.

Figure 8 shows the data points for the Lorenz curves in 1983 and 2000, as computed by the IRS (Strudler et al. 2003). Drăgulescu and Yakovenko (2001b) analytically derived the Lorenz curve formula $y = x + (1 - x)\ln(1 - x)$ for a purely exponential distribution $P(r) = c\exp(-r/T_r)$. This formula is shown by the red line in Fig. 8 and describes the 1983 data reasonably well.

However, for 2000, it is essential to take into account the fraction f of income in the upper tail, which modifies the Lorenz formula as follows (Drăgulescu and Yakovenko 2003; Yakovenko and Silva 2005; Silva and Yakovenko 2005):

$$y = (1 - f)[x + (1 - x)\ln(1 - x)] + f\Theta(x - 1). \tag{20}$$

The last term in (20) represent the vertical jump of the Lorenz curve at $x = 1$, where a very small percentage of the population in the upper class accounts for a substantial fraction f of the total income. The blue curve representing (20) fits the 2000 data in Fig. 8 very well.

The deviation of the Lorenz curve from the straight diagonal line in Fig. 8 is a certain measure of income inequality. Indeed, if everybody had the same income, the Lorenz curve would be a diagonal line, because the fraction of income would be

Statistical Mechanics Approach to Econophysics, Fig. 8 Lorenz plots for income distribution in 1983 and 2000. The data points are from the IRS (Strudler et al. 2003), and the theoretical curves represent (20) with f from Fig. 7. *Inset*: The *closed circles* are the IRS data 113 for the Gini coefficient G, and the *open circles* show the theoretical formula $G = (1 + f)/2$. (Reproduced from Silva and Yakovenko (2005))
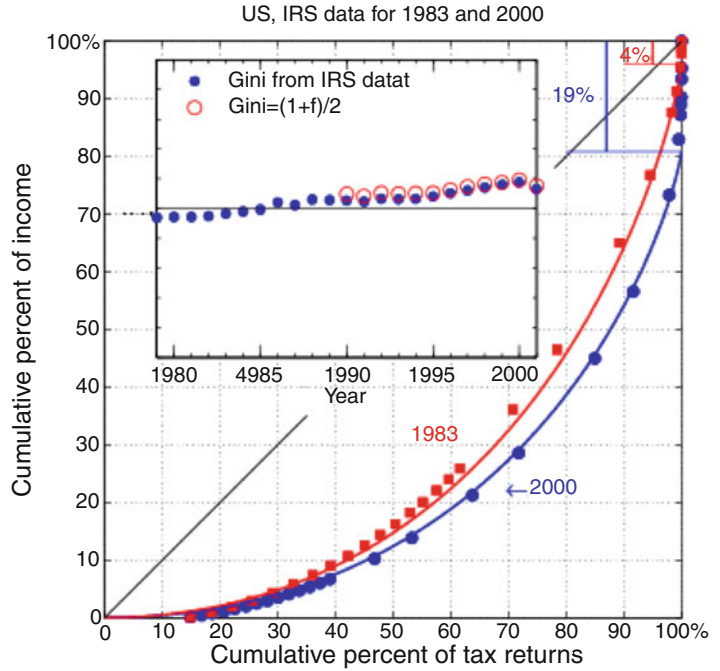
proportional to the fraction of the population. The standard measure of income inequality is the so-called Gini coefficient $0 \leq G \leq 1$, which is defined as the area between the Lorenz curve and the diagonal line, divided by the area of the triangle beneath the diagonal line (Kakwani 1980). Time evolution of the Gini coefficient, as computed by the IRS (Strudler et al. 2003), is shown in the inset of Fig. 8. Drăgulescu and Yakovenko (2001b) derived analytically the result that $G = 1/2$ for a purely exponential distribution. In the first approximation, the values of G shown in the inset of Fig. 8 are indeed close to the theoretical value 1/2. If we take into account the upper tail using (20), the formula for the Gini coefficient becomes $G = (1 + f)/2$ (Silva and Yakovenko 2005). The inset in Fig. 8 shows that this formula is a very good fit to the IRS data for the 1990s using the values of f deduced from Fig. 7. The values $G < 1/2$ for the 1980s cannot be captured by this formula, because the Lorenz data points are slightly above the theoretical curve for 1983 in Fig. 8. Overall, we observe that income inequality has been increasing for the last 20 years, because of swelling of the Pareto tail, but decreased in 2001 after the stock market crash.

