ALLEN STAIRS*

12. KRISKE, TUPMAN AND QUANTUM LOGIC: THE QUANTUM LOGICIAN'S CONUNDRUM

ABSTRACT

Almost thirty years ago, Saul Kripke gave a talk in which he offered an extended critique of quantum logic. Neither that talk nor any commentary on it appear in the published literature. Today, there is much less interest in quantum logic as an interpretive program in the foundations of quantum mechanics. Nonetheless, Kripke's critique raises interesting issues about what it might mean to contemplate a change in logic. Set against the larger background of the literature at that time, the lecture also provides an interesting springboard for exploring a number of issues about realism and quantum mechanics of the sort that Jeff Bub has wrestled with over his career. This paper will present an extended summary of a related critique by one P. Kriske, and will proceed from there to a discussion of the larger questions that must be addressed in order to provide an adequate reply to Kriske.

I've known Jeff Bub for over thirty years as a teacher, colleague and friend, and I'm delighted to be able to contribute to this volume in his honor. What I plan to do, however, is start with some unpublished material from a dissertation that I wrote almost 30 years ago and that I had not even held in my hands for almost that long. It's a bit like talking to a ghost. As it turns out, that may be appropriate; ultimately, the problem I want to worry is the peculiarly elusive nature of the attempt to interpret quantum theory. We'll begin, however, with a quasi-mythical episode in the history of quantum logic. The episode has its own interest, but it will also serve as a segue into a broader discussion.

1 QUANTUM LOGIC AND REALIST DREAMS

When I wrote my dissertation in 1978, some of us at Western Ontario saw quantum logic as the key to a realist interpretation of quantum mechanics. One important part of what we meant by "realist" was that in the ideal case, measurement should merely reveal the pre-existing values of physical quantities: if the measuring instrument said that the y-spin was $+1/2$, that was supposed to be because it really had that value before the measurement was made. But since we can perform any measurement we like, that implied that all quantities would have to have simultaneous values – whether we could measure the quantities simultaneously or not.

* Dept. of Philosophy, University of Maryland, College Park, MD 200742 USA; Email: stairs@umd.edu.

W. Demopoulos and I. Pitowsky (eds.), Physical Theory and its Interpretation, 253–272.

Whether or not this was a reasonable understanding of realism, the difficulty is clear: results such as Kochen and Specker's[1] apparently show that the physical quantities couldn't possibly all have values at once. If two quantities Q and Q' share an eigenspace S, the K&S theorem assumes that the values of the quantities are accordingly related: either both have a value that goes with S or neither does. If "$Q = q$" and "$Q' = q'$" represent the same proposition when they are associated with the same subspace, claiming that the finite set of K&S quantities all have values at once amounts to a classical contradiction.

One way to provide for definite values is to reject the K&S constraints and adopt a contextual hidden variable theory. Unfortunately, this comes at a price: what we're measuring "here" must either be influenced or partly constituted by what's being measured "there"; otherwise, we run afoul of Bell's theorem. Quantum logic proposed a way around this problem: identify propositions as the Hilbert space suggests. Since propositions are identified in a context-free way, local quantities are locally constituted. If a quantity Q has possible values $q_1, q_2, \ldots, q_n$ then the quantum logical disjunction

$$Q = q_1 \vee Q = q_2 \vee \cdots \vee Q = q_n$$

is true. The conjunction of all these disjunctions yields a classical contradiction, but logic isn't classical and properties don't mesh as classical logic says they do.[2] That would allow the serpentine Kochen and Specker "contradiction" to be a truth – a truth that supposedly says of each quantity that it takes one of its possible values. If we add the claim that measurement simply reveals pre-existing values, then not only are quantities locally defined, but local measurement results don't depend for their outcomes on distant measurements.

2 INTRODUCING PROFESSORS KRISKE AND TUPMAN

There are two thoughts here. One is that quantum logic allows us to say that all quantities have value, revealed by measurement. The other is that changes in physics might rightly induce us to accept changes in logic. In 1974, Saul Kripke gave a lecture at the University of Pittsburgh in which he offered a critique of quantum logical *value-definiteness*, the thesis that all quantum mechanical quantities have values, which measurement simply reveals. He also called into question the very idea that logic might change in response to empirical discoveries. That lecture was the subject of rumors, myths and much conversation in corridors. It was also the topic of the first two papers of my dissertation. I'd like to discuss what Kripke said, but this presents a problem. He never published his talk, which is why I never published the relevant portions of the dissertation. Worse still, the tape I once possessed is long since lost. All I have are the quotes and paraphrases in the dissertation. There's also the matter of propriety. Since Kripke's talk never appeared in print, it isn't part of the public record of positions he's committed himself to. He might well object to being saddled with what he said back then.

I propose the following solution. I'll discuss a position that wouldn't have occurred to me if I had never heard the tape of Kripke's talk, but I won't promise that the position is Kripke's. I'll attribute it to the fictitious philosopher Paul Kriske – Kriske for short. If I attribute something to Kriske, you may assume that I didn't think of it myself. But you may *not* assume that it's an accurate reflection of Kripke's view. It may, for all I'm willing to claim, be based on a complete misunderstanding of what Kripke actually said. Since my tape no longer exists, and since I have nothing close to a full transcript of it, you should take this possibility very seriously.

The paper that Kriske singles out for his critique is Putnam's 1968 essay "Is Logic Empirical?"[3] But just as fairness led to the introduction of Kriske into our discussion, it's also fair to ask if what this Kriske has to say about Putnam is true to the real-life Putnam. Since Putnam exegesis is not my concern, I will introduce another character into our drama, Prof. Tupman, who is the subject of Kriske's criticism.

