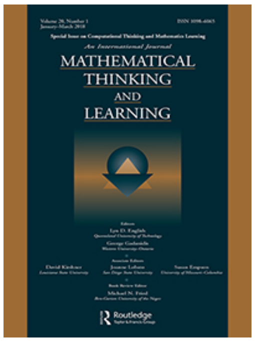

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Cultivating Computational Thinking Practices and Mathematical Habits of Mind in Lattice Land

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ABSTRACT

There is a great deal of overlap between the set of practices collected under the term "computational thinking" and the mathematical habits of mind that are the focus of much mathematics instruction. Despite this overlap, the links between these two desirable educational outcomes are rarely made explicit, either in classrooms or in the literature. This paper presents Lattice Land, a computational learning environment and accompanying curriculum designed to support the development of mathematical habits of mind and promote computational thinking practices in high-school mathematics classrooms. Lattice Land is a mathematical microworld where learners explore geometrical concepts by manipulating polygons drawn with discrete points on a plane. Using data from an implementation in a low-income, urban public high school, we show how the design of Lattice Land provides an opportunity for learners to use computational thinking practices and develop mathematical habits of mind, including tinkering, experimentation, pattern recognition, and formalizing hypothesis in conventional mathematical notation. We present Lattice Land as a restructuration of geometry, showing how this new and novel representational approach facilitates learners in developing computational thinking and mathematical habits of mind. The paper concludes with a discussion of the interplay between computational thinking and mathematical habits of mind, and how the thoughtful design of computational learning environments can support meaningful learning at the intersection of these disciplines.

Introduction

Technology has long been seen as having transformative potential in the context of education. As early as the 1970s, Papert and colleagues envisioned a system of education that would integrate *computational thinking* into everyday life (Papert, 1972; Papert, 1980). Though at the time, technology was not widely accessible enough to realize this vision, advances in computing in the last half-century have led to the growing need to make computational thinking a part of every students' educational experience. Putting this idea into practice, Papert and colleagues probed how constructionist math environments could drive this integration and allow learners of all levels and differing interests to engage deeply in activities that are both mathematically and computationally rich (e.g., Abelson & DiSessa, 1986; Cuoco & Goldenberg, 1996; Eisenberg, 2002; Harel & Papert, 1990; Noss & Hoyles, 1987; Papert, 1980, 1990, 1996; Resnick et al., 2009; Wilensky, 1995).

The contemporary conversation pushing to bring computational thinking to all learners was sparked by Jeannette Wing's (2006) piece titled Computational Thinking, where she argued: "to

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Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/hmtl.

reading, writing, and arithmetic, we should add computational thinking to every child's analytical ability" (Wing, 2006, p. 33). Since its publishing, this piece has generated a great deal of debate, much of which relates to the nature of computational thinking, defining its constituent skills, and figuring out how it relates to other long-standing educational goals, like problem-solving, critical thinking, and mathematical habits of mind. One way forward to understand the nature of computational thinking and its relationship to other disciplines is through the creation of blended curricula that infuse computational thinking practices into topics from other domains. In doing so, the destination domain can serve as a context for developing and using computational thinking practices, while also presenting learners with new, computationally facilitated approaches for exploring disciplinary ideas.

In this paper, we argue that computational thinking practices and mathematical habits of mind are distinct, yet mutually supportive constructs and that through the careful design of computational learning environments and curricula, the two have a home in contemporary mathematics classrooms. As Wilensky (1995) notes, new technologies "expand mathematics *content* beyond the boundaries circumscribed by school…to make abstract mathematical concepts concrete, to explore areas of mathematics previously inaccessible and to create new mathematics" (p. 257) To make this argument, we present data from a classroom study conducted with Lattice Land, a mathematical microworld designed to allow learners to explore geometrical concepts, develop and employ computational thinking practices, and foster mathematical habits of mind. Through their work with Lattice Land, students confront foundational mathematics questions related to the nature of proofs, the role of mathematical formalisms, and engage in authentic mathematical practices of inquiry, experimentation, and questioning. We draw from the mathematical habits of mind outlined by Cuoco, Goldenberg, and Mark (1996) and used for the Connected Geometry curriculum (Goldenberg, 1999), which resonates with the Lattice Land curriculum in its focus on mathematical practices over content and its use of geometry to connect to many other fields of mathematics. At the same time, the computational context of the inquiry and the design of the activities allow learners to employ numerous computational thinking practices, including debugging, identifying patterns, generating abstractions, and using computers to solve problems. To ground these practices in the computational thinking literature, we use the Computational Thinking in Math and Science Practices Taxonomy (Weintrop et al., 2016). By using these frameworks as interpretive lenses on the classroom data, we begin to see how technological tools can be brought to bear in mathematics classrooms to foster mathematical habits of mind alongside the development of computational thinking practices.

The structure of the paper is as follows: we begin by reviewing the relevant literature that informed the design of Lattice Land and shapes our conceptualization of computational thinking and mathematical habits of mind. Next, we present the design of Lattice Land, the curriculum used, and details of the study that was conducted. The findings section follows, focusing on two specific activities that allowed learners to employ computational thinking practices in service of developing mathematical habits of mind. The paper concludes with a discussion of Lattice Land, how its design facilitated the learning that was observed, and a larger discussion on the relationship between computational thinking and mathematics.