Thus far we have discussed the distribution of individual income. An interesting related question is the distribution $P_2(r)$ of family income $r = r_1 + r_2$, where r_1 and r_2 are the incomes of spouses. If individual incomes are distributed exponentially $P(r) \propto \exp(-r/T_r)$, then

$$P_2(r) = \int_0^r dr' P(r') P(r - r') = cr \exp(-r/T_r),$$

(21)

where c is a normalization constant. Figure 9 shows that (21) is in good agreement with the family income distribution data from the US Census Bureau (Drăgulescu and Yakovenko 2001b). In (21), we assumed that incomes of spouses are uncorrelated. This simple approximation is indeed supported by the scatter plot of incomes of spouses shown in Fig. 10. Each family is represented in this plot by two points (r_1, r_2) and (r_2, r_1) for symmetry. We observe that the density of points is approximately constant along the lines of constant family income $r_1 + r_2 = \text{const}$, which indicates that incomes of spouses are approximately uncorrelated. There is no significant clustering of points along the diagonal $r_1 = r_2$, that is,

Statistical Mechanics Approach to Econophysics, Fig. 9 Probability distribution of family income for families with two adults (US Census Bureau data). *Solid line*: fit to (21). (Reproduced from Drăgulescu and Yakovenko (2001b))
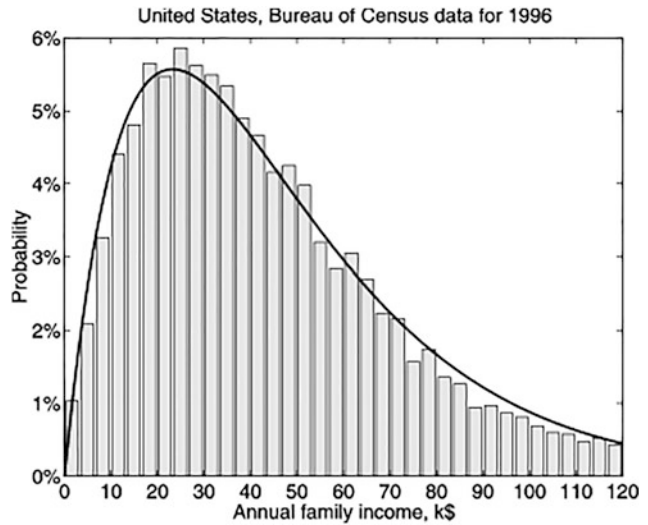
Statistical Mechanics Approach to Econophysics, Fig. 10 Scatter plot of the spouses' incomes (r_1, r_2) and (r_2, r_1) based on the data from the Panel Study of Income Dynamics (PSID). (Reproduced from Drăgulescu and Yakovenko (2003))
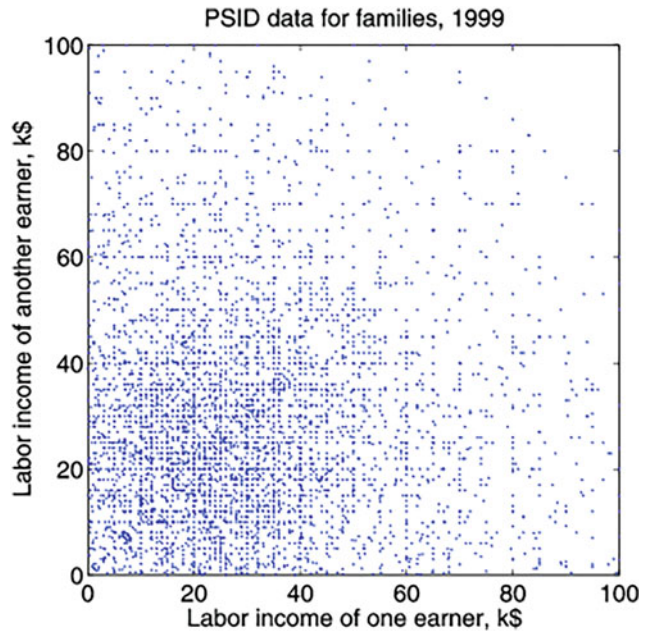
no strong positive correlation of spouses' incomes.

The Gini coefficient for the family income distribution (21) was calculated in Drăgulescu and Yakovenko (2001b) as $G = 3/8 = 37.5\%$. Figure 11 shows the Lorenz quintiles and the Gini coefficient for 1947–1994 plotted from the US Census Bureau data. The solid line, representing the Lorenz curve calculated from (21), is in good agreement with the data.

The systematic deviation for the top 5% of earners results from the upper tail, which has a less pronounced effect on family income than on individual income, because of income averaging in the family. The Gini coefficient, shown in the inset of Fig. 11, is close to the calculated value of 37.5%. Moreover, the average G for the developed capitalist countries of North America and western Europe, as determined by the World Bank

Statistical Mechanics Approach to Econophysics, Fig. 11 Lorenz plot for family income calculated from (21), compared with the US Census data points. *Inset*: The US Census data points for the Gini coefficient for families, compared with the theoretically calculated value 3/8=37.5%. (Reproduced from Drăgulescu and Yakovenko (2001b))
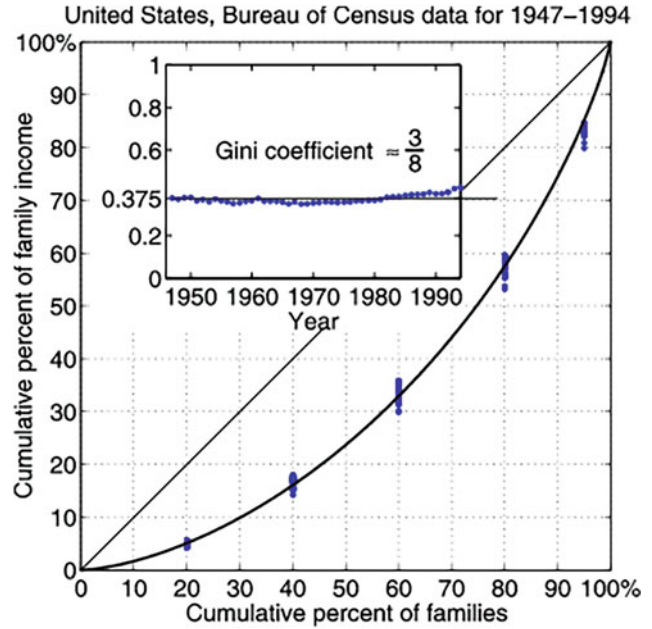
(Drăgulescu and Yakovenko 2003), is also close to the calculated value of 37.5%.