3 KRISKE ON TUPMAN

Suppose that A and B are two non-commuting operators, each with eigenvalues 1 and 2. Tupman wants to say that both of these statements are true as ordinarily understood, and before any measurements are made:

(1) $A = 1$ or $A = 2$ (that is, $A = 1 \vee A = 2$)
(2) $B = 1$ or $B = 2$ (that is, $B = 1 \vee B = 2$)

Nonetheless, Tupman also wants to say that all of these are true as well:

(3) $\sim (A = 1 \wedge B = 1)$
(4) $\sim (A = 1 \wedge B = 2)$
(5) $\sim (A = 2 \wedge B = 1)$
(6) $\sim (A = 2 \wedge B = 2)$

You might have thought that for A and B to have values, one of the following would have to be true:

(3′) $A = 1 \wedge B = 1$
(4′) $A = 1 \wedge B = 2$
(5′) $A = 2 \wedge B = 1$
(6′) $A = 2 \wedge B = 2$

Tupman's claim is that so long as (1) and (2) are true, we have all we need for value-definiteness. He frames the issue in terms of the distributive law. In classical logic,

$$(\mathrm{W} \vee \mathrm{X}) \wedge (\mathrm{Y} \vee \mathrm{Z})$$

implies

$$(\mathrm{W} \wedge \mathrm{Y}) \vee (\mathrm{W} \wedge \mathrm{Z}) \vee (\mathrm{X} \wedge \mathrm{Y}) \vee (\mathrm{X} \wedge \mathrm{Z})$$

by the distributive law. But the distributive law doesn't hold in the lattice of subspaces of a vector space, and according to Tupman, that lattice reflects the correct logic. Hence, we can't move from the conjunction of (1) and (2) to (3) through (6).

To make this more palatable, Tupman offers an analogy with geometry. Before relativity and non-Euclidean geometry, the idea that two *straight* lines might be a constant distance apart over some interval but intersect further along would be an intuitive contradiction. As it turns out, however, this "contradiction" about space might well be true. The moral? Don't trust intuition – not even in cases where ignoring it feels like a contradiction.

As Kriske reads Tupman, the distributive law amounts to an axiom of classical logic, and is up for grabs once rival systems are on the table. With the right sorts of empirical pressures, we might abandon one formal system for another. In the case of quantum mechanics, Tupman sees this as the smoothest course to follow. Give up the distributive law and adopt quantum logic; the payoff for the intuitive pain is a realist interpretation of quantum theory.

Kriske points out that we *seem* to be able to knock this view down with a simple argument. Tupman says that A and B have values that show up when we make a measurement. Suppose we measure A and find that

(7) $A = 1$.

Tupman says that B has one of the two values 1 or 2; that's what (2) above tells us. But now reason by cases. If B has the value 1, then $A = 1$ and $B = 1$, and that contradicts (3) above; if B has the value 2, then $A = 1$ and $B = 2$, which contradicts (4). Since there aren't any more cases, there's no way for A and B both to have values.

Kriske thinks that's as complete a refutation of Tupman as we could hope for, but he assumes that he'll be accused of begging the question. His refutation of Tupman, so his opponent will say, called on the distributive law, which is what the reasoning by cases amounts to here. But the distributive law is exactly what's at issue.

Kriske's reply reveals the heart of his position. His refutation depended on reasoning from

(7) $A = 1$

and

(2) $B = 1$ or $B = 2$

to the conclusion

C: $(A = 1$ and $B = 1)$ or $(A = 1$ and $B = 2)$.

However, that conclusion is just what Tupman rejects when he says that

(3) $\sim (A = 1 \wedge B = 1)$

and

(4) $\sim (A = 1 \wedge B = 2)$

are both true. Apparently, then, Tupman doesn't think that C follows from (7) and (2). Apparently Tupman thinks that to get C from (1) and (2), we need an extra premise, the distributive law, which is what Kriske has begged.

Kriske disagrees. He doesn't think he needs an extra premise to get from (7) and (2) to C. The reasoning by cases helps us *see* that the argument is valid, but it doesn't *add* anything to its validity. If you say otherwise, Kriske thinks, you are begging the question against him.

4 CAN LOGIC BE CHANGED?

In fact, Kriske argues, the very idea of "adopting" a logic is incoherent. Tupman thinks of "logics" as systems of axioms that can be treated as hypotheses to be accepted or rejected on the basis of empirical considerations. However, Kriske maintains that we couldn't possibly *adopt* the logic we already have. The inspiration for his argument comes from Quine's "Truth by Convention"[4] and from Lewis Carroll's famous discussion of Achilles and the Tortoise.[5] One way to put Lewis Carroll's point is that if someone didn't already reason according to *modus ponens*, adding it as an explicit axiom wouldn't help. Suppose someone accepts

A

and also accepts

If *A* then *B*

but for some reason doesn't see that *B* follows. Imagine offering him the following as an explicit principle:

MP: If "*A*" is true and "If *A* then *B*" is true, then "*B*" is true.

Unless the person already grasps *modus ponens*, this won't do any good. He accepts that "*A*" is true and he also accepts that "If *A* then *B*" is true. Let's add, although it's not as trivial as it seems in this context, that he also accepts "'*A*' is true and 'if *A* then *B*' is true." Suppose he also agrees, perhaps accepting your authority, that MP is correct. The problem is that MP is a conditional. To conclude that "*B*" is true, he'll have to reason by *modus ponens*, which is precisely what he wasn't able to do in the first place.

Kriske points out, following Quine, that the same difficulty comes up for universal instantiation. If someone didn't already see, for example, that "All ravens are black" commits her to "Jake (a particular raven) is black," then adding that "All universal statements imply their instances" wouldn't help. The principle itself is a universal statement, and to apply it to a particular case, we would have to infer an instance of it. We run into the same trouble with the rule of conjunction and the principle of non-contradiction; details are left as exercises.

So much for treating logical laws as hypotheses that we might adopt or reject based on the fecundity of their consequences. If by "logic" we mean what we use when we reason, then there's no neutral ground outside logic where we can stand and make judgments about how to draw those consequences.

Kriske briefly discusses two cases in which it might seem that we have allowed or at least considered changes in logic. One is intuitionism. The other is the rejection of the Aristotelian principle that "All P are Q" implies "Some P are Q." In the case of intuitionism, Kriske maintains that the intuitionists didn't reject the rules that applied to the old connectives but rather introduced new connectives. The classical negation of a mathematical statement, in the intuitionists' view, is not guaranteed to be a mathematical statement. Intuitionistic negation can be explained by way of notions we already understand, and it keeps us within mathematics when we reason mathematically. As for Aristotelian logic, modern logicians say that we can get from "All P are Q" to "Some P are Q" *if and only if we assume that there are P's*. But there are cases where "All P are Q" is true even though there aren't any P's. Seeing this isn't "changing logic"; it's recognizing a mistake simply by using ordinary reasoning.