Literature review

Microworlds and dynamic geometry software

In mathematics education, Papert (1980) pioneered the idea of a virtual "Mathland" where learners could explore and express mathematical ideas in a native language. This idea of a *microworld* "centers on the creation of a rule-governed environment made accessible for manipulation and exploration by the learner" (Edwards, 1998, p. 64). Since its inception, the concept of a microworld has been leveraged to create countless active learning environments, particularly in mathematics.

One area of mathematics for which novel curricula and microworlds of this kind have been designed is geometry. Geometry is deeply connected to other mathematical topics (e.g., number theory, algebra) as well as non-mathematical topics (e.g., art, astronomy, design), and therefore offers an excellent entry point into student thinking (Goldenberg, 1999). In particular, there are numerous examples of *dynamic geometry software* (DGS), which allow users to create geometric constructions with ease, to replicate with speed, and to transform images in real time (Sinclair & Jackiw, 2010). These computational tools opened completely new ways of looking at canonical Euclidean geometry (Hohenwarter & Fuchs, 2004; Scher, 1999), allowing students to act more like practicing mathematicians in the ways in which they discussed with peers and reported findings amongst each other (Yerushalmy & Houde, 1986). DGS add the ability for students to construct knowledge in ways that are not possible in non-computational settings; they "permit activities that need high-level thinking" (Edwards & Jones, 2006, p. 30). These activities resonate strongly with canonical computational thinking practices (Grover & Pea, 2013; Weintrop et al., 2016). For example, visualizations that vary dynamically promote tinkering and debugging, and highlight abstract interdisciplinary connections. DGS have grown tremendously from Geometric Supposer (Yerushalmy & Houde, 1986), to modern-day Cabri (Laborde, 2002), Geometer's Sketchpad (Jackiw, 1991), and GeoGebra (Hohenwarter & Fuchs, 2004), and their success has resulted in broad adoption in classrooms across the world. DGS have been used to shift the focus from low-level details to high-level thinking, to develop *mathematical habits of mind* (Cuoco & Goldenberg, 1996).

Mathematics habits of mind

As new technological tools are reshaping the way scientists and mathematicians understand the world, students must also learn and think in increasingly computational ways (Orton et al., 2016). However, though experts recognize the importance of both *content knowledge* and *mathematical practices*, the former receives far more attention (Achieve, 2008; CCSS-M, 2010; NCTM, 2000). We developed a curriculum focused on practices, on *mathematical habits of mind*—skills honed through thoughtful engagement with mathematics (Cuoco et al., 1996). These practices are not limited to a single domain but "students can apply it in the context of geometry, trigonometry, calculus, data and statistics, or other advanced courses" (Achieve, 2008, p. 4). Mathematical habits of mind are, fundamentally, the ways in which mathematicians think about mathematics (Lim & Selden, 2009).

It is especially important, in this age of rapid technological growth, that we prepare students to be adaptable and creative problem solvers. This includes fostering mathematical habits of mind such as creating visualizations, making strategic guesses, tinkering and systematic experimentation, using precise language and definitions, making logical connections (e.g., recognizing patterns) and looking for invariants, reasoning about discrete processes and continuity, and using varied perspectives to approach a problem (Cuoco et al., 1996). While some of these practices resonate particularly well with geometry and others more with algebra, this framework does not attempt to privilege specific content. More broadly speaking, such habits of mind transcend subject levels and are crucial to our understanding of the world (Cuoco et al., 1996).

Computational thinking

The recent push for computational thinking as a focus of educational efforts stems from the idea that knowledge and skills from the field of computer science have far-reaching applications from which everyone can benefit: "[Computational thinking] represents a universally applicable attitude and skill set everyone, not just computer scientists, would be eager to learn and use" (Wing, 2006, p. 33). As computation plays a growing role in our world, this argument is becoming more salient; however, it is not new. Papert (1980), a pioneer in the use of computers as a tool for learning, argued that "computer presence could contribute to mental processes not only instrumentally but in more essential, conceptual ways, influencing how people think even when they are far removed from physical contact with a computer" (p. 4). He even went so far as to foresee the rise of the contemporary conceptualization of computational thinking, stating: "Computer languages that simultaneously provide a means of control over the computer and offer new and powerful descriptive languages for thinking will undoubtedly be carried into the general culture" (Papert, 1980, p. 98).

The contemporary discussion of computational thinking begins with Wing's (2006) piece. In the wake of its publishing, a pair of National Research Council meetings were convened to discuss the scope, nature, and educational implications of computational thinking. The report that emerged from those meetings included a list of over 20 high-level skills and practices that computational thinking might include, such as problem abstraction and decomposition, heuristic reasoning, search strategies, and knowledge of computer science concepts like parallel processing, machine learning, and recursion (NRC, 2010). A review of the state of computational thinking identified much scholarship contributing to the emergence and conceptualization of computational thinking, but little in the way of consensus on a single definition (Grover & Pea, 2013).

