HW4, due Tuesday, November 11
Math 403, Fall 2013
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1. Suppose X is a topological space and U is an open subset of X. Write h_U for the sheaf on X consisting of continous sections of the étalé space $U \to X$. So $h_U(V)$ is the empty set if V is not contained in U and $h_U(V)$ is a singleton if V is contained in U. Suppose $\mathscr{F}$ is a sheaf of sets on X. Show that there is a canonical isomorphism $\mathscr{F}(U) \to \mathrm{Hom}(h_U, \mathscr{F})$ sending a map $\sigma : h_U \to \mathscr{F}$ to the section of $\mathscr{F}(U)$ which is the image of the singleton $h_U(U)$.

2. In a category $\mathscr{C}$, a morphism $f : X \to Y$ is said to be a *monomorphism* if for all objects Z, the induced map $\mathrm{Hom}(Z,X) \to \mathrm{Hom}(Z,Y)$ is injective. Using the previous exercise, show that a morphism $\mathscr{F} \to \mathscr{G}$ in the category of sheaves on a topological space X is a monomorphism if and only if, for every open U in X, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

3. Suppose X is a topological space and $i : \{x\} \to X$ is the inclusion of a point of X. For a set S, we get a sheaf $i_*(S)$ by pushforward. Show that, for any sheaf $\mathscr{F}$ on X, $\mathrm{Hom}(\mathscr{F}, i_*(S)) = \mathrm{Hom}(\mathscr{F}_x, S)$.

4. In a category $\mathscr{C}$, a morphism $f : X \to Y$ is said to be an *epimorphism* if the opposite map to f is a monomorphism in the opposite category to $\mathscr{C}$. Explicitly, this means that f is an epimorphism, if for all Z in $\mathscr{C}$, the map $\mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ is injective. Using the previous exercise, show that a morphism $\mathscr{F} \to \mathscr{G}$ of sheaves on a topological space X is an epimorphism if and only if the map $\mathscr{F}_x \to \mathscr{G}_x$ is surjective.

5. Show that a map of sheaves on X is a monomorphism iff the map $\mathscr{F}_x \to \mathscr{G}_x$ is a monomorphism for all $x \in X$.

6. Show that a morphism of sheaves of sets is an isomorphism if and only if it is an epimorphism and a monomorphism.

7. Write **Rings** for the category of rings. Any scheme X defines a functor h_X : **Rings** $\to$ **Sets** sending a ring R to $h_X(R) := \mathrm{Hom}(\mathrm{Spec}\, R, X)$. Usually we write $X(R)$ for $h_X(R)$ and call $X(R)$ the *R-valued points of X*. Show that the natural map $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(h_X, h_Y)$ is an isomorphism. Conclude that X and Y are isomorphic if and only if h_X and h_Y are. We say a functor F : **Rings** $\to$ **Sets** is *representable* if it is isomorphic to a functor of the form h_X for X a scheme.

8. GW Problem 3.1.

9. GW Problem 3.5.

10. GW Problem 3.6.