5 KRISKE CONSIDERED

So far, Kriske has argued that

(1) Tupman treats logic as though it were just another theory – just another set of propositions that we accept or reject on the basis of their consequences. However,
(2) that can't be right because it suggests that logic is "up for grabs," when in fact we couldn't consider the consequences of the supposed theory unless we already had logic to do it with – that is, unless we already could reason. Furthermore,
(3) looking at cases like *modus ponens* and universal instantiation makes clear that the very idea of adopting a logic makes no sense. These principles aren't hypotheses; we couldn't adopt them unless we already grasped them. Finally,
(4) there are no good examples of changing logic. In particular, the rejection of Aristotelian logic doesn't count. It's a case of using intuitive reasoning to spot a fallacy.

There are two issues before us. One is whether Kriske is right to think that Tupman's defense of value-definiteness is unsustainable. I think he is, and I will simply assume that from now on. The second issue is whether Kriske has really shown that empirical discoveries couldn't rightly lead us to revise our logical opinions.

5.1 Logic and doxastic practices

If by "logic" we mean something like "correct reasoning," then it would make no sense to think of logic as "just another theory." We need to be able to reason in order to think about anything at all. That said, one suspects that Kriske and the quantum logician may be talking past one another. When Kriske talks about logic, he is talking about a *doxastic practice* in William Alston's sense[6] – a socially established practice of forming and criticizing beliefs. Kriske points out that we don't have a choice about engaging in the practice, and that we have to use logic to justify or criticize logical beliefs. However, Alston reminds us that this is so of other important doxastic practices. We can't avoid using sense experience to form beliefs about the world, but any attempt to justify or criticize either the practice itself or the results of using it will call for relying on things that we learned from the senses – from using the very practice at issue. In spite of that, specific claims based on sense experience can be treated as hypotheses that could be revised, even though we have to use sense experience to justify the revisions.

This suggests a way to think about challenging logical claims. We need to distinguish between *reasoning* – a doxastic practice – and the theory or discipline in which we attempt to state logical truths and spell out correct forms of inference explicitly. Let's call the output of this discipline "Logic" with a capital "L." Logic in this sense isn't a substitute for the practice of reasoning, but the claims of Logic can be true or false, correct or incorrect and even, perhaps, fecund or barren. Perhaps Tupman could say: we can't put the whole doxastic practice of reasoning up for grabs at once. Nonetheless, we *can* call some of the basic deliverances of reasoning into question, even if we have to reason to do so.

We've already abandoned the hope that we can defend value-definiteness by appeal to Tupman-style quantum logic, but for what comes later it will be helpful to bracket that concession and reconsider the exchange that Kriske imagines himself having with Tupman. Kriske argues, reasoning by cases, that *A* and *B* can't both have values. He imagines Tupman accusing him of begging the question – of omitting a premise (the distributive law) that he needs for his argument to be valid. Kriske replies that Tupman would be begging the question against him; as Kriske sees it, he doesn't need the extra premise. But consider the case of Aristotelian logic. Suppose we insist that that the principle of subalternation is false – that from "All dogs are mammals," it doesn't follow that some dogs are mammals. We insist that the conclusion calls for an extra premise: dogs exist. Imagine the Aristotelian replying that he needs no such premise and hasn't begged any question. Subalternation is valid, he claims, and we modern logicians are begging the question against him if we claim the he needs an extra premise. How would the debate proceed?

The first point is that it actually *could* proceed. The Aristotelian might insist that in cases where "P" is empty, "All P are Q" isn't true. After all, both "All Martians are Americans" and "All Martians are non-Americans" sound odd, even though modern logicians say that both are true. Of course "All Jedi Knights have superhuman powers" seems to be true, but so does "Some Jedi Knights turned to the dark side of the Force." Since the latter hardly entails that there really are Jedi Knights, the Aristotelian could argue that "All Jedi Knights have superhuman powers" was never *literally* true in the first place. If so, it doesn't count against the claim that universal categoricals are false if their subject terms are empty.

The debate could continue. We could point out to the Aristotelian that if "Some P are Q" entails the existence of Ps, as he presumably would agree, then he will have to give up either the principle of conversion for universal negatives or the principle of obversion. (Hint: start with the banal truth "No Canadians are Martians." Then convert, obvert and take the subaltern). Modern logicians have decided that things go more smoothly if we adopt the Boolean interpretation of categorical statements. Nonetheless, no matter what solution we settle on, it will make for some intuitive strain. If there's something to negotiate in the case of the principle of subalternation, Tupman might insist that we can also negotiate in the case of the distributive law.

5.2 Self-presupposing principles

There's a particular difficulty with this reply that we'll get to below. Meanwhile, we come up against the third of Kriske's four points: we seem to be assuming that the distributive law is a hypothesis that can be adopted or rejected based on its consequences. But the discussion of *modus ponens*, universal instantiation and so on was meant to show that the idea of adopting a logical law or logical axiom makes no sense to begin with.

The self-presupposing quality of these principles is striking. However, Tupman could point out that the distributive law doesn't have this peculiarity. It's hard to see that no one could adopt it unless he already grasped it. Furthermore, even if it weren't possible to *adopt* the distributive law, *rejecting* it might still be possible. Though the examples are controversial, it has been argued (most famously by Van McGee[7]) that *modus ponens* doesn't hold in all cases. Whatever one makes of the examples, it's no reply to point out that no one could adopt *modus ponens* if he didn't already grasp it. Likewise, for all the Carroll/Quine/Kriske examples show, the distributive law may be a principle that we could reject. Tupman would say that empirical discoveries have uncovered exceptions to what had looked like a logical truth.

5.3 Internal vs. external

Kriske would insist that we've missed the point. The issue over subalternation is entirely an in-house squabble that never takes us outside the doxastic practice of reasoning. Tupman's case against the distributive law is extramural. He isn't arguing that there's an intuitive objection to the distributive law. He's claiming that if we give it up, we gain a certain *extra-logical* benefit: a realist interpretation of quantum mechanics. That, Kriske would insist, misses the point that logic is all about *reasoning*.

6 QUANTUM *LOGIC?*

Kriske's view of logic is something that we might call *Intuitivism*: claims about logical truth and logical consequence must be grounded in intuitive reasoning. And though just what might count as an intuitive consideration isn't easy to say, appeals to contingent empirical facts don't make the grade.