The study presented in this work draws most directly from research investigating the nature of computational thinking specifically within mathematics and scientific contexts. A growing body of work is finding that embedding computational thinking into math and science contexts is productive (Hambrusch, Hoffmann, Korb, Haugan, & Hosking, 2009; Lee, Martin, & Apone, 2014; Orton et al., 2016; Settle, Goldberg, & Barr, 2013; Wilensky, Brady, & Horn, 2014; Yadav, Zhou, Mayfield, Hambrusch, & Korb, 2011). To provide structure to this approach, Weintrop et al. (2016) propose a Computational Thinking in Math and Science Practices Taxonomy divided into four over-arching categories: data, modeling and simulation, computational problem-solving, and systems thinking, with each category being further subdivided into specific, concrete practices. This taxonomy was used to inform our analytic approach for this work, which we discuss below.

Methods

Meet lattice land

Lattice Land is a collection of math microworlds and an accompanying curriculum based in a discrete geometry. It is intended to serve as a low-threshold entry to lattice geometry while also having a high ceiling by not constraining learners to narrow forms of interaction or usage patterns. While Lattice Land may be used to support open-ended exploration of non-traditional geometry, in this work we use it in an inquiry-based mathematics curriculum. The content is based on Sally & Sally's (2011) work with lattice geometry and Pick's Theorem for teachers and the processes by which the students construct their knowledge in the spirit of Goldenberg's (1999) Connected Geometry.

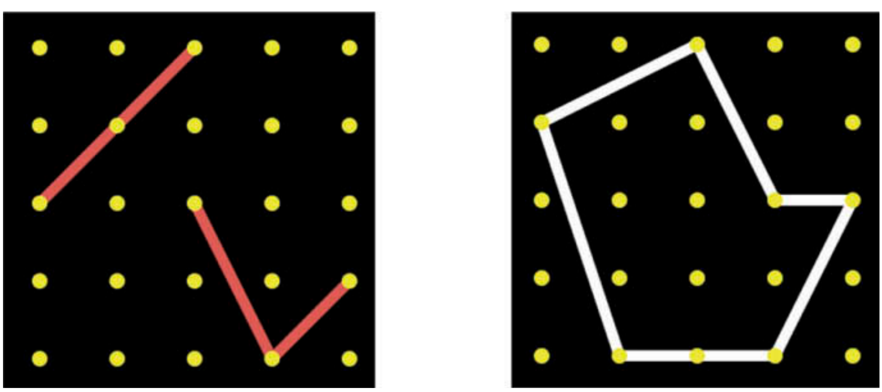


Figure 1. Lattice segments and a lattice polygon in Lattice Land.

The *lattice* is an array of dots on a plane, such that there is one dot at each coordinate *(x,y)*, where *x* and *y* are integers. Each *dot* on the lattice is one unit away from each of its four closest neighbors (see Figure 1.) *Lattice segments* must begin at one dot on the lattice and end at another; thus, *lattice polygons* may only have vertices on dots of the lattice. Lattice Land is built in NetLogo (Wilensky, 1999) which enables learners to interact directly with the dots of the lattice and the segments and polygons they create. Each microworld within Lattice Land is defined by unique constraints and features. Lattice Land is comprised of a growing number of microworlds, three of which we present in this study. The two Triangle microworlds, which are based on the popular math manipulative Geoboard (Gattegno, 1988), allow users to click and drag the three vertices of a triangle to any point on the lattice and see its measurements change dynamically. The Explore microworld allows users to define the size of the world, up to a finite point, construct lattice polygons, and calculate their areas and perimeters at the click of a button.

The Lattice Land curriculum is comprised of a series of lessons that focus on one lattice-based microworld at a time to probe a set of questions that span many mathematical practices. For this study, we implemented a set of lessons around the concept of area. Research has shown that both students and teachers share false beliefs about area (Ma, 1999), and DGS such as Geometer's Sketchpad have been used successfully to disprove these false notions (Stone, 1994). Students began with a Lattice Land microworld called Lattice Triangles Explore, which generates a random triangle and allows the learner to move its vertices to any dot on the lattice, and check the area and perimeter. The next microworld, Lattice Triangles Draw, gave students tools to draw segments over their triangle, creating various dissections of the given space. Finally, students spent one class making constructions of their own choice in Lattice Land Explore, which allows the user to define the size of the lattice, and construct complex polygons and multi-colored line segments.

Since the Lattice Land curriculum is designed to foreground mathematical habits of mind and provide opportunities for students to use computational thinking practices, it reverses the usual teaching order; asking students to experiment, collect data, and build conjectures with empirically based *inductive* reasoning, rather than *deductive* reasoning. Students were encouraged to ask questions, share observations, and answer questions posed by others. Microworlds were presented to the students with open-ended questions and served as a focal point for group work, which resulted in each group of students following slightly different directions. At points during the lessons, the instructor would help to refocus on some key ideas—definitions of geometric terms, area, and methods for finding the areas of polygons—to encourage students to tinker with tools, look for patterns, and develop their own mathematical formalisms.

