Error Propagation and Roundoff Error

In general our problem has a certain number of input values \( x_1, \ldots, x_n \) and a certain number of output values \( y_1, \ldots, y_m \), and some (possibly complicated) formulas describe how the output values depend on the input values. Let us consider the simplest case of one input and one output value where we have \( y = f(x) \).

**Error propagation**

If we only have an approximation \( \tilde{x} \) of our input value \( x \) available (e.g., because of measurement errors), the best thing we can do is to compute \( y := f(\tilde{x}) \). For the resulting relative error we obtain

\[
\varepsilon_y := \frac{\tilde{y} - y}{y} = \frac{f(\tilde{x}) - f(x)}{f(x)} \approx \frac{(\tilde{x} - x) f'(x)}{f(x)} = \frac{x f'(x)}{f(x)} \cdot \frac{\tilde{x} - x}{x} = c_f \cdot \varepsilon_x
\]

where the magnification factor \( c_f(x) := \frac{x f'(x)}{f(x)} \) is called the condition number of the function \( f \) at \( x \). The condition number determines how sensitive a problem is to small perturbations of input values. If \( |c_f| \) is not much larger than 1 we call the problem well-conditioned, in the case of \( |c_f| \gg 1 \) we call the problem ill-conditioned.

**Example 1:** The function \( f(x) = \frac{1}{x} \) has the condition number \( c_f(x) = \frac{x}{x^2} = -1 \) and is therefore well conditioned for all \( x \). E.g., for \( x = 2 \) and \( \tilde{x} = 1.96 \) we obtain \( \varepsilon_y \approx -\varepsilon_x \):

\[
\begin{align*}
  x &= 2 \\
  y &= f(2) = \frac{1}{2} = .5 \\
  \tilde{x} &= 1.96 \\
  \tilde{y} &= f(\tilde{x}) = \frac{1}{1.96} \approx .5102 \\
  \varepsilon_x &= \frac{\tilde{x} - x}{x} = -.02 \\
  \varepsilon_y &= \frac{\tilde{y} - y}{y} \approx .0204
\end{align*}
\]

For the function \( f(x) = x^\alpha \) we obtain the condition number \( c_f(x) = \frac{x \cdot \alpha x^{\alpha-1}}{x^\alpha} = \alpha \). This is therefore well conditioned unless \(|\alpha| \) is huge.

**Example 2:** The function \( f(x) = \ln x \) has the condition number \( c_f(x) = \frac{x - 1}{\ln x} = \frac{1}{\ln x} \). For \( x = 1.01 \) the function is ill conditioned: we obtain the condition number

\[
c_f(x) = \frac{1}{\ln x} \approx \frac{1}{1.01} = 100
\]

using the Taylor approximation \( \ln x \approx 0 + 1 \cdot (x - 1) \) for \( x \) close to 1. E.g., for \( x = 1.01 \) and \( \tilde{x} = 1.02 \) we obtain \( \varepsilon_y \approx 100 \varepsilon_x \):

\[
\begin{align*}
  x &= 1.01 \\
  \tilde{x} &= 1.02 \\
  \varepsilon_x &= \frac{\tilde{x} - x}{x} \approx .0099 \\
  y &= f(x) \approx .00995 \\
  \tilde{y} &= f(1.02) \approx .0198 \\
  \varepsilon_y &= \frac{\tilde{y} - y}{y} \approx .99
\end{align*}
\]

Note that \( x \) and \( \tilde{x} \) are close to a zero of \( \ln x \). Since \( \ln x \approx x - 1 \) for \( x \) close to 1 we have \( y \approx .01 \) and \( \tilde{y} \approx .02 \) which corresponds to a relative error of 1 whereas \( \frac{\tilde{x} - x}{x} \approx .01 \).