There's a related point. If Kriske is insisting that by its nature, logic is *a priori* (a matter of "reasoning" and "intuition") then quantum logic seems excluded from the start. Quantum logicians are making claims about physical reality, but they don't claim that the structure of physical reality is something we can know *a priori.*

That's surely right; we can't figure out the structure of the world just by reasoning. Nonetheless, I think it still may be possible to meet Kriske on his own terms. Interestingly enough, his discussion of Aristotelian logic provides a hint. According to Kriske, the Aristotelian's mistake was to overlook something: the possibility that a universal categorical might be true even though its subject term is empty. What the quantum logician must say is that the classical logician has also overlooked some possibility.

6.1 Minimal Quantum Logic

Consider the following three theses:

> *I* The propositions of Quantum Logic (call them Q-propositions) are ascriptions of values to quantum mechanical quantities or logical constructions of such propositions.
> *II* Not every quantum mechanical quantity has a value
> *III* When a quantum mechanical quantity lacks a value, there are true disjunctive Q-propositions whose disjuncts are not true.

I is a stipulation. It says that this is what Quantum Logic, as understood here, will be constructed from. *II* is widely accepted even by people who want nothing to do with Quantum Logic. *III* is the most contentious of the three theses. Though we'll need to say more, we can use an example to provide some motivation. Suppose that a spin-one particle is in the state $|Sz = 0\rangle$. In that case, Sx doesn't have a value; none of the propositions

$$Sx = +1, Sx = 0, Sx = -1$$

is true – or so it's reasonable to believe. However, there's a case to be made for saying that $(Sx)^2$ does have a value – a value of 1 – even though neither "$Sx = +1$" nor "$Sx = -1$" is true. On the view we're considering, the fact that $(Sx)^2 = +1$ will be the same fact as the one expressed by the disjunction

$$Sx = +1 \vee Sx = -1.$$

If one is true, so is the other.

6.1.1 The distributive law revisited Our theses *I* through *III* don't give us full-blown Quantum Logic, but they're enough to make sense of how someone might think that the distributive law could fail. Suppose that P is a true Q-proposition. Suppose that $Q \vee R$ is also a true Q-proposition, but with disjuncts that aren't true (whether or not we say that they're false is another matter; more on that below.) In this case, the conjunction

$$P \wedge (Q \vee R)$$

will be true, but neither of the propositions

$$P \wedge Q, P \wedge R$$

will be. (We'll leave aside for the moment the question of whether these "propositions" are even well-defined.) The "intuitive" explanation is that the Kriske-style classical logician has overlooked something: the possibility of a true disjunction with disjuncts that aren't true.

It's worth stressing that this account of how the distributive law fails isn't what Tupman, let alone Putnam, had in mind. The value-definiteness thesis is gone. That means that some of Kriske's criticisms of Tupman are no longer relevant. However, Kriske might still insist that

$$(P \wedge Q) \vee (P \wedge R)$$

simply follows from

$$P \wedge (Q \vee R)$$

He might also say that the "possibility" he's accused of overlooking – that a disjunction might be true when neither of its disjuncts is – doesn't deserve to be taken seriously. Given the sketchiness of the defense we've offered for *III*, this wouldn't be unreasonable, though we'll have more to say later. But *I* through *III* are not the central claims of Quantum Logic. What's really at stake lies a little deeper.

6.2 *The deeper level*

Quantum mechanics represents physical quantities in a striking way. The particular feature of structure that Quantum Logic focuses on is the family of relations of necessary equivalence, necessary exclusion, and entailment that quantum mechanics seems to embody. We can illustrate with a familiar example from Kochen and Specker: a spin-one particle and the components of spin in three orthogonal directions x, y and z. Each of the spin matrices Sx, Sy, Sz has three eigenvalues, -1, 0 and $+1$, corresponding to the three possible results of a measurement of the spin component. The squares of each of these matrices, $(Sx)^2$, $(\mathbf{S}y)^2$ and $(Sz)^2$, each have eigenvalues 0 and 1. The distinctive part of the story begins when we introduce the operator

$$H_S = a(Sx)^2 + b(Sy)^2 + c(Sz)^2$$

whose eigenvalues are $x_0 = b + c$, $y_0 = a + c$, and $z_0 = a + b$. Here the vector $|x_0\rangle$ is also an eigenvector of Sx and of $(\mathbf{S}x)^2$, with eigenvalue 0. Corresponding remarks apply to $|y_0\rangle$ and $|z_0\rangle$. A contextualist would say that

$$H_S = x_0, Sx = 0, (Sx)^2 = 0$$

represent distinct propositions that might differ in truth value. Quantum Logic treats these propositions as necessarily equivalent – as picking out the same possible state of affairs. As for necessary exclusion, the vectors

$$|x_0\rangle, |y_0\rangle, |z_0\rangle$$

are mutually orthogonal. The contextualist would say that in spite of this, it's possible for Hs to take the value x_0 and Sy to take the value 0 at the same time. Once again, Quantum Logic treats these propositions as necessarily excluding one another – as denoting states of affairs such that if it's true that one obtains, it's false that the other does. Finally, the vector $|z_0\rangle$, for example, is a superposition of $|x_+\rangle$ and $|x_-\rangle$. That means that the subspace corresponding to "$Sz = 0$" lies within the subspace corresponding to "$Sx = +1 \vee Sx = -1$". Quantum Logic take the truth of "$Sz = 0\rangle$" to necessitate the truth of "$Sx = +1 \vee Sx = -1$."

Putting all this together gives us a fourth thesis:

> *IV* Each Q-proposition is associated with a subspace of a Hilbert space. (i) If two propositions are associated with the same subspace, the propositions are necessarily equivalent. (ii) If two propositions are associated with orthogonal subspaces, then the truth of one proposition entails the falsity of the other. (iii) If the subspace associated with a Q-proposition Q lies within the subspace associated with the subspace associated with Q', then the truth of Q entails the truth of Q'.

This is the heart of Quantum Logic. The algebraic structure of the theory suggests a particular network of relations among quantum mechanical properties. According to Quantum Logic, these relations are reflected in logical relations among propositions that ascribe properties to the system. Taken in small handfuls, the relations don't lead to any conflict with classical logic. For example, we could describe a classical structure that exhibits the relations of equivalence and exclusion among Hs, Sx, Sy, Sz, $(Sx)^2$, $(Sy)^2$ and $(Sz)^2$. However, as the network grows, we reach a point where a classical structure can't make room for the relations. If we held onto the view that every quantity always has one of its values, this tipping point would be a collapse into incoherence. Quantum Logic tells another story.

