Math 602 Term 2 2012 at UMD
Homework 1
Due Friday, March 8.

Problem 1. Do Exercise 2.4.4 and 2.5.3 in Weibel's book.

Problem 2. Let $\mathcal{A}$ denote an abelian category, and let $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ consisting of objects X such that $X_n = 0$ for all integers $n > 0$. Show that the inclusion functor $i : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ is the left-adjoint of the functor $\tau_{\geq 0} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. To do this, show that, for $Y \in \mathrm{Ch}(\mathcal{A})$ and $X \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$, the inclusion $\tau_{\geq 0} Y \to Y$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, \tau_{\geq 0} Y) \to \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, Y).$$

Problem 3. Let F be a field and let $\mathcal{A}$ denote the category consististing of pairs (V, N) where V is a finite dimensional vector space over F and $N \in \mathrm{End}_F(V)$ is a nilpotent operator. A morphism from $\varphi : (V, N) \to (V', N')$ is an F-linear transformation $\varphi_V : V \to V'$ such that $N' \circ \varphi = \varphi \circ N$.

(1) Show that $\mathcal{A}$ is an abelian category.
(2) Show that the functor $D : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ given by $(V, N) \mapsto (V^*, N^*)$ is an equivalence.
(3) An object (V, N) is *cyclic* if there exists a $v \in V$ such that V is the smallest subobject of (V, N) containing v. Show that every object in $\mathcal{A}$ is a direct sum of cyclic objects.
(4) Define functors $T_i : \mathcal{A}$ to the category Vect_F of F-vector spaces by setting $T_0(V, N) = \ker N$, $T_1(V, N) = \mathrm{coker}\, N$ and $T_i(V, N) = 0$ for any integer $i > 1$. Explain why the snake lemma applied to an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ gives rise to a natural transformation $d : T_0 \to T_1$ making $(T_i)_{i \in \mathbb{Z}}$ into a cohomological δ-functor.
(5) Show that T_1 is effaceable, and conclude that (T_i) is the universal cohomological δ-functor.

Problem 4. Suppose $T : \mathcal{A} \to \mathcal{B}$ is an additive functor between additive categories. Show that $T(0) = 0$. (**Hint:** Remember that, by definition, functors have to take identity morphisms to identity morphisms.)

Problem 5. Suppose X and Y are objects in an additive category $\mathcal{A}$. Let $s_X : X \to X \oplus Y$ and $p_X : X \oplus Y \to X$ be the canonical morphisms arising respectively from viewing $X \oplus Y$ as the coproduct and the product of X and Y. Define $s_Y : X \to X \oplus Y$ and $p_Y : X \oplus Y \to Y$ similarly. Show that $s_X p_X + s_Y p_Y$ is the identity on $X \oplus Y$.