Methodology and analytic approach

The Lattice Land curriculum was implemented in four sections of a geometry class in an urban, public school in the American Midwest. The study population consisted of second-year high-school students (15-years-old on average) who were interested in potentially participating in the school's new International Baccalaureate Program. The ethnic and socioeconomic demographics in the four classes roughly match those of the school (81% Hispanic/Latino, 9% African-American, 7% White, 2% Asian, 1% mixed race/other; 95% of students come from economically disadvantaged households). Each class had approximately 30 students, and the unit took place over the course of 6 school days, for one 50-minute class period each day. Students in these classes typically work in groups of at most 4. All four classes were taught by the same teacher and worked through the same curricular materials. Two of the classes were selected for data collection. Video was captured for two groups of four students in each class (a total of 16 students). There were no exclusion criteria beyond students choosing to participate in the research.

In addition to the recordings of the class, we conducted one-on-one pre- and post-semi-structured interviews for the students in the four focus groups, resulting in a total of 14 pairs of pre-/post-interviews.

The interviews included content questions intended to elicit computational thinking practices and mathematical habits of mind as well as questions designed to capture students' epistemological beliefs about the purpose and importance of mathematics and their attitudes toward mathematics. The pre- and post-interview questions were the same with one exception: the pre-interview asked students to talk about themselves, while the post-interview asked students to report on their thoughts and feelings with regard to Lattice Land.

The analytic lens we bring to this work is informed by existing work on mathematical habits of mind and computational thinking. To make these connections explicit, we draw on two frameworks that provide structure to these two concepts. Weintrop et al.'s (2016) computational thinking in mathematics and science practices taxonomy delineates specific computational thinking practices that are authentic to mathematics contexts that we can attend to through our analysis. Cuoco, Goldenberg and Mark's (1996) framework for mathematical habits of mind attend to higher order ways of thinking about and with mathematics that we will show are supported by computational tools. These frameworks informed what we were looking for in analyzing the data and provide a language to describe the computational thinking that was taking place. In bringing these frameworks to the Lattice Land data corpus, we have chosen specific activities and episodes from the curriculum that are rich with computational thinking and mathematical habits of mind.

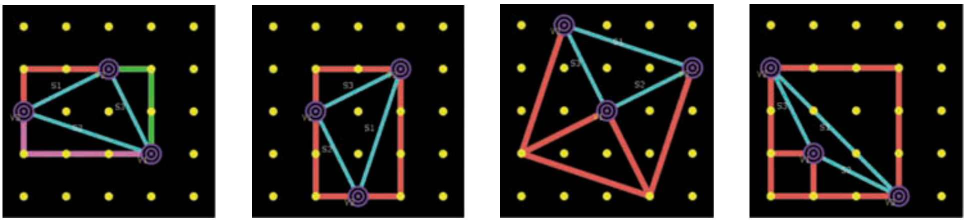
Findings

In this section, we present two analyses of Lattice Land activities highlighting how students engaged in mathematical habits of mind and employed computational thinking as they explored different concepts related to lattice geometry. In each section, we first present the data and the results of our analysis, followed by a short discussion of the mathematical habits of mind and computational thinking practices on display. We used open coding to see what mathematical habits of mind emerged, and if and how they aligned with the Computational Thinking Taxonomy and the mathematical habits of mind framework.

Calculating areas and problem decomposition

The first activity we present focuses on students working in the Lattice Triangles Explore microworld and asks them to find at least one example of every possible triangle area in the given 4 × 4 lattice (there are 16 possible areas). This activity gives students the opportunity to explore shapes and size of lattice triangles and discover (or re-discover) some unexpected results. For example, while tinkering with differently shaped triangles, one student noticed that as she moved one vertex back and forth along the dots in the same horizontal row that the area did not change. At first, this surprised her as the shape of the triangle was changing, but then she remembered that "the area is just one-half base times height," which provided an explanation for why triangles with different shapes would have the same area. Here we see the student making a connection between the open-ended, exploratory Lattice Triangles Explore microworld and the formalism they traditionally use in math class.

A second example of a surprising result came when students realized that in a 4 × 4 lattice, some triangle areas, such as 2.5 square units, cannot be achieved using a triangle whose base is parallel to the x-axis. However, one can create a triangle with an area of 2.5 square units that has sides whose lengths are not whole numbers (e.g., the blue triangles in Figure 2a, 2b, and 2c). The fact that a triangle comprised of line segments whose lengths had long decimal tails could still produce an area with a round number like 2.5 was unexpected and initially hard for students to make sense of. This led to a class discussion about area, aided by the introduction of the Lattice Triangles Draw microworld and a "draw segment" tool (see Figure 2). This concluded when Bobby "boxed" the triangle into a rectangle and subtracted the areas of excess until only the original triangle remained (see Figure 2a). His finding led to a whole new method for calculating areas by inscribing the triangle



(a) (b) (c) (d)

