LINES IN $\mathbb{P}^3$

Points in $\mathbb{P}^3$ correspond to (projective equivalence classes) of nonzero vectors in $\mathbb{R}^4$. That is, the point in $\mathbb{P}^3$ with homogeneous coordinates $[X : Y : Z : W]$ is the line $[\mathbf{v}]$ spanned by the nonzero vector

$$\mathbf{v} := \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \in \mathbb{R}^4.$$

Similarly, planes in $\mathbb{P}^3$ correspond to (projective equivalence classes) of covectors

$$\phi := \begin{bmatrix} a & b & c & d \end{bmatrix} \in (\mathbb{R}^4)^*,$$

where $[\phi] = [\![a : b : c : d]\!]$ is the hyperplane defined in homogeneous coordinates by $\phi(\mathbf{v}) = 0$, that is,

$$aX + bY + cZ + dW = 0. \tag{1}$$

That is, the point $[X : Y : Z : W]$ lies on the plane $[\![a : b : c : d]\!]$ if and only if (1) is satisfied.

Thus lines and planes in $\mathbb{P}^3$ are defined in homogeneous coordinates by vectors in the vector space $\mathsf{V} := \mathbb{R}^4$ and covectors in its dual vector space $\mathsf{V}^* = (\mathbb{R}^4)^*$. Moreover, the orthogonal complement $\mathbf{v}^\perp$ of the line $\mathbb{R}\mathbf{v} \in \mathbb{R}^4$ is the hyperplane in $\mathbb{R}^4$ defined by the covector $\mathbf{v}^\dagger$, which is the *transpose* of $\mathbf{v}$.

How can you describe *lines* in $\mathbb{P}^3$ in a similar way by homogeneous coordinates?

Exterior Outer Products

Recall that $\mathfrak{so}(n)$ denotes the set of $n \times n$ *skew-symmetric* matrices, that is $X \in \mathsf{Mat}_n$ such that $X + X^\dagger = 0$. The *exterior outer product* is the alternating bilinear map:

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathfrak{so}(n) \\ (\mathbf{v}, \mathbf{w}) &\longmapsto \mathbf{v} \wedge \mathbf{w} := \mathbf{w}\mathbf{v}^\dagger - \mathbf{v}^\dagger\mathbf{w}. \end{aligned}$$

Date: December 12, 2013.

The following facts are easy to verify:

(1)
$$(\mathbf{u}\wedge\mathbf{v}) : \mathbf{w} \longmapsto (\mathbf{v}\cdot\mathbf{w})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{w}$$
(2) If $n = 3$, then $(\mathbf{u}\wedge\mathbf{v})(\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

(3) $\mathbf{w}$ and $\mathbf{v}$ are linearly dependent if and only if $\mathbf{w}\wedge\mathbf{v} = 0$.

(4) If $\mathbf{w}$ and $\mathbf{v}$ are linearly independent, then the projective equivalence class $[\mathbf{w}\wedge\mathbf{v}] \in \mathsf{P}\big(\mathfrak{so}(n)\big)$ depends only the plane $\mathbb{R}\langle\mathbf{w},\mathbf{v}\rangle$ spanned by $\mathbf{w},\mathbf{v}$.

(5) The orthogonal complement of the plane $\mathbb{R}\langle\mathbf{w},\mathbf{v}\rangle \subset \mathsf{V}$ lies in the kernel $\mathsf{Ker}(\mathbf{w}\wedge\mathbf{v})$:
$$\mathbb{R}\langle\mathbf{w},\mathbf{v}\rangle^{\perp} \subset \mathsf{Ker}(\mathbf{w}\wedge\mathbf{v}).$$

Because of (4) every 2-dimensional linear subspace L (plane through the origin) of $\mathbb{R}^n$ determines an element of the projective space $\mathsf{P}\big(\mathfrak{so}(n)\big)$; the corresponding homogeneous coordinates are called the *Plücker coordinates* of the plane, or the corresponding projective line $\mathsf{P}(L) \subset \mathsf{P}(\mathbb{R}^n)$.

Plücker coordinates in $\mathbb{P}^3$

Let $\mathsf{V} = \mathbb{R}^4$ and $\Lambda = \mathfrak{so}(4)$ the 6-dimensional vector space of 4×4 skew-symmetric matrices. Then lines in $\mathbb{P}^3 = \mathsf{P}(\mathsf{V})$ correspond to 2-dimensional linear subspaces of V, which in turn correspond to projective equivalence classes of certain nonzero elements $\mathbf{v}\wedge\mathbf{w}\in\Lambda$. Which elements of Λ correspond to lines in $\mathbb{P}^3$?

Since $\dim(\mathsf{V}) = 4$, the plane $\mathbb{R}\langle\mathbf{v},\mathbf{w}\rangle \neq \mathsf{V}$ so there exists a nonzero vector $\mathbf{n}$ *normal* to this plane. By the above, $\mathbf{n}$ lies in the kernel of the skew-symmetric matrix $\mathbf{v}\wedge\mathbf{w}$. Thus lines in $\mathbb{P}^3$ determine nonzero *singular* matrices in Λ.