Income distribution has been examined in econophysics papers for different countries: Japan (Ferrero 2004; Souma 2001, 2002; Fujiwara et al. 2003; Aoyama et al. 2003; Souma and Nirei 2005, Nirei and Souma 2007; Ferrero 2005), Germany (Clementi and Gallegati 2005a; Clementi et al. 2007), the UK (Ferrero 2004, 2005; Richmond et al. 2006b; Clementi and Gallegati 2005a; Clementi et al. 2007), Italy (Clementi et al. 2007; Clementi and Gallegati 2005b; Clementi et al. 2006), the USA (Clementi and Gallegati 2005a; Rawlings et al. 2004), India (Sinha 2006), Australia (Di Matteo et al. 2004; Clementi et al. 2006; Banerjee et al. 2006), and New Zealand (Ferrero 2004, 2005). The distributions are qualitatively similar to the results presented in this section. The upper tail follows a power law and comprises a small fraction of the population. To fit the lower part of the distribution, the use of exponential, Gamma, and log-normal distributions was reported in different papers. Unfortunately, income distribution is often reported by statistical agencies for households, so it is difficult to differentiate between one-earner and two-earner income distributions.

Some papers reported the use of interpolating functions with different asymptotic behavior for low and high incomes, such as the Tsallis function (Ferrero 2005) and the Kaniadakis function (Clementi et al. 2007). However, the transition between the lower and upper classes is not smooth for the US data shown in Figs. 6 and 7, so such functions would not be useful in this case. The special case is income distribution in Argentina during the economic crisis, which shows a time-dependent bimodal shape with two peaks (Ferrero 2005).

Theoretical Models of Income Distribution

Having examined the empirical data on income distribution, let us now discuss theoretical models. Income r_i is the influx of money per unit time to an agent i. If the money balance m_i is analogous to energy, then the income r_i would be analogous to power, which is the energy flux per unit time. So, one should conceptually distinguish between the distributions of money and income. While money is regularly transferred from one agent to another in pairwise transactions, it is not typical for agents to trade portions of their income. Nevertheless, indirect transfer of income may occur when one employee is promoted and

another demoted, while the total annual budget is fixed, or when one company gets a contract, whereas another one loses it, etc. A reasonable approach, which has a long tradition in the economics literature (Gibrat 1931; Kalecki 1945; Champernowne 1953), is to treat individual income r as a stochastic process and study its probability distribution. In general, one can study a Markov process generated by a matrix of transitions from one income to another. In the case where income r changes by a small amount Δr over a time period Δt, the Markov process can be treated as income diffusion . Then one can apply the general Fokker-Planck equation (Lifshitz and Pitaevskii 1981) to describe evolution in time t of the income distribution function $P(r, t)$ (Silva and Yakovenko 2005):

$$
\frac{\partial P}{\partial t} = \frac{\partial}{\partial r}\left[AP + \frac{\partial(BP)}{\partial r}\right],
$$
$$
A = -\frac{\langle \Delta r \rangle}{\Delta t}, \quad B = \frac{\langle (\Delta r)^2 \rangle}{2\Delta t}. \tag{22}
$$

The coefficients A and B in (22) are determined by the first and second moments of income changes per unit time. The stationary solution $\partial_t P = 0$ of (22) obeys the following equation with the general solution:

$$
\frac{\partial(BP)}{\partial r} = -AP,
$$
$$
P(r) = \frac{c}{B(r)}\exp\left(-\int^r \frac{A(r')}{B(r')}\,dr'\right). \tag{23}
$$

For the lower part of the distribution, it is reasonable to assume that Δr is independent of r, that is, the changes of income are independent of income itself. This process is called additive diffusion (Silva and Yakovenko 2005). In this case, the coefficients in (22) are constants A_0 and B_0. Then (23) gives the exponential distribution $P(r) \propto \exp(-r/T_r)$, with the effective income temperature $T_r = B_0/A_0$. Notice that a meaningful stationary solution (23) requires that $A > 0$, that is, $\langle \Delta r \rangle < 0$. The coincidence of this result with the Boltzmann-Gibbs exponential law (1) and (7) is not accidental. Indeed, instead of considering

pairwise interaction between particles, one can derive (1) by considering energy transfers between a particle and a big reservoir, as long as the transfer process is "additive" and does not involve a Maxwell-demon-like discrimination. Stochastic income fluctuations are described by a similar process. So, although money and income are different concepts, they may have similar distributions, because they are governed by similar mathematical principles. It was shown explicitly in Drăgulescu and Yakovenko (2000), Slanina (2004), and Cordier et al. (2005) that the models of pairwise money transfer can be described in a certain limit by the Fokker-Planck equation.