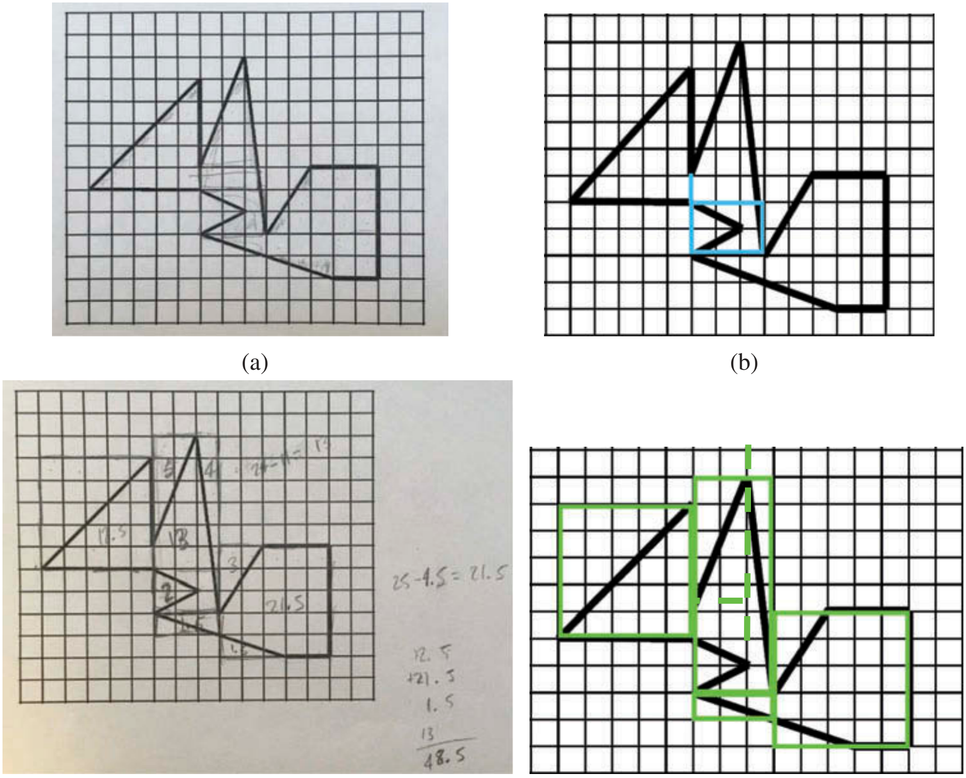
Figure 2. Student discovered strategies for calculating the area of a triangle in Lattice Land.

in a rectangle, then calculating the area of the three right triangles that emerged and subtracting it from the easily calculated area of the rectangle. This realization opened the door for new avenues of exploration related to approaches for calculating areas for the class to pursue. As the class progressed, other methods emerged. For example, one student realized that for some triangles you could make three copies of the triangle and combine them to make a square where the triangle is one-fourth the area (Figure 2c). As students continued to explore, they realized the "boxing" approach would not work for all triangles, as it is possible to draw a triangle which cannot be contained in a rectangle whose sides intersect all three vertices. This led to further class-wide investigation, whereupon students realized they could use the same approach just with a few modifications. As shown in Figure 2d, students realized they could still create a rectangle to calculate area, only they needed to further break down the rectangle into more, easily calculated shapes in order to derive the area of the triangle of interest.

As the curriculum continued, students were challenged to calculate the area of more complex and less regular polygons using the strategies they had developed in earlier lessons focused on triangles. This strategy for calculating areas was one of the goals of the curriculum as it employs the desirable computational thinking practice of problem decomposition and reinforces productive mathematical habits of mind including experimentation and building upon known concepts toward mathematical generalizations. To further understand if and how Lattice Land was able to develop these skills in students, we included an activity in the pre- and post-interviews that would elicit these skills. Specifically, students were asked to calculate the area of the polygon shown by the thick black line in the images in Figure 3. We constructed this polygon on an unmarked grid to mimic the look of Lattice Land and ensured that students would need to dissect the figure in different ways to derive a complete solution. Below we present a detailed presentation of a single student's pre- and post-interviews, then present results across the full set of students to show the typicality of this interview.

Figure 3a shows Alice's solution to the polygon problem in the pre-interview; next to it, we have reproduced her response, making the dissections she proposed clearer (Figure 3b). The first thing Alice noticed about the polygon was how it included a right triangle on the left side, so her first step was to "cut" it from the rest of the figure. However, she abandoned this attempt when she realized, "I don't know the formula for the area of a triangle," leading her to conclude: "I don't know how to do this." When pressed to try to reason about what the area might be, she discussed areas of figures she did know, including the formula for the area of a square. That drew her attention away from the triangle, toward the rectangular jutting on the right side of the figure. At this point, she changed her strategy and started counting whole-unit squares. The light shading that can be seen in Figure 3a was her way of marking off all non-unit squares. Additionally, she boxed off a small area where there were partial squares and tried to make an estimate by eyeballing the size of the space.

Figure 3c and 3d show Alice's solution from the post-interview. Like her pre interview, Alice started with the large right triangle on the left side of the polygon and realized she still could not recall the mathematical formula for calculating area; however unlike the pre interview, this time she



(c) (d)

Figure 3. Alice's polygon solution in the pre-interview (top row, images a and b) and the post-interview (bottom row, images c and d).

drew a square around it as her first step toward figuring out its area, then dividing it by two. Having this piece of the polygon figured out, she systematically worked from left-to-right, decomposing the polygon into smaller, more regular shapes, writing in the area of these subsections as she progressed. Her decisions for how to divide the polygon are shown with the green lines in Figure 3d. Using both additive and subtractive logic, Alice kept track of the parts that composed the original figure to determine her final answer.