7 GLEASON'S THEOREM

Consider a finite algebra B of propositions that obey classical logic. The algebra will be Boolean, and it will contain atoms – maximally informative non-contradictory elements. If we assign the value 1 (i.e., true) to an atom, then the truth value of every other proposition in the algebra is determined. Furthermore, these truth values amount to a measure on the algebra B, with values in the interval [0,1].

Talking about the whole interval [0,1] is a bit coy, but the reason is probably obvious. Suppose that A is an algebra of Q-propositions associated with a finite-dimensional Hilbert space of dimension 3 or greater. Then A also has atoms, and if we assign the truth value 1 to one of these atoms, there is a unique measure on A that assigns each proposition a value in the interval [0,1].[8] This is a consequence of Gleason's theorem[9], and we'll call such values the *Gleason measures* of the propositions. The difference, however, is that in the classical case, all the values are in the set {0,1}; in the quantum case, they fill up the whole interval [0,1].

Let H be a Hilbert space of dimension 3 or greater, and let A be the associated algebra of propositions. (Whether A is a lattice or a partial Boolean algebra is something we don't need to decide at this point.) Suppose that R is a ray in H, and that R is the associated proposition. Now suppose that R is true and let S be a sphere containing R. By IV (iii), the proposition S associated with S is true, but S has many representations. In particular, there are infinitely many disjunctions $S_1 \vee S_2 \vee S_3$, where the S_i correspond to orthogonal rays, and where $S_1 \vee S_2 \vee S_3$ is equivalent to S. By IV (ii), if one of the disjuncts is true, then the others are false. Could each such disjunction be true by virtue of the truth of one of its disjuncts?

It's an immediate consequence of Gleason's theorem – or of Kochen and Specker's – that the answer is no. But since R implies each of these disjunctions, this tells us that if Quantum Logic is correct, there must be true disjunctions without true disjuncts.

This need not mean that all the disjuncts are false. Consider

V There are Q-propositions that are neither true nor false.

It seems wrong to say that the propositions corresponding to the components of a superposition are true. However, interference effects are real; the possibilities that correspond to the components of a superposition seem to have an influence on the actual that would be strange if these propositions were simply false. The thought behind V is not that we need to make room for vagueness or linguistic indecision, but for the strange way in which the components of a quantum superposition bear on the world.

In any case, if we say that some propositions are neither true nor false, we avoid an unpleasant consequence: true disjunctions all of whose disjuncts are false. But if not true and also not false, then what? Perhaps we don't need a firm answer, but here is one possibility. We could take the Gleason measures induced by the truth of an atom to be *truth-values*. When a proposition's Gleason measure is close to 1, its contribution to the superposition all but swamps the contributions of the other components; when its Gleason measure is close to 0, it makes all but no contribution – and so on. And of course, if the Gleason measures are truth-values, then we can say more about the truth of disjunction. In classical logic, the truth value of an exclusive disjunction is the sum of the truth values of its components. The same would be true for Gleason-measure truth values. In particular, if $P \vee Q$ is a true quantum disjunction with mutually exclusive disjuncts, then the "Gleason truth-values" of the disjuncts will sum to 1. Also, the more complicated rules that apply to non-exclusive disjunctions will be borne out as well, provided the components of the disjunction all belong to a common Boolean algebra. (This, by the way, seems like a reason for preferring partial Boolean algebras to lattices.)

We will remain agnostic about whether Gleason measures are truth values; the issues would take us too far afield. Nonetheless, Gleason measures do encode real features of the system. We'll say more about this later.

8 ANSWERING KRISKE

We have the materials for answering Kriske on his own terms. The claim is that if we rest with classical logic, we've overlooked something. We described it as the possibility that disjunctions might be true even though their disjuncts aren't. The more basic notion is incompatibility or, as I prefer, incommensurability. Two propositions are incommensurable if they don't belong to a common Boolean algebra. This general notion applies in principle to a broader class of situations than the purely quantum mechanical. If we concentrate on quantum mechanics, and if we agree that some Q-propositions are neither true nor false, then incommensurability amounts to this: two Q-propositions are incommensurable if (a) neither implies the other, and (b) the truth of one rules out the falsity of the other. Notice that (b) isn't possible classically unless (a) is false; in classical logic, if a proposition X rules out the falsity of a proposition Y, then X implies Y.

For Q-propositions, (a) and (b) can be restated in terms of Gleason measures: P and Q are incommensurable if (a) there are Gleason measures that assign 1 to P but not to Q and vice-versa, and (b) every Gleason measure that assigns 1 to P assigns Q a value strictly greater than 0 and vice-versa.

The claim, then, is that the classical logician has overlooked the possibility that propositions can be incommensurable. However, Kriske's model of a case in which a logician has overlooked something is the rejection of Aristotelian logic, where what was overlooked could be uncovered simply by reasoning. Is this notion of incommensurability likewise something that we could have come up with simply by reasoning?

Perhaps. We can imagine a story like the one in Paper Four of *Interpreting the Quantum World*.[10] There, Jeff imagines a bright student who invents quantum mechanics as a thought experiment while thinking about Hilbert space. We could tell a similar tale for the interpretation we're offering here. On such a story, the empirical discovery relevant to Quantum Logic would not be the discovery that incommensurability is a coherent notion, but rather the discovery that there actually *are* empirically significant incommensurable propositions. This is something that couldn't have been known *a priori*, and so it couldn't have been known *a priori* that Classical Logic is inadequate for describing physical reality. But just as it could be and arguably was known *a priori* that geometry didn't *have* to be Euclidean, so it could have been, though wasn't, known *a priori* that *I* through *V* could all be true. Or so the quantum logician would say.