On the other hand, for the upper tail of the income distribution, it is reasonable to expect that $\Delta r \propto r$, that is, income changes are proportional to income itself. This is known as the proportionality principle of Gibrat (1931), and the process is called multiplicative diffusion (Silva and Yakovenko 2005). In this case, $A = ar$ and $B = br^2$, and (23) gives the power-law distribution $P(r) \propto 1/r^{\alpha + 1}$, with $\alpha = 1 + a/b$.

Generally, lower-class income comes from wages and salaries, where the additive process is appropriate, whereas upper-class income comes from bonuses, investments, and capital gains, calculated in percentages, where the multiplicative process applies (Milaković 2005). However, the additive and multiplicative processes may coexist. An employee may receive a cost-of-living rise calculated in percentages (the multiplicative process) and a merit rise calculated in dollars (the additive process). In this case, we have $A = A_0 + ar$ and $B = B_0 + br^2 = b(r_0^2 + r^2)$, where $r_0^2 = B_0/b$. Substituting these expressions into (23), we find

$$
P(r) = c\,\frac{e^{-\left(\frac{r_0}{T_r}\right)\arctan\left(\frac{r}{r_0}\right)}}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{\frac{1+a}{2b}}}. \tag{24}
$$

The distribution (24) interpolates between the exponential law for low r and the power law for high r, because either the additive or the multiplicative process dominates in the corresponding

limit. The crossover between the two regimes takes place at $r \sim r_0$, where the additive and multiplicative contributions to B are equal. The distribution (24) has three parameters: the "income temperature" $T_r = A_0/B_0$, the Pareto exponent $\alpha = 1 + a/b$, and the crossover income r_0. It is a minimal model that captures the salient features of the empirical income distribution shown in Fig. 6. A mathematically similar, but more economically oriented model was proposed in Souma and Nirei (2005) and Nirei and Souma (2007), where labor income and asset accumulation are described by the additive and multiplicative processes correspondingly. A general stochastic process with additive and multiplicative noise was studied numerically in Takayasu et al. (1997), but the stationary distribution was not derived analytically. A similar process with discrete time increments was studied by Kesten (1973). Recently, a formula similar to (24) was obtained in Fiaschi and Marsili (2007).

To verify the multiplicative and additive hypotheses empirically, it is necessary to have data on income mobility, that is, the income changes Δr of the same people from one year to another. The distribution of income changes $P(\Delta r| r)$ conditional on income r is generally not available publicly, although it can be reconstructed by researchers at the tax agencies. Nevertheless, the multiplicative hypothesis for the upper class was quantitatively verified in Fujiwara et al. (2003) and Aoyama et al. (2003) for Japan, where tax identification data is published for the top taxpayers.

The power-law distribution is meaningful only when it is limited to high enough incomes $r > r_0$. If all incomes r from 0 to ∞ follow a purely multiplicative process, then one can change to a logarithmic variable $x = \ln(r/r_*)$ in (22) and show that it gives a Gaussian time-dependent distribution $P_t(x) \propto \exp(-x^2/2\sigma^2 t)$ for x, that is, the log-normal distribution for r, also known as the Gibrat distribution (Gibrat 1931). However, the width of this distribution increases linearly in time, so the distribution is not stationary. This was pointed out by Kalecki (1945) a long time ago, but the log-normal distribution is still widely used for fitting income distribution, despite this fundamental

logical flaw in its justification. In a classic paper, Champernowne (1953) showed that a multiplicative process gives a stationary power-law distribution when a boundary condition is imposed at $r_0 \neq 0$. Later, this result was rediscovered by econophysicists (Levy and Solomon 1996; Sornette and Cont 1997). In our (24), the exponential distribution of the lower class effectively provides such a boundary condition for the power law of the upper class. Notice also that (24) reduces to (18) in the limit $r_0 \to 0$, which corresponds to purely multiplicative noise $B = br^2$.

There are alternative approaches to income distribution in the economics literature. One of them, proposed by Lydall (1959), involves social hierarchy. Groups of people have leaders, who have leaders of a higher order, and so on. The number of people decreases geometrically (exponentially) with the increase of the hierarchical level. If individual income increases by a certain factor (i.e., multiplicatively) when moving to the next hierarchical level, then income distribution follows a power law (Lydall 1959). However, the original argument of Lydall can be easily modified to produce an exponential distribution. If individual income increases by a certain amount, that is, income increases linearly with the hierarchical level, then income distribution is exponential. The latter process seems to be more realistic for moderate incomes below \$ 100,000. A similar scenario is the Bernoulli trials (Feller 1966), where individuals have a constant probability of increasing their income by a fixed amount. We see that the deterministic hierarchical models and the stochastic models of additive and multiplicative income mobility represent essentially the same ideas.

