1. We are given the following information about $f(x)$:

$$f(0) = 2, \quad f(1) = 1, \quad f(3) = 0, \quad f(4) = 1$$

(a) Write down the divided difference table. Find interpolating polynomial in Newton form (i) for the nodes in the order $0, 1, 3, 4$, (ii) for the nodes in the order $4, 3, 1, 0$.

Divided difference table: $x_0 = 0, x_1 = 1, x_2 = 3, x_4 = 4$

$$
\begin{align*}
  f[x_1] &= 2, \quad f[x_1, x_2] = -1 \\
  f[x_2] &= 1, \quad f[x_2, x_3] = -\frac{1}{2} \\
  f[x_3] &= 0, \quad f[x_3, x_4] = 1 \\
  f[x_4] &= 1
\end{align*}
$$

$p(x) = f[x_1] + f[x_1, x_2](x-x_1) + f[x_1, x_2, x_3](x-x_1)(x-x_2) + f[x_1, x_2, x_3, x_4](x-x_1)(x-x_2)(x-x_3)$

$$
= 2 + (-1) \cdot (x-0) + \frac{1}{6}(x-0)(x-1) + \frac{1}{12} \cdot (x-0)(x-1)(x-3)
$$

$p(x) = f[x_4] + f[x_3, x_4](x-x_4) + f[x_2, x_3, x_4](x-x_3)(x-x_4) + f[x_1, x_2, x_3, x_4](x-x_2)(x-x_3)(x-x_4)$

$$
= 1 + 1 \cdot (x-4) + \frac{1}{2}(x-4)(x-3) + \frac{1}{12} \cdot (x-4)(x-3)(x-1)
$$

(b) Assume we know that the 4th derivative satisfies $|f^{(4)}(x)| \leq 10$ for $x \in [0, 4]$. Find an upper bound for $|f(2) - p(2)|$.

Let $\tilde{x} = 2$. The error formula states that there exists $t \in (0, 4)$ such that

$$
|f(\tilde{x}) - p(\tilde{x})| = \left| \frac{f^{(4)}(t)}{4!} \right| |(\tilde{x} - x_1)(\tilde{x} - x_2)(\tilde{x} - x_3)(\tilde{x} - x_4)|
$$

$$
\leq \frac{10}{24} \cdot 2 \cdot 1 \cdot 2 \cdot 2 = \frac{5}{3}
$$

2. Consider the $(x, y)$ data points $(-1, 2), (1, 1), (2, 0)$. We want to fit the data with a function $g(x) = c_1 + c_2x^2$

(a) Find the best least squares fit by hand.

Here $y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. With $\phi_1(x) = 1$ and $\phi_2(x) = x^2$ we have $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}$. The normal equations are $A^T Ac = A^T y$:

$$
\begin{bmatrix}
3 & 6 \\
6 & 18
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
3
\end{bmatrix}
$$

which gives $c_1 = 2, c_2 = -\frac{1}{2}$.

(b) Write a Matlab program which the backslash command solve this problem.

```matlab
x = [-1;1;2]; y=[2;1;0];
A = [x.^0,x.^2];
c = A\y;
```

3. We want to find $x$ such that $x + x^5 = 3$.

(a) Perform one step of the bisection method with $a_0 = 1, b_0 = 2$. Find $k$ such that $|b_k - a_k| \leq 10^{-6}$.

$f(x) = x^5 + x - 3, f(a_0) = -1 < 0, f(b_0) = 31 > 0, c_0 = (a_0 + b_0)/2 = 1.5, f(1.5) = 1.55 + 1.5 - 3 = 1.5^5 - 1.5 > 0$, hence $[a_1, b_1] = [a_0, c_0] = [1, 1.5]$. So we have $x_* \in (1, 1.5)$.

We have $|b_k - a_k| = 2^{-k} |b_0 - a_0| = 2^{-k}$. We have

$$
2^{-k} \leq 10^{-6} \iff (-k) \log 2 \leq \log(10^{-6}) \iff k \geq \frac{\log(10^6)}{\log 2} = 19.93,
$$

hence we need $k \geq 20$.

(b) Perform one step of the secant method with $x_0 = 1, x_1 = 2$ to find $x_2$.

$$
x_2 = x_1 - \frac{f(x_1) x_1 - x_0}{f(x_1) - f(x_0)} = 2 - 31 \frac{2 - 1}{31 - (-1)} = 2 - \frac{31}{32} = 1 + \frac{1}{32}
$$
(c) Will the Newton method converge if we start with \( x_0 \) sufficiently close to the solution \( x_* \)? Explain.
We showed: If \( f, f', f'' \) are continuous and \( f'(x_*) \neq 0 \), then the Newton method converges for \( x_0 \) sufficiently close to \( x_* \). Here \( f(x) = x^5 + x - 3 \), \( f'(x) = 5x^4 + 1 \). We know from the intermediate value theorem that there is a root in the interval \((1, 2)\). Since \( f'(x) \geq 1 > 0 \) there is a unique root, and we must have \( f'(x_*) > 0 \). So all the assumptions of the theorem are satisfied.

4. Consider the nonlinear system
\[
\begin{align*}
x_1 + x_1x_2 + x_2 &= 2, \\
x_1 - x_2 - x_1x_2^2 &= 0
\end{align*}
\]

(a) Perform one step of the Newton method starting with initial guess \( x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).
We have \( f(x) = \begin{bmatrix} x_1 + x_1x_2 + x_2 - 2 \\ x_1 - x_2 - x_1x_2^2 \end{bmatrix} \) with the Jacobian matrix \( f'(x) = \begin{bmatrix} 1 + x_2 & x_1 + 1 \\ 1 - x_2^2 & -1 - 2x_1x_2 \end{bmatrix} \). For \( x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) we get \( y = f(x^{(0)}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) and \( A = f'(x^{(0)}) = \begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix} \). Solving the linear system \( Ad = -y \) gives \( d = \begin{bmatrix} -1/6 \\ -1/3 \end{bmatrix} \) and the new approximation \( x^{(1)} = x^{(0)} + d = \begin{bmatrix} 5/6 \\ 2/3 \end{bmatrix} \).

(b) Write a Matlab program which uses the Newton method to find a solution, starting with initial guess \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). The program should print out the approximation for \( x \) after each iteration.

```matlab
f = @(x) [ x(1)+x(1)*x(2)+x(2)-2 ; x(1)-x(2)-x(1)*x(2)^2 ];
fp = @(x) [ 1+x(2) , x(1)+1; 1-x(2)^2 , -1 -2*x(1)*x(2)];

x = [1;1];
while 1
    b = f(x); A = fp(x);
    d = -A\b;
    x = x + d \text{ \% prints out } x \text{ for each iteration}
    if norm(d)<1e-14
        break
    end
end
```