By the spectral theorem for real skew-symmetric matrices, the eigenvalues are purely imaginary and occur in complex conjugate pairs. For example, when $n = 3$, every element in $\mathfrak{so}(3)$ must have a zero eigenvalue (if zero occurs with higher multiplicity the matrix itself must be zero). This implies that every element of $\mathsf{SO}(3)$ is a rotation, for example.

When $n = 4$, then if 0 is not an eigenvalue, then the set of eigenvalues must be of the form
$$\{r_1 i, -r_1 i, r_2 i, -r_2 i\},$$
where $r_1, r_2 \in \mathbb{R}$ are nonzero. In particular such a matrix has determinant $r_1^2 r_2^2 > 0$. Since $\mathsf{Det}(\mathbf{v}\wedge\mathbf{w}) = 0$, but $\mathbf{v}\wedge\mathbf{w}\neq 0$, the multiplicity of 0 as an eigenvalue is exactly *two,* so the matrix $\mathbf{v}\wedge\mathbf{w}$ has *rank* 2.

When n is even, skew-symmetric matrices in $\mathfrak{so}(n)$ have the following curious property. In general the determinant of an $n\times n$ is a degree

n polynomial in its entries. When n is even, there is a degree $n/2$ polynomial $\mathcal{P}$ on $\mathfrak{so}(n)$ (called the *Pfaffian*) such that if $M \in \mathfrak{so}(n)$, then

$$\mathsf{Det}(M) = \mathcal{P}(M)^2$$

for $M \in \mathfrak{so}(n)$. That is, in even dimensions, the determinant of a skew-symmetric matrix is a *perfect square.* For example, when $n = 2$, the general skew-symmetric matrix is

$$M = \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix}$$

which has determinant y^2. Thus $\mathcal{P}(M) = y$, for example.

When $n = 4$, the Pfaffian is a quadratic polynomial. The general element of $\mathfrak{so}(4)$ is:

$$M \ := \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0. \end{bmatrix}$$

which has determinant

$$\mathsf{Det}(M) = \big(m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34}\big)^2$$

so the Pfaffian is (up to a choice of -1):

$$\mathcal{P}(M) = m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34}.$$

The vector space Λ has dimension 6, with coordinates

$$m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}.$$

Thus projective equivalence classes of nonzero 4×4 skew-symmetric matrices is the projective space

$$\mathsf{P}(\Lambda) \cong \mathbb{P}^5$$

with homogeneous coodinates

$$[m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}].$$

The nonzero singular matrices (namely, those of rank two), are those for which $\mathcal{P}(M) = 0$, which is just the homogeneous quadratic polynomial condition:

$$m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} = 0$$

This defines a *quadric hypersurface* $\mathcal{Q}$ in $\mathbb{P}^5$. Since it is defined by one equation in a 5-dimensional space, this quadratic has dimesion 4.

Intuitively, we would expect that the space of lines in $\mathbb{P}^3$ has dimension 4. A generic line $\ell \subset \mathbb{P}^3$ is not ideal and does not pass through the origin. In that case there is a point

$$p(\ell) \in \mathbb{R}^3 \setminus \{0\}$$

closest to the origin $0 \in \mathbb{R}^3$. These points form a 3-dimensional space $\mathbb{R}^3 \setminus \{0\}$.

Any point $p \in \mathbb{R}^3 \setminus \{0\}$ is the closest point $p(\ell)$ for some ℓ. Namely, look at the plane $W(p)$ containing p and normal to the vector from 0 to p. Any line ℓ on $W(p)$ passing through p satisfies $p(\ell) = p$. The set of all lines ℓ with $p(\ell) = p$ forms a $\mathbb{P}^1$, which is one-dimensional. Thus lines in $\mathbb{P}^3$ are parametrized by a $4 = 3 + 1$-dimensional space.

This space is the quadric $\mathcal{Q}$ defined above.

Just as quadric surfaces in $\mathbb{P}^3$ can be parametrized as tori $S^1 \times S^1$, the 4-dimensional quadric hypersurface in $\mathbb{P}^5$ can be parametrized by $S^2 \times S^2$. Namely make the elementary linear substitution

$$\begin{aligned} X &:= (m_{14} + m_{23})/2, \quad A := (m_{14} - m_{23})/2, \\ Y &:= (m_{13} - m_{24})/2, \quad B := (m_{13} + m_{24})/2, \\ Z &:= (m_{12} + m_{34})/2, \quad C := (m_{12} - m_{34})/2. \end{aligned}$$

so that

$$\begin{aligned} \mathcal{P}(M) &= m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} \\ &= X^2 - A^2 + Y^2 - B^2 + Z^2 - C^2. \end{aligned}$$

Thus $\mathcal{Q}$ is the quadric in $\mathbb{P}^5$ consisting of points with homogeneous coordinates $[X : Y : Z : A : B : C]$ satisfying

$$X^2 + Y^2 + Z^2 = A^2 + B^2 + C^2.$$

Since the coordinates are real at least one is nonzero, this common sum-of-squares is positive. By rescaling we may suppose that that $X^2 + Y^2 + Z^2 = 1$ and $A^2 + B^2 + C^2 = 1$. Each of these equations describes a unit sphere in a 3-dimensional Euclidean space. Furthermore the coordinates (A, B, C) and (X, Y, Z) are independent of one another (we are looking at a *direct-sum decomposition* of $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$), so that the quadric $\mathcal{Q}$ looks like $S^2 \times S^2$.