Other Applications of Statistical Physics

Statistical physics was applied to a number of other subjects in economics. Because of lack of space, only two such topics are briefly discussed in this section.

Economic Temperatures in Different Countries

As discussed in section "Empirical Data on Money and Wealth Distributions" and "Empirical Data on Income Distribution," the distributions of money, wealth, and income are often described by exponential functions for the majority of the population. These exponential distributions are characterized by the parameters T_m, T_w, and T_r, which are mathematically analogous to temperature in the Boltzmann-Gibbs distribution (1). The values of these parameters, extracted from the fits of the empirical data, are generally different for different countries, i.e., different countries have different economic "temperatures." For example, Drăgulescu and Yakovenko (2001a) found that the US income temperature was 1.9 times higher than the UK income temperature in 1998 (using the exchange rate of dollars to pounds at that time). Also, there was ±25% variation between income temperatures of different states within the USA (Drăgulescu and Yakovenko 2001a).

In physics, a difference of temperatures allows one to set up a thermal machine. In was argued in Drăgulescu and Yakovenko (2000) that the difference of money or income temperatures between different countries allows one to extract profit in international trade. Indeed, as discussed at the end of section "Empirical Data on Money and Wealth Distributions," the prices of goods should be commensurate with money or income temperature , because otherwise people cannot afford to buy those goods. So, an international trading company can buy goods at a low price T_1 in a "low-temperature" country and sell them at a high price T_2 in a "high-temperature" country. The difference of prices $T_2 - T_1$ would be the profit of the trading company. In this process, money (the analog of energy) flows from the "high-temperature" to the "low-temperature" country, in agreement with the second law of thermodynamics, whereas products flow in the opposite direction. This process very much resembles what is going on in the global economy now. In this framework, the perpetual trade deficit of the USA is the consequence of the second law of thermodynamics and the difference of temperatures between the USA and "low-temperature"

countries, such as China. Similar ideas were developed in more detail in Mimkes and Aruka (2005) and Mimkes (2006a), including a formal Carnot cycle for international trade.

The statistical physics approach demonstrates that profit originates from statistical non-equilibrium (the difference of temperatures), which exists in the global economy. However, it does not answer the question what is the origin of this difference. By analogy with physics, one would expect that the money flow should reduce the temperature difference and, eventually, lead to equilibrization of temperatures. In physics, this situation is known as the "thermal death of the universe." In a completely equilibrated global economy, it would be impossible to make profit by exploiting differences of economic temperatures between different countries. Although globalization of the modern economy does show a tendency toward equilibrization of living standards in different countries, this process is far from straightforward, and there are many examples contrary to equilibrization. This interesting and timely subject certainly requires further study.

Society as a Binary Alloy

In 1971, Thomas Schelling (Schelling 1971) proposed the now-famous mathematical model of segregation. He considered a lattice, where the sites can be occupied by agents of two types, for example, blacks and whites in the problem of racial segregation. He showed that if the agents have some probabilistic preference for the neighbors of the same type, the system spontaneously segregates into black and white neighborhoods. This mathematical model is similar to the so-called Ising model, which is a popular model for studying phase transitions in physics. In this model, each lattice site is occupied by a magnetic atom, whose magnetic moment has only two possible orientations, up or down. The interaction energy between two neighboring atoms depends on whether their magnetic moments point in the same or in the opposite directions. In physics language, the segregation found by Schelling represents a phase transition in this system.

Another similar model is the binary alloy, a mixture of two elements which attract or repel

each other. It was noticed in Mimkes (1995) that the behavior of actual binary alloys is strikingly similar to social segregation. In the following papers (Mimkes 2000, 2006b), this mathematical analogy was developed further and compared with social data. Interesting concepts, such as the coexistence curve between two phases and the solubility limit, were discussed in this work. The latter concept means that a small amount of one substance dissolves in another up to some limit, but phase separation (segregation) develops for higher concentrations. Recently, similar ideas were rediscovered in Jego and Roehner (2007), Stauffer and Schulze (2007), and Dall'Asta et al. (2007). The vast experience of physicists in dealing with phase transitions and alloys may be helpful for practical applications of such models (Lim et al. 2007).

Future Directions, Criticism, and Conclusions

The statistical models described in this review are quite simple. It is commonly accepted in physics that theoretical models are not intended to be photographic copies of reality, but rather to be caricatures, capturing the most essential features of a phenomenon with a minimal number of details. With only few rules and parameters, the models discussed in section "Statistical Mechanics of Money Distribution," "Statistical Mechanics of Wealth Distribution," and "Data and Models for Income Distribution" reproduce spontaneous development of stable inequality, which is present in virtually all societies. It is amazing that the calculated Gini coefficients, $G = 1/2$ for individuals and $G = 3/8$ for families, are actually very close to the US income data, as shown in Figs. 8 and 11. These simple models establish a baseline and a reference point for development of more sophisticated and more realistic models. Some of these future directions are outlined below.