Looking across the full set of 14 students who participated in both pre- and post-interviews, only two students in pre-interviews chose to dissect the polygon into triangles and rectangles and also knew how to find the areas of those shapes. In post-interviews, 10 of the 14 students employed this strategy. It is worth noting that the post-interview occurred more than a week after the conclusion of the unit on triangle dissections. Another interesting thing to note about the post-interview responses is that every student dissected the polygon in a different way. There were no duplicate solutions. This suggests that learners were not employing a rote solution they had learned in class and were able to apply a general heuristic in a way they saw as productive.

Mathematical habits of mind

In using dissection as a mechanism for calculating area in Lattice Land, students employed several important mathematical habits of mind drawn from Cuoco et al. (1996) framework: (1) They *visualized* and *re-visualized* a geometric object in different ways. (2) They *tinkered* with potential solutions, as they broke down a bigger problem into parts they could understand and compute. (3) Some students also *generalized* to a more formulaic approach to working with area, akin to familiar area formulas, but situated in their own understanding. The discrete constraint of Lattice Land forced students to construct triangles in atypical orientations, yet also allowed students to avoid calculations with irrational-valued lengths they had constructed and reason about higher level concepts (formula for triangle area, dissection) using integer values.

In Alice's attempt at the polygon problem in her pre-interview, we see her first reverting to the kind of content-based math she was used to using in math class––seeking formulas that can be applied. However, when she abandoned this approach, she relied on educated *guessing*, which is an important habit of mind. In her post-interview, Alice exhibited greater comfort with taking on the problem through *experimentation, problem decomposition*, and *reframing* an unknown problem into a series of problems for which the answer (or at least an approach) is known. All of these are important mathematical habits of mind (Cuoco et al., 1996).

Computational thinking practices

In the above episodes, we see a number of computational thinking practices being employed, both during the computational exploration activities in class and in the interview's pen-and-paper activity. While students were working with the Lattice Triangles Draw microworld, we see students using the microworld as a way to explore and develop an understanding of a concept, specifically the relationship between the shape of a triangle (and the length of its sides) and the resulting area. The practice of using a computational tool to understand a concept is a computational thinking practice in the Modeling and Simulation portion of Weintrop et al.'s (2016) taxonomy. By being able to use the dynamic and interactive characteristics of the computational medium, students could see how moving a vertex along points in the lattice would change the shape of the triangle yet keep the area fixed. This characteristic can then be linked to the formula for a triangle, which relies on the triangle's height and base but does not factor in the measure of the angles, and thus the shape, of the triangle. Lattice Land, with its computational context, makes this relationship clear to the user in a way that is not possible (or at least not as accessible) in a non-computational setting.

In the post-survey, we see a second computational thinking practice being employed, this time in a non-computational context. To calculate the area of the polygon, Alice takes a large problem (finding the area of an irregular polygon) and breaking it down into a series of sub-problems (finding the area of triangles and rectangles), each of which is more easily solved. This can clearly be seen in how she dissected the polygon in Figure 3d. The practice of breaking problems down into sub-problems is another practice from the taxonomy. Additionally, this approach, which Grover and Pea (2013) call "structured problem decomposition," is commonly cited as a core computational thinking practice and has widespread application across both computational and non-computational settings.

Deriving mathematical formulae with data and pattern recognition

The second activity we present came later in the curriculum and focuses on a series of classroom activities in which students were challenged to derive Pick's Theorem, which defines the relationship between the lattice points associated with a lattice polygon and the area of that polygon. Specifically, using the number of lattice points intersected by the *boundary* (B) of a lattice polygon and the number of points on the *interior* (I) of the lattice polygon, the *area* (A) of the lattice polygon is as follows: $A = \frac{1}{2}B + I - 1$. For example, the polygon in Figure 2a has two interior points and three boundary points, which using the formula, gives us the same area of 2.5 that students' inscription and dissection approach yielded. Students were never shown this formula; instead, they were challenged to try and figure out a universal approach to calculating the area of a lattice polygon (as opposed to the "boxing," or dissection strategy discussed above). There was a natural impetus to generalize this formula, since one student, Grace, had conjectured while using the Triangle micro-worlds, "A lattice triangle that has no points inside of it and no points crossing through its line segments must have an area of 0.5 square units." Her peers were mystified that it seemed to hold across all the microworlds

The first step in this activity was for students to draw lattice polygons in Lattice Land Explore, then record the number of interior points, the number of boundary points, and the area (which was provided by the microworld). Students did this independently and entered their results into a spreadsheet that was shared by the class. Once each group had submitted their entries to the spreadsheet, the focus of the activity shifted to looking for patterns in the data students had just collected. By sorting the data according to interior points, the class first focused only on polygons with no interior points. Two students pointed out that an increase of 0.5 in area corresponded with an increase of 1 boundary point. Students suggested many potential formulae to reflect this relationship—including $A + 0.5 = B + 1$ and $A = 0.5B$—which fit some of the data collected but did not work for all polygon data students had collected. These early formulae suggestions are encouraging as it showed that students were reasoning about the relationship between two variables in generative ways and were able to move from the non-numerical Lattice Land to algebraic formalisms of conventional math curricula. After multiple trials, two students working together proposed the formula: $A = \frac{B-2}{2}$. Again, this formula fit for polygons with no interior points but did not work for all of the data. The class then moved on to start looking for patterns in the data for polygons that did have interior points. One student recognized that the formula proposed by the two students earlier could be updated to match more of the data: $A = \frac{B-2}{2} + I$. This formula has a slightly different form but is logically equivalent to the formal for Pick's Theorem presented at the start of this section. After multiple checks against the data they had collected, and against new polygons the students created to test their conjecture, the class collectively agreed upon this formula. In post-interview with the teacher, she expressed surprise that her students made these discoveries, because "usually they don't *see patterns*."