In actual fact, the idea that Classical Logic might be inadequate wasn't dreamt up as an exercise in abstract mathematics. It was a response to empirical discoveries rather than a speculation that guided them. However, the relevant question for answering Kriske is whether the quantum logician's proposal amounts to a coherent thesis. This is a conceptual question, even though it almost certainly would never have arisen but for certain scientific developments. Given Tupman's understanding of Quantum Logic, Kriske was right to accuse him of incoherence. However, Kriske's specific criticisms were directed at the claim that whenever a disjunctive Q-proposition is true, one of its disjuncts is true. Those criticisms have no force here. His more general line of attack was that the idea of "adopting a logic" is incoherent and that logic is ultimately a matter of reasoning or "intuition." The reply is that there is an issue for "reasoning" or "intuition," or perhaps better, philosophical reflection here: whether the quantum logical proposal is coherent. Perhaps it's not. But it won't do, for instance, to insist that from $P \wedge (Q \vee R)$ it follows that one of $(P \wedge Q)$, $(P \wedge R)$ is true. The quantum logician claims that this overlooks a real possibility: the possibility that the pairs of propositions (P, Q) and (P, R) might be incommensurable.

8.1 Disjunction defined?

Some may think (Tim Maudlin, for example[11]) that disjunction is *defined* by the requirement that for a disjunction to be true, at least one of its disjuncts must be true. However, this definition operates against the background assumption that all propositions have truth values. When that assumption falls away, it's no longer so clear that this is the best understanding of disjunction. Notice that there will be a "truth-maker" for a quantum disjunction. It will be the truth of the proposition that implies it. Also, there will be a true disjunction that *is* true by virtue of the truth of one of its disjuncts, and that is equivalent to the anomalous disjunction with its non-true disjuncts. Furthermore, if either disjunct of our anomalous disjunction *were* true, then that disjunct would be a truth-maker for this disjunction. In other words, the quantum disjunction can be made true by one of its disjuncts; in the right circumstances, it behaves like a classical disjunction. What the quantum logician adds is that there are also circumstances not hitherto dreamt of in our philosophies.

9 FROM LOGIC TO THE LAB

What's been said so far about the coherence of Quantum Logic is at best a sketch of a defense. However, suppose the sketch could be filled in. We come now to a harder question. Suppose we allow that the logical relationships among properties *could* be as the quantum logician says. How could we know – or at least reasonably believe – that there really are systems with that sort of structure?

The answer to that question surely turns at least in part on another: what would we expect to see if we encountered a system whose property structure fit Quantum Logic? Even if it's possible for propositions to be incommensurable, it's not necessary that any actually are; the non-classical features of Quantum Logic *could* fail to fit the real world. To have reason to believe that the world has quantum-logical features, we would have to have reason to think that those features would make a detectable difference to the way things behave. And so we need to ask: what would that difference be?

At this level of generality, there is no clear answer. To find out anything about a system, we have to interact with it, and unless we know something about the sorts of interactions that can take place, we have no basis for any expectations. What would we need to assume about systems supposedly described by Quantum Logic for empirical questions about them to have any content? I will take it for granted that we assume each Boolean subalgebra to correspond to an observable. We would also need to assume that we can prepare systems in such a way that certain Q-propositions are true of them – that we can prepare states, in effect. And to have any assurance of that, we would also have to assume that *if* a system is prepared with a certain property, there are reliable ways of making it display the property. All of this needs more spelling out. For the sake of brevity, I will gesture to the assumptions that Simon Saunders makes use of in the first three sections of his "Derivation of the Born Rule from Operational Assumptions."[12] However, a tricky issue remains.

9.1 *Two kinds of contextualism*

A Q-proposition will typically belong to many Boolean subalgebras. Quantum logic is non-contextual in that it counts the proposition as picking out the same state of affairs regardless of which Boolean subalgebra we associate it with. This is, as it were, *ontological non-contextualism*. However, there is a further *empirical* issue about contextualism.

Suppose Q is a quantity with distinct values $q_1, q_2 \ldots q_n$ and that R is another quantity with values $r_1, r_2, \ldots r_n$. Suppose that none of the propositions "$Q = q_i$", "$R = r_i$" are true and none are false. (On the standard picture, this would amount to supposing that the state vector is given by

$$|\psi\rangle = \Sigma_i c_i |q_i\rangle = \Sigma_i d_i |r_i\rangle$$

where none of the coefficients are zero.) Finally, suppose that $|q_1\rangle\langle q_1| = |r_1\rangle\langle r_1|$.

First, consider a measurement of Q. What should we expect?

We know what to expect *in fact*: repeated measurements of Q in state $|\psi\rangle$ should yield distributions of results that accord with the Born rule. However, the question is what we should expect if we look at things with an eye to figuring out what a quantum logical world would be like.

Suppose we can expect to get some result or other – a macroscopic event that betokens one of the eigenvalues q_i. And suppose we want to assign probabilities to such results. How should we do it?

It won't do simply to appeal to the Born Rule. Our assumption is that the various Q-propositions are related as Quantum Logic says they are. But Quantum Logic is an account of relations of equivalence, exclusion and implication. That doesn't immediately tell us anything about experimental probability.

It might seem that we can easily bridge the gap. As we have already pointed out, the quantum logical structure , together with the assumption that the proposition associated with $|\psi\rangle\langle\psi|$ is true (call it P), yields a Gleason measure on all the propositions. This measure is unique; there is no other way to assign numbers in [0,1] simultaneously to all the Q-propositions in such a way that the numbers yield measures on each of the Boolean subalgebras. Moreover, the Gleason measure of a proposition will be the very number given by the Born Rule. In this case, the Gleason measure of proposition $Q = q_1$ will be $|\langle\psi|q_1\rangle|^2$, and since $|q_1\rangle = |r_1\rangle$, this will also be the Gleason measure of $R = r_1$. Can't the quantum logician simply treat this as a probability?

Not without further assumptions. Let's grant that when P is true, the system has some feature represented by the fact that $Q = q_1$ has Gleason measure $|\langle\psi|q_1\rangle|^2$ – in this case, the same feature as the one represented by the fact that $R = r_1$ has the Gleason measure $|\langle\psi|r_1\rangle|^2$. The question, however, is what understanding of this feature Quantum Logic is entitled to. It's not hard to see how it might *fail* to be a probability.