Future Directions

Agents with a Finite Lifespan

The models discussed in this review consider immortal agents who live forever, like atoms. However, humans have a finite lifespan. They enter the economy as young people and exit at an old age. Evolution of income and wealth as functions of age is studied in economics using the so-called overlapping-generations model. The absence of the age variable was one of the criticisms of econophysics by the economist Paul Anglin (Anglin 2005). However, the drawback of the standard overlapping-generations model is that there is no variation of income and wealth between agents of the same age, because it is a representative-agent model. It would be best to combine stochastic models with the age variable. Also, to take into account inflation of average income, (22) should be rewritten for relative income, in the spirit of (17). These modifications would allow one to study the effects of demographic waves, such as baby boomers, on the distributions of income and wealth.

Agent-Based Simulations of the Two-Class Society

The empirical data presented in section "Empirical Data on Income Distribution" show quite convincingly that the US population consists of two very distinct classes characterized by different distribution functions. However, the theoretical models discussed in section "Statistical Mechanics of Money Distribution" and "Statistical Mechanics of Wealth Distribution" do not produce two classes, although they do produce broad distributions. Generally, not much attention has been paid in the agent-based literature to simulation of two classes. One exception is Wright (2005), in which spontaneous development of employers and employees classes from initially equal agents was simulated (Wright 2007). More work in this direction would be certainly desirable.

Access to Detailed Empirical Data

A great amount of statistical information is publicly available on the Internet, but not for all types

of data. As discussed in section "Empirical Data on Money and Wealth Distributions," it would be very interesting to obtain data on the distribution of balances on bank accounts, which would give information about the distribution of money (as opposed to wealth). As discussed in section "Theoretical Models of Income Distribution," it would be useful to obtain detailed data on income mobility, to verify the additive and multiplicative hypotheses for income dynamics. Income distribution is often reported as a mix of data on individual income and family income, when the counting unit is a tax return (joint or single) or a household. To have a meaningful comparison with theoretical models, it is desirable to obtain clean data where the counting unit is an individual. Direct collaboration with statistical agencies would be very useful.

Economies in Transition

Inequality in developed capitalist countries is generally quite stable. The situation is very different for former socialist countries making a transition to a market economy. According to the World Bank data (Drăgulescu and Yakovenko 2003), the average Gini coefficient for family income in eastern Europe and the former Soviet Union jumped from 25% in 1988 to 47% in 1993. The Gini coefficient in the socialist countries before the transition was well below the equilibrium value of 37.5% for market economies. However, the fast collapse of socialism left these countries out of market equilibrium and generated a much higher inequality. One may expect that, with time, their inequality will decrease to the equilibrium value of 37.5%. It would be very interesting to trace how fast this relaxation takes place. Such a study would also verify whether the equilibrium level of inequality is universal for all market economies.

Relation to Physical Energy

The analogy between energy and money discussed in section "Conservation of Money" is a formal mathematical analogy. However, actual physical energy with low entropy (typically in the form of fossil fuel) also plays a very important role in the modern economy, as the basis of current human technology. In view of the looming energy and climate crisis, it is imperative to find realistic ways for making a transition from the current "disposable" economy based on "cheap" and "unlimited" energy and natural resources to a sustainable one. Heterogeneity of human society is one of the important factors affecting such a transition. Econophysics, at the intersection of energy, entropy, economy, and statistical physics, may play a useful role in this quest (Defilla 2007).

Criticism from Economists

As econophysics is gaining popularity, some criticism has appeared from economists (Anglin 2005), including those who are closely involved with the econophysics movement (Lux 2005; 2008; Gallegati et al. 2006). This reflects a longstanding tradition in economic and social sciences of writing critiques on different schools of thought. Much of the criticism is useful and constructive and is already being accommodated in econophysics work. However, some criticism results from misunderstanding or miscommunication between the two fields and some from significant differences in scientific philosophy. Several insightful responses to the criticism have been published (McCauley 2006; Richmond et al. 2006c; Rosser 2006a); see also (Stauffer 2004; Rosser 2006b). In this section, we briefly address the issues that are directly related to the material discussed in this review.

Awareness of Previous Economics Literature

One complaint of Anglin (2005), Lux (2005), (2008), and Gallegati et al. (2006) is that physicists are not well aware of the previous economics literature and either rediscover known results or ignore well-established approaches. To address this issue, it is useful to keep in mind that science itself is a complex system, and scientific progress is an evolutionary process with natural selection. The sea of scientific literature is enormous, and nobody knows it all. Recurrent rediscovery of regularities in the natural and social world only confirms their validity. Independent rediscovery usually brings a different perspective, broader

applicability range, higher accuracy, and better mathematical treatment, so there is progress even when some overlap with previous results exists. Physicists are grateful to economists for bringing relevant and specific references to their attention. Since the beginning of modern econophysics, many old references have been uncovered and are now routinely cited.