Mathematical habits of mind

The derivation of Pick's Theorem highlights a fundamental design choice of Lattice Land—the use of a discrete plane. When first asked for a relationship between the number of boundary points and area of the resulting triangle, students responded with prolonged silence. That silence eventually turned into speculation and testing, because it was easy to test hypotheses. They were, after all, constructing figures with a software package that facilitated counting in a systematic way. Repetition led to pattern recognition, which gave way to the discovery of a useful functional relationship. The discrete framework and interactive interface of the microworld made it possible for students to look for a functional relationship between variables without getting bogged down in the details of calculation.

Students balanced several seemingly contradictory mathematical habits of mind. For example: (1) By switching between formulating a relationship between variables and testing unique polygons, students were *mixing deduction and experimentation*, "seeing the interdependence of interpreting experiments and making theories" (Goldenberg, 1999). (2) Likewise, students attended to *patterns* in the numbers but were also on the lookout for *invariants* which might disprove a proposed formula. (3) They used more *precise language* because they needed to get their ideas across, recalling both familiar geometry terminology as well as new words they had recently defined such as "boxing."

Computational thinking practices

This activity shows students engaging in a number of computational thinking practices identified as productive in mathematical disciplines. The first group of practices comes from the Data Practices category of Weintrop et al.'s (2016) taxonomy. The first step in deriving Pick's Theorem involved collecting data and recording it in a standardized format. The next step was to manipulate this dataset via sorting and analyzing the data in search of patterns. Throughout these steps, students used computational tools to facilitate these data-driven computational thinking practices. A second way that computational thinking practices appear in the above vignette is in the process of formalizing the data in the form of a mathematical formula. We view this process as a form of creating an abstraction. In the process of translating a set of data into a concise and precise mathematical formula, the students created an abstraction that would allow them to calculate the area of any lattice polygon, regardless of its shape or size. Furthermore, this abstraction is in a form expressed in such a way that a computational device can evaluate it, which means learners can now bring computational power to bear in solving future problems related to calculating the area of a lattice polygon. While this may seem like an exaggeration of the utility of creating this abstraction given the simplicity of the formula and the ease with which it can be manually computed, it is not difficult to imagine larger more complex formulae that do not share that characteristic. Going one step further, the larger practice of creating abstractions that can be evaluated or executed by computational tools is a far-reaching and powerful computational thinking practice that learners are developing in their mathematics classroom.

Discussion

The mutually supportive nature of mathematics and computational thinking

This paper has documented one example of a computationally mediated mathematics learning environment where students develop both mathematical habits of mind and computational thinking practices. Lattice Land, with its open-ended constructionist design and underlying mathematical ideas, serves as a case study for the mutually supportive nature of mathematical learning and computational thinking. Computational thinking is argued to be a broad skill set that has widespread applicability across contexts and domains. While this can be seen as a strength of the notion of computational thinking, it also presents challenges as, on its own, it does not have a specific context in which to situate the concepts and practices. In the above example, we show how geometry, algebra, and a high-school mathematics classroom create a compelling context for engaging in mathematical explorations and provide the needed context to allow learners to meaningfully engage in computational thinking. Additionally, while computer science classrooms might seem the most natural environment for computational thinking, embedding computational thinking practices in mathematics gives this work potential for broader impacts. In America, computer science courses count toward graduation credit in only 30 of 50 states (Code.org, 2015), whereas math is a core subject taught at all levels of school. At the same time, by designing learning tools and curricula around powerful computational thinking practices like problem decomposition, abstraction generation, and data collection and analysis, computational thinking practices can serve as a compelling way to motivate learners to develop mathematical habits of mind and engage in powerful ideas of mathematic. For example, the challenge of discovering Pick's Theorem served as a context for interacting with dynamic computational models while collecting, manipulating and analyzing data with the help of computational tools, which are important computational thinking practices in math and science (Weintrop et al., 2016). At the same time, the model exploration and data analysis was in service of hypothesis testing, inductive reasoning, and creating mathematical generalizations, which are productive mathematical habits of mind (Cuoco et al., 1996).

The design of Lattice Land, which encourages tinkering and exploration and uses the computational medium as a way to facilitate hypothesis generation and testing, and inquiry around mathematical concepts shows the potential for computational thinking's role in service of mathematics learning. Lattice Land promotes many proof-based habits of mind as existing dynamic geometry systems (DGS) do, but unlike conventional DGS, the focus on discrete geometry resonates more strongly with computational thinking and taps into our earliest experiences with counting. Additionally, the curriculum uses evidence-based inductive approaches to reasoning, which aligns with real-world data collection, and which Schoenfeld (1986) has shown strengthens deductive approaches to geometry and helps to break students' tendencies to compartmentalize knowledge.