A measurement of Q might elicit the property associated with $Q = q_1$. Furthermore, since this is the same property as the one associated with $R = r_1$, that would also count as eliciting the property associated with $R = r_1$. Parallel comments apply to a measurement of R. But even though Quantum Logic is ontologically non-contextual, two different probabilistic ways of eliciting one and the same property *could* yield two different probabilities. Put another way, even though Quantum Logic treats quantum quantities as ontologically non-contextual, it doesn't rule out the possibility that they are *empirically* contextual.[14] Given that we've rejected the value-definiteness thesis, Quantum Logic will have to say that a measurement typically doesn't just reveal something; it induces a change in the system. There's nothing incoherent in the thought that the way in which the change is induced might affect the probabilities for one and the same micro-event to occur.

9.2 Trimming the context tree

In order to know what to expect if Quantum Logic is correct, we need to assume more than that quantum mechanical propositions are related as Quantum Logic says they are. One obvious additional assumption is that for ideal measurements, the only properties that bear on the empirical probabilities are the ones encoded in the Quantum Logical algebra of propositions – that those are what ideal measuring instruments respond to. This is hardly an *ad hoc* move. Making such an assumption amounts to assuming that the relations embodied in the Quantum Logical algebra of propositions are fundamental for determining how quantum systems will behave; there are no further "hidden variables." This fits with the idea that quantum theory is a principle theory whose fundamental constraints are given by the Quantum Logical algebra of propositions – an idea that has long been part of Quantum Logic.[14] If we make this assumption, then Gleason's theorem guarantees that the only possible assignments of probabilities are the ones that accord with the Born rule. Though more needs to be said, Quantum Logic at least offers the promise of a coherent, attractive foundation for thinking about quantum probability. The probabilities emerge from the most basic features of the quantum quantities: their logical relationships to one another.

10 THE QUANTUM LOGICIAN'S CONUNDRUM

We've described a version of Quantum Logic that avoids the incoherence of Tupman's approach but still counts as realist: it sees Quantum Logic as an hypothesis about the way the world is structured. It also takes seriously Kriske's challenge that logic must be grounded in *reasoning*. It meets the challenge by claiming that classical logic has overlooked a coherent possibility: a disjunction could be true even though none of its disjuncts are. As a mere abstract claim, this would have little to recommend it. However, we pursued the idea that at its heart, Quantum Logic is a view about the way in which quantum-mechanical properties are related to one another. The thesis about disjunction is grounded in this deeper picture. Furthermore, Quantum Logic offers the beginnings of an appealing treatment of probabilities.

All of this *seems* to add up to an answer to the question "How would things behave if Quantum Logic were correct?" The answer seems to be: they would behave the way that quantum theory, as usually understood, says they would. In fact, this answer is problematic.

10.1 How Quantum Mechanical is a Quantum Logical world?

What has been said so far leaves some large questions. For one thing, nothing has been said about dynamics. Measurement aside, dynamical transformations are usually thought of in Quantum Logic as automorphisms on the algebra of propositions; every unitary transformation on a Hilbert space induces such an automorphism. However, we can't leave measurement aside, and Quantum Logic as presented here can't avoid the measurement problem. Measurements are stochastic changes in the properties of the systems, and they can't be modeled by automorphisms on the algebra. Quantum Logic has nothing to say about what induces those changes. Worse still, if the explanation for non-unitary change is some additional variable, it will no longer be clear that empirical probabilities should depend only on which Q-propositions are true before the measurement and on what's encoded in the algebra of propositions. This puts Quantum Logic's account of quantum probability at risk.

Of course, it's not clear that quantum mechanics itself has much to say about what why measurements have results. The measurement problem, after all, is the problem of explaining how *quantum mechanics* can provide a satisfactory account of measurement. Perhaps the Quantum Logician can punt on this issue. It's not clear that in order to be viable, Quantum Logic has to answer all interpretive questions. All the quantum logician need claim is that the structures Quantum Logic posits are *part* of the story of why quantum systems behave as they do.

There's an obvious related issue. Since Quantum Logic posits indefinite values, the problem of Schrödinger's cat looms on the horizon. Once again, the difficulty isn't peculiar to Quantum Logic, but the rejection of value-definiteness means that Quantum Logic can't dodge the problem in any easy or obvious way. Still, we might say, although Quantum Logic must be consistent with some acceptable solution to these problems, it needn't contain the solution itself.

And then there's locality. If a pair of electrons is in the singlet state, then the quantum logician is committed to saying that none of the local spin quantities have values. However, after a spin measurement on one of the systems, what was once indefinite on the distant system will become definite. Something has changed "there" because of something that happened "here." Quantum Logic may be able to avoid ontological non-locality, but it's hard to see how it can steer clear of non-local causal influences. Furthermore, we have the familiar problem of selecting the hyperplane on which the change occurs.

Once again, we have a difficulty that's hardly unique to Quantum Logic. But that excuse may be wearing thin. Quantum Logic appears to have nothing to contribute to the problem of explaining why measurements have results; it practically ensures that we will face the problem of Schrödinger's cat; and it seems to be on a collision course with special relativity. But measurements *do* have results, superposed cats appear to be mythical beasts, and conflict with special relativity is to be avoided where possible.

Although we pointed out that there are lingering issues about contextualism, the area where Quantum Logic seems to show the most promise is in understanding quantum probability. However, it's not clear that Quantum Logic has any real advantage here. Recent work by Deutsch[15] and Wallace[16] on probability in the Everett interpretation has been extended by Simon Saunders[17] to all interpretations that treat different ways of performing measurements as equivalent whenever they are unitarily equivalent. Saunders shows, generalizing Deutsch's result, that with this assumption, we can derive the Born rule from what he refers to as operational assumptions. The proof is compact and elegant; no need for Gleason's theorem.

We've arrived at the conundrum. For it to be plausible that Quantum Logic is a coherent thesis, there has to be a good answer to the question of what the world would be like if it were quantum logical. We know that the world acts the way that *quantum mechanics* says it does, and we know that Quantum Logic fits neatly into the standard mathematical apparatus that quantum mechanics uses. But quantum mechanics is not just its mathematics; it's also a set of techniques and practices for applying the math. We know that trying to think of that mathematical apparatus as a depiction of the world leads to the frustratingly hard problems of interpretation that have kept workers in the foundations of physics employed for decades. It may be that the usual mathematical apparatus is nothing but the guts of a highly successful prediction machine that *can't* be taken at face value. And it may be that Quantum Logic is the purest expression of what makes the standard theory so hard to interpret!