However, not all old references are relevant to the new development. For example, Gallegati et al. (2006) complained that the econophysics literature on income distribution ignores the so-called Kuznets hypothesis (Kuznets 1955). The Kuznets hypothesis postulates that income inequality first rises during an industrial revolution and then decreases, producing an inverted-U-shaped curve. Gallegati et al. (2006) admitted that, to date, the large amount of literature on the Kuznets hypothesis is inconclusive. Kuznets (1955) mentioned that this hypothesis applies to the period from colonial times to the 1970s; however, the empirical data for this period are sparse and not very reliable. The econophysics literature deals with reliable volumes of data for the second half of the twentieth century, collected with the introduction of computers. It is not clear what is the definition of industrial revolution and when exactly it starts and ends. The chain of technological progress seems to be continuous (steam engine, internal combustion engine, cars, plastics, computers, Internet), so it is not clear where the purported U-curve is supposed to be placed in time. Thus, the Kuznets hypothesis appears to be, in principle, unverifiable and unfalsifiable. The original paper by Kuznets (1955) actually does not contain any curves, but it has one table filled with made-up, imaginary data! Kuznets admits that he has "neither the necessary data nor a reasonably complete theoretical model" (p. 12 in Kuznets (1955)). So, this paper is understandably ignored by the econophysics community. In fact, the data analysis for 1947–1984 shows amazing stability of income distribution (Levy 1987), consistent with Fig. 11. The increase of inequality in the 1990s resulted from growth of the upper class relative to the lower class, but the relative inequality within the lower class remains very stable, as shown in Fig. 7.

Reliance on Visual Data Analysis

Another complaint of Gallegati et al. (2006) is that econophysicists favor graphic analysis of data over the formal and "rigorous" testing prescribed by mathematical statistics, as favored by economists. This complaint goes against the trend of all sciences to use increasingly sophisticated data visualization for uncovering regularities in complex systems. The thick IRS publication 1304 (Internal Revenue Service 1999) is filled with data tables, but has virtually no graphs. Despite the abundance of data, it gives a reader no idea about income distribution, whereas plotting the data immediately gives insight. However, intelligent plotting is the art with many tools, which not many researchers have mastered. The author completely agrees with Gallegati et al. (2006) that too many papers mindlessly plot any kind of data on a log-log scale, pick a finite interval, where any smooth curved line can be approximated by a straight line, and claim that there is a power law. In many cases, replotting the same data on a log-linear scale converts a curved line into a straight line, which means that the law is actually exponential.

Good visualization is extremely helpful in identifying trends in complex data, which can then be fitted to a mathematical function; however, for a complex system, such a fit should not be expected with infinite precision. The fundamental laws of physics, such as Newton's law of gravity or Maxwell's equations, are valid with enormous precision. However, the laws in condensed matter physics, uncovered by experimentalists with a combination of visual analysis and fitting, usually have much lower precision, at best 10% or so. Most of these laws would fail the formal criteria of mathematical statistics. Nevertheless these approximate laws are enormously useful in practice, and everyday devices engineered on the basis of these laws work very well for all of us.

Because of the finite accuracy, different functions may produce equally good fits. Discrimination between the exponential, Gamma, and lognormal functions may not be always possible (Banerjee et al. 2006). However, the exponential function has fewer fitting parameters, so it is

preferable on the basis of simplicity. The other two functions can simply mimic the exponential function with a particular choice of the additional parameters (Banerjee et al. 2006). Unfortunately, many papers in mathematical statistics introduce too many fitting parameters into complicated functions, such as the generalized beta distribution mentioned in Gallegati et al. (2006). Such overparameterization is more misleading than insightful for data fitting.

Quest for Universality

Gallegati et al. (2006) criticized physicists for trying to find universality in economics data. They also seemed to equate the concepts of power law, scaling, and universality. These are three different, albeit overlapping, concepts. Power laws usually apply only to a small fraction of data at the high ends of various distributions. Moreover, the exponents of these power laws are usually nonuniversal and vary from case to case. Scaling means that the shape of a function remains the same when its scale changes. However, the scaling function does not have to be a power-law function. A good example of scaling is shown in Fig. 7, where income distributions for the lower class collapse on the same exponential line for about 20 years of data. We observe amazing universality of income distribution, unrelated to a power law. In a general sense, the diffusion equation is universal, because it describes a wide range of systems, from dissolution of sugar in water to a random walk in the stock market.

Universalities are not easy to uncover, but they form the backbone of regularities in the world around us. This is why physicists are so interested in them. Universalities establish the first-order effect, and deviations represent the second-order effect. Different countries may have somewhat different distributions, and economists often tend to focus on these differences. However, this focus on details misses the big picture that, in the first approximation, the distributions are quite similar and universal.

Theoretical Modeling of Money, Wealth, and Income

It was pointed out in Anglin (2005), Gallegati et al. (2006), and Lux (2008) that many econophysics papers confuse or misuse the terms for money, wealth, and income. It is true that terminology is sloppy in many papers and should be refined. However, the terms in Drăgulescu and Yakovenko (2000) and Chakraborti and Chakrabarti (2000) are quite precise, and this review clearly distinguishes between these concepts in section "Statistical Mechanics of Money Distribution," "Statistical Mechanics of Wealth Distribution," and "Data and Models for Income Distribution."