Furthermore, as shown above, computational thinking is part of the learning activity, but certainly not all of it; the practices learners employ and ideas they generate serve as a launch point for classroom discussion and other conventional mathematics exercises that themselves are not necessarily computational in nature. In other words, computational thinking need not be the sole focus of a mathematics curriculum; instead, it can be viewed as one component in the design of a larger mathematics learning experience. By situating mathematical learning in a computational microworld, and encouraging learners to employ computational thinking practices in pursuit of mathematical understanding, we demonstrate the mutually supportive nature of computational thinking and the mathematical habits of mind that are the goal of mathematics classrooms.

A restructuration of classic geometry

There is a growing body of research documenting the relationship between the representational infrastructure used to express the ideas of a domain and the nature of the knowledge itself (Kaput, Noss, & Hoyles, 2002; Noss, Healy, & Hoyles, 1997; Wilensky & Papert, 2010). Wilensky and Papert's (2010) theory of *restructurations* captures this relationship and describes various criteria along which different *structurations* can be evaluated. An example of such a comparison can be seen in Sherin's (2001) work investigating the use of conventional algebraic representations compared to programmatic representations in physics courses. He found that the different representational forms had different affordances with respect to students learning physics concepts and, as result, affects their conceptualization of the material learned. Other examples include the shift from Roman Numerals to Hindu–Arabic notation (Swetz, 1989; Wilensky & Papert, 2010), the emergence of algebraic notation replacing natural language (diSessa, 2000), and agent-based modeling as an alternative to differential equations (Wilensky & Rand, 2014). While often assumed to be static, Wilensky and Papert show that the structurations that underpin a discipline can, and sometimes should, change as new technologies and ideas emerge. While geometry has been the focus of restructurations in the past, most notably with Logo (Abelson & DiSessa, 1986) and DGS (Cuoco & Goldenberg, 1996), we argue that Lattice Lands serve as another productive way to rethink the way learners represent, and thus, conceptualize and interact with the powerful ideas of geometry. Furthermore, it accomplishes this through the context of lattice geometry, a little-explored component of the field. In this way, we view Lattice Land as a restructuration of continuous Euclidean geometry that makes it possible for students to see and reason about complex geometrical patterns.

While a lot of work has been done with DGS such as Cabri, Geometer's Sketchpad, and GeoGebra, the focus is on supporting canonical Euclidean geometry curriculum (Hohenwarter & Fuchs, 2004; Jackiw & Finzer, 1993; Laborde, 2002). When it comes to geometry, the American math canon focuses almost exclusively on classical Euclidean geometry. As one of the foundations of Western mathematical thought, Euclidean geometry has historically been the foundation for proof and reasoning. However, there are important reasons to consider teaching math practices using a wholly different structure of geometry with a different underlying representational infrastructure.

One feature of Lattice Land serving as a restructuration of Euclidian geometry is that, at times, is at odds with the prior geometry instruction students have received. For example, it is impossible to construct an equilateral triangle in Lattice Land. Such contradictions with prior mathematical knowledge provide opportunities for questioning of presupposed rules of mathematics from the students, and require the use of many mathematical habits of mind—visualizing, using precise language, tinkering, proving—in order to define and navigate new mathematical worlds (Goldenberg, 1999). It has also been shown that students' reasoning about discrete processes as fundamentally connected with algebra, showing how mathematical habits of mind have no content specificity (Goldenberg, 1999).

Furthermore, we found that students expressed a great deal of creativity in their construction of computationally mediated mathematical objects. These students exhibited how *visualizing* a problem in different ways led to interesting findings. Students became *relative experts*, and some showed greater willingness to *explore* and *share* interesting findings. Lattice Land has the potential to bring a bit of whimsy into computational and mathematical learning. It bridges mathematical and computational ways of thinking and crosses into the realm of math-art and play. We hope that it challenges teachers and learners to change their ideas about what learning can or should look like.

Conclusion

Computation is playing an increasing role in the lives of learners, both inside the classroom and beyond. Along with these tools comes the ability to express ideas in new ways and interact with concepts through a media and new representational tools. Increasingly, educators and parents are recognizing the need to help students develop skills associated with productively using these powerful tools, but open questions remain as to how to accomplish this and where it might fit within existing educational infrastructure. At the same time, the nature of mathematics and the goals of mathematics instruction, while fixed for a long time, are experiencing a shift in terms of the tools available to instructors and the potential ways to support learners in exploring the powerful ideas of the discipline. In this paper, we argue for the synergy between computational thinking practices and mathematical habits of mind and provide an example of one tool that lives at the intersection of these two learning goals. By giving learners an accessible and expressive palette to explore novel geometric ideas, and linking that experience with both the computational thinking practices it engenders and the mathematical habits of mind it promotes, we show the potential of merging these two desirable educational outcomes. In bringing together mathematical habits of mind and computational thinking in the form of Lattice Land, this work serves as one example of the larger class of learning experiences that blends domain content with the computational practices learners will need to succeed in an increasingly technological world.

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