11 INCONCLUSIVE CONCLUDING THOUGHTS

The version of Quantum Logic under consideration here is an attempt to follow the realist instinct that motivated Western Ontario-style quantum logic thirty odd years ago. What's gone are the twin commitments to definite values and to the thesis that measurements simply reveal. The realism that remains consists in two things: first, the claim that quantum properties really embody the logical relations that Quantum Logic says they do, and second the claim that this fact helps explain why quantum systems behave as they do. And while bivalence is gone, this version of Quantum Logic takes the Gleason measures of propositions to be real features of the system. If we were to take Gleason measures as truth values, then even though bivalence itself would be gone, we would have a definite though non-standard realist understanding of truth.

This is by no means the only approach to quantum logic (lower-case to indicate the generic.) Much work done under the heading "quantum logic" is frankly operational and makes no radical claims about the structure of properties.[18] More recently, William Demopoulos has offered an understanding of quantum logic according to which the *logical* relations among quantum propositions aren't represented by the structure referred to here as Quantum Logic, but by a much less constraining structure that allows every quantum mechanical proposition to be determinately true or false. However, on Demopoulos's view complete knowledge of a quantum system is impossible in principle; the structure that we have been calling Quantum Logic has epistemic rather than alethic significance. It represents constraints on our *knowledge* of the quantum world.[19]

Quantum mechanics is strange business and whatever the true story of the quantum world may be, it's safe to say it's weird. Wildly different interpretations abound; none can claim wide allegiance. Worse still, it's far from clear how we should even go about deciding among the competitors. Bohmian mechanics is consistent, far as I know. Is it true? How would we decide? The Everett interpretation is probably consistent. It *may* even be able to make sense of the probabilities, though I have my doubts.[20] But even if such doubts can be resolved, many of us find it hard to imagine actually believing that the picture is correct. However – and not helpfully for getting at the truth – all of this may be a matter of taste. Quantum Logic invokes its own incredulous and irrefutable stares.[21]

So there we are. Perhaps for purely sentimental reasons, I'd like to think that Quantum Logic, understood in a realist way, is a coherent conjecture, and that it can make some genuine explanatory contribution to our understanding of quantum systems. I'd like to think this; I'm not quite ready to say it isn't so. But if the True Believers were asked to stand, I fear I'd be huddled in the corner with those of flickering faith.

NOTES

1 Kochen, S. and Specker, E. P., "The problem of hidden variables in Quantum Mechanics," *Journal of Mathematics and Mechanics* **17** (1967) 59–67.

2 Our way of writing the disjunction is a bit misleading. The statement $Q = q_i$ will correspond to the same subspace as various other statements, e.g., $R = r_j$, and for the contradiction to emerge, each of these statements must be taken to represent the same proposition and expressed accordingly.

3 Putnam, Hilary, "Is logic empirical?" in R. Cohen and M. P. Wartofski (eds.), *Boston Studies in the Philosophy of Science* **5** (Dordrecht, Holland: D. Reidel, 1968).

4 Quine, Willard van Orman *The Ways of Paradox and Other Essays* (Revised and Expanded Edition), Cambridge: Harvard University Press, 1976. 102, ff.

5 Carroll, Lewis, "What the Tortoise Said to Achilles," *Mind* **4** (1985) 278–280.

6 Alston, William, *Perceiving God*, Ithaca NY: Cornell University Press, 1991.

7 McGee, Van, "A Counterexample to Modus Ponens." *Journal of Philosophy* (September 1985), 82: 462–471.

8 By calling this a measure, we mean that it is defined on subspaces of the Hilbert space, that the measure of the whole space is 1, and that if S and S' are orthogonal subspaces, then the measure of their span $S \vee S'$ is the sum of the measures of S and S'.

9 Gleason, Andrew M. "Measures on the closed subspaces of a Hilbert space," *Journal of Mathematics and Mechanics*, **6** (1957) 885–893.

10 Bub, Jeffrey, *Interpreting the Quantum World*, Cambridge: Cambridge University Press, 1997.

11 Maudlin, Timothy, "The Tale of Quantum Logic" (forthcoming).

12 Saunders, Simon, "Derivation of the Born Rule from Operational Assumptions," *Proceedings of the Royal Society, London* **A 460** (2004) 1–18.

13 This distinction is rather like the distinction that Heywood and Redhead make between ontological and environmental contextualism. One notable difference is that Heywood and Redhead's distinction is set in the context of a discussion of local hidden variables. See Heywood, P. and Redhead, M. L. G., "Non-locality and the Kochen-Specker paradox," *Foundations of Physics* **13** (1983) pp. 481–499. See also Redhead, Michael, *Incompleteness, Nonlocality and Realism*. Oxford: Clarendon Press (1987) ch. 6.

[14] See, for example, Bub, Jeffrey, *The Interpretation of Quantum Mechanics*. Dodrecht, Hollad; Reidel (1974) p. viii, 143.

[15] Deutsch, D. "Quantum Theory of probability and decisions." *Proceedings of the Royal Society of London* **A** 455 (1999) 3129–3137.

[16] Wallace, David, "Everettian rationality." *Studies in History and Philosophy of Modern Physics* 34 (2003) 87–105.

[17] Loc. cit.

[18] For a review of various approaches to quantum logic, see Wilce, Alexander, "Quantum Logic and Probability Theory, *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/entries/qt-quantlog/#2.

[19] Demopoulos, William, "Elementary propositions and essentially incomplete knowledge: a framework for the interpretation of quantum mechanics," *Noûs* **38**: 1 (2004) 86–109. The logical relations, on Demopoulos's view, are represented by the free partial Boolean algebra whose structure is isomorphic to the partial Boolean algebra of subspaces of two-dimensional Hilbert space.

[20] I don't question the mathematics of David Wallace's improvements on Deutsch's argument. (See the pieces by Deutsch and Wallace cited above.) I do, however, harbor grave doubts about whether a rational agent is in any way constrained to treat possibilities that have very different consequences for how the branching will unfold as though they were equivalent for purposes of probability.

[21] With apologies to David Lewis, who famously remarked, when confronted with certain critics of his modal realism, that it's hard to refute an incredulous stare. For Lewis's methodological discussion of the incredulous stare, see Lewis, David K., *On the Plurality of Worlds*. New York: Blackwell (1986) p. 133 ff.