One contentious issue is about conservation of money. Gallegati et al. (2006) agree that "transactions are a key economic process, and they are necessarily conservative," that is, money is indeed conserved in transactions between agents. However, Anglin (2005), Gallegati et al. (2006), and Lux (2008) complain that the models of conservative exchange do not consider production of goods, which is the core economic process and the source of economic growth. Material production is indeed the ultimate goal of an economy, but it does not violate conservation of money by itself. One can grow coffee beans, but nobody can grow money on a money tree. Money is an artificial economic device that is designed to be conserved. As explained in section "Statistical Mechanics of Money Distribution," the money transfer models implicitly assume that money in transactions is voluntarily paid for goods and services generated by production for the mutual benefit of the parties. In principle, one can introduce a billion variables to keep track of every coffee bean and other product of the economy. What difference would it make for the distribution of *money*? Despite the claims in Anglin (2005) and Gallegati et al. (2006), there is no contradiction between models of conservative exchange and the classic work of Adam Smith and David Ricardo. The difference is only in the focus: We keep track of money, whereas they keep track of coffee beans, from production to consumption. These approaches address different questions, but do not contradict each other. Because money constantly circulates

in the system as payment for production and consumption, the resulting statistical distribution of money may very well not depend on what exactly is produced and in what quantities.

In principle, the models with random transfers of money should be considered as a reference point for developing more sophisticated models. Despite the totally random rules and "zero intelligence" of the agents, these models develop well-characterized, stable, and stationary distributions of money. One can modify the rules to make the agents more intelligent and realistic and see how much the resulting distribution changes relative to the reference one. Such an attempt was made in Lux (2005) by modifying the model of Silver et al. (2002) with various more realistic economic ingredients. However, despite the modifications, the resulting distributions were essentially the same as in the original model. This example illustrates the typical robustness and universality of statistical models: Modifying details of microscopic rules does not necessarily change the statistical outcome.

Another misconception, elaborated in Lux (2005), (2008), is that the money transfer models discussed in section "Statistical Mechanics of Money Distribution" imply that money is transferred by fraud, theft, and violence, rather than voluntarily. One should keep in mind that the catchy labels "theft-and-fraud," "marriage-and-divorce," and "yard-sale" were given to the money transfer models by the journalist Brian Hayes (2002) in a popular article. Econophysicists who originally introduced and studied these models do not subscribe to this terminology, although the early work of Angle (1986) did mention violence as one source of redistribution. In the opinion of the author, it is indeed difficult to justify the proportionality rule (8), which implies that agents with high balances pay proportionally greater amounts in transactions than agents with low balances. However, the additive model of Drăgulescu and Yakovenko (2000), where money transfers Δm are independent of money balances m_i of the agents, does not have this problem. As explained in section "The Boltzmann-Gibbs Distribution of Money," this model simply means that all agents pay the same

prices for the same product, although prices may be different for different products. So, this model is consistent with voluntary transactions in a free market.

McCauley (2006) argued that conservation of money is violated by credit. As explained in section "Models with Debt," credit does not violate conservation law, but creates positive and negative money without changing net worth. Negative money (debt) is as real as positive money. McCauley (2006) claimed that money can be easily created with the tap of a computer key via credit. Then why would an employer not tap the key and double salaries, or a funding agency double research grants? Because budget constraints are real. Credit may provide a temporary relief, but sooner or later it has to be paid back. Allowing debt may produce a double-exponential distribution as shown in Fig. 3, but it does not change the distribution fundamentally.

As discussed in section "Conservation of Money," a central bank or a central government can inject new money into the economy. As discussed in section "Statistical Mechanics of Wealth Distribution," wealth is generally not conserved. As discussed in section "Data and Models for Income Distribution," income is different from money and is described by a different model (22). However, the empirical distribution of income shown in Fig. 6 is qualitatively similar to the distribution of wealth shown in Fig. 5, and we do not have data on money distribution.

Conclusions

The "invasion" of physicists into economics and finance at the turn of the millennium is a fascinating phenomenon. The physicist Joseph McCauley proclaims that "Econophysics will displace economics in both the universities and boardrooms, simply because what is taught in economics classes doesn't work" (Ball 2006; McCauley 2004; Farmer et al. 2005; Samanidou et al. 2007; Econophysics forum 2008). Although there is some truth in his arguments (McCauley 2006), one may consider a less radical scenario. Econophysics may become a branch of

economics, in the same way as games theory, psychological economics, and now agent-based modeling became branches of economics. These branches have their own interests, methods, philosophy, and journals. The main contribution from the infusion of new ideas from a different field is not in answering old questions, but in raising new questions. Much of the misunderstanding between economists and physicists happens not because they are getting different answers, but because they are answering different questions.

The subject of income and wealth distributions and social inequality was very popular at the turn of another century and is associated with the names of Pareto, Lorenz, Gini, Gibrat, and Champernowne, among others. Following the work by Pareto, attention of researchers was primarily focused on the power laws. However, when physicists took a fresh, unbiased look at the empirical data, they found a different, exponential law for the lower part of the distribution. The motivation for looking at the exponential law, of course, came from the Boltzmann-Gibbs distribution in physics. Further studies provided a more detailed picture of the two-class distribution in a society. Although social classes have been known in political economy since Karl Marx, the realization that they are described by simple mathematical distributions is quite new. Demonstration of the ubiquitous nature of the exponential distribution for money, wealth, and income is one of the new contributions produced by econophysics.

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