HW2, due Tuesday, September 24
Math 606, Fall 2013
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1. Let X denote the affine algebraic subset of $\mathbb{A}^2(\mathbb{C})$ defined by the equation $y^2 = x^3$. Let U denote the complement of the point $(1,1)$ in X. Show that the rational function $f(x,y) = x^2/(y-x)$ is regular on U.

Solution. Clearly $f(x,y)$ is regular outside of the locus where $y = x$. Thus f is regular on $U \setminus \{0,0\}$. On the other hand, in the fraction field $K(X)$,

$$\begin{aligned} f = \frac{x^2}{y-x} &= \frac{x^2}{y-x} \cdot \frac{y+x}{y+x} \\ &= \frac{x^2(y+x)}{y^2-x^2} = \frac{x^2(y+x)}{x^3-x^2} \\ &= \frac{y+x}{x-1}. \end{aligned}$$

Since the value of $x-1$ is -1 at $(0,0)$, $x-1$ is not in the maximal ideal $\mathfrak{m}_{(0,0)}$. So f is regular at $(0,0)$.

2. Can f be written as g/h with $g,h \in \Gamma(X)$ and h non-zero on U?

Solution. No. First note that the ideal $(y^2 - x^3) \in \mathbb{C}[x,y]$ is irreducible. In fact, it follows from Gauss's lemma that $y^2 - x^3$ is irreducible as a polynomial in one variable over the field $\mathbb{C}(x)$. So $\Gamma(X) = \mathbb{C}[x,y]/(y^2 - x^3)$ is an integral domain. Now set $Y = \mathbb{A}^1(\mathbb{C})$ and let $\varphi : \Gamma(X) \to \mathbb{C}[t] = \Gamma(Y)$ be the $\mathbb{C}$-algebra homomorphism given by $x \mapsto t^2, y \mapsto t^3$. This is a well-defined ring homomorphism because the ideal $(y^2 - x^3)$ is sent to 0. So it defines a morphism $p : Y \to X$ of irreducible, affine algebraic sets.

Set $V := X \setminus \{(0,0)\}$ and $W := Y \setminus \{0\}$. Define $\sigma : V \to W$ by $(x,y) \mapsto y/x$. Since $1/x$ is regular on V, V is a morphism. Moreover, $p \circ \sigma = \mathrm{id}_V$ and $\sigma \circ (p_{|W}) = \mathrm{id}_W$. It follows that $K(X) = K(V) = K(W) = K(Y) = \mathbb{C}(t)$. So the ring homomorphism $\varphi : \Gamma(X) \to \Gamma(Y)$ is injective. Clearly the image of φ is $\mathbb{C}[t^2,t^3] \subset \Gamma(Y) = \mathbb{C}[t]$. In other words, $\varphi(\Gamma(X)) = \{P \in \mathbb{C}[t] : P'(0) = 0\}$.

Now suppose $h \in \Gamma(X)$ is non-zero on U. Then $P := \varphi(h) = h \circ p$ is non-zero on $p^{-1}(U) = Y \setminus \{1\}$. Thus $P = \alpha(t-1)^n$ for some non-zero $\alpha \in \mathbb{C}$ and some non-negative integer n. But then $P' = n\alpha(t-1)^{n-1}$. So $P'(0) = n\alpha$. It follows that $n = 0$. So P, and thus, h is constant.

So f can be writen as g/h with $g.h \in \Gamma(X)$ and h non-zero on U iff $f \in \Gamma(X)$. But $\varphi(f) = t^4/(t^3 - t^2) = t^2/(t-1)$. So $\varphi(f)$ is not in $\mathbb{C}[t]$. Therefore, f is not in $\Gamma(X)$.