Orthogonal Complement and Involution

Since $\mathcal{P}$ is a homogeneous quadratic function on the vector space Λ, it arises from a symmetric bilinear form $\mathcal{P}$ on Λ by the usual correspondences:

$$\mathcal{P}(X) = \mathcal{P}(X, X),$$
$$\mathcal{P}(X, Y) := \frac{1}{2}\big(\mathcal{P}(X+Y) - \mathcal{P}(X) - \mathcal{P}(Y)\big)$$

Explicitly,

$$\mathcal{P}(M, N) = \frac{1}{2}(m_{14}n_{23} + m_{23}n_{14} - m_{13}n_{24} - m_{24}n_{13} + m_{12}n_{34} + m_{34}n_{12}).$$

The usual inner product (dot product) on $\mathfrak{so}(4)$ is given by

$$\begin{aligned} M \cdot N &= -\frac{1}{2}\mathsf{tr}(MN) \\ &= m_{12}n_{12} + m_{13}n_{13} + m_{14}n_{14} + m_{23}n_{23} + m_{24}n_{24} + m_{34}n_{34} \end{aligned}$$

Since the bilinear forms $\mathcal{P}$ and the above dot product define linear isomorphisms $\Lambda \xrightarrow{\cong} \mathsf{V}^*$, they are related by a linear isomorphism $\Lambda \xrightarrow{\mathcal{I}} \Lambda$ defined by:

$$\Lambda \xrightarrow{\mathfrak{I}} \Lambda$$
$$M \longmapsto \begin{bmatrix} 0 & m_{34} & -m_{24} & m_{23} \\ -m_{34} & 0 & m_{14} & -m_{13} \\ m_{24} & -m_{14} & 0 & m_{12} \\ -m_{23} & m_{13} & -m_{12} & 0. \end{bmatrix},$$

that is,

$$\mathcal{I}(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) := (m_{34}, -m_{24}, m_{14}, m_{14}, -m_{13}, m_{23})$$

Clearly $\mathcal{I} \circ \mathcal{I} = I$; such a transformation is called an *involution.*

Geometrically, if $M \in \mathcal{Q}$ corresponds to a 2-dimensional linear subspace $L \subset \mathsf{V}$, then $\mathcal{I}(M)$ corresponds to its orthogonal complement $L^{\perp} \subset \mathsf{V}$.

If $p \in \mathbb{P}^3$ is a point corresponding to a 1-dimensional linear subspace $L \subset \mathsf{V}$, then its dual plane $p^* \subset \mathbb{P}^3$ corresponds to the orthogonal complemenr $L^{\perp}$. (The homogeneous coordinates of p^* form the *transpose* of the vector formed by the homogneous coordinates of p.) Then $\mathcal{I}$ maps lines through p to the lines contained in the plane p^*.

Here is a basic example. Take p to be the origin $(0,0,0)$ in the standard affine patch; then p^* is the ideal plane. The line through 0 in the direction (a, b, c) has Plücker coordinates

$$M := \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ -a & -b & -c & 0 \end{bmatrix}.$$

Its dual is the ideal line, which in the ideal plane $\mathbb{P}^2_\infty$ has homogeneous coordinates $[\![a : b : c]\!]$ (that is, the line defined in homogeneous coordinates $aX + bY + cZ = 0$). In $\mathbb{P}^3$ this line has Plücker coordinates:

$$\mathcal{I}(M) := \begin{bmatrix} 0 & c & -b & 0 \\ -c & 0 & a & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Relation to Orth

In an earlier handout was defined the alternating trlinear function Orth which is a kind of four-dimensional cross product. It can be defined in terms of the involution $\mathcal{I}$ and exterior outer product $\wedge$:

$$\mathsf{Orth}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathcal{I}\big((\mathbf{w} \wedge \mathbf{u})(\mathbf{v})\big)$$

Three points $[\mathbf{u}], [\mathbf{v}], [\mathbf{w}] \in \mathbb{P}^3$ (where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathsf{V}$ are nonzero vectors are colinear if and only if $\mathsf{Orth}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$. Otherwise they span a plane in $\mathbb{P}^3$ represented by $[\mathsf{Orth}(\mathbf{u}, \mathbf{v}, \mathbf{w})^\dagger]$.

Dually, suppose $\phi, \psi, \xi \in \mathsf{V}^*$ are nonzero covectors. The corresponding planes

$$[\phi], [\psi], [\xi] \subset \mathbb{P}^3$$

meet in a single point if and only if ϕ, ψ, ξ are linearly independent, in which case

$$[\phi] \cap [\psi] \cap [\xi] = [\mathsf{Orth}(\phi^\dagger, \psi^\dagger, \xi^\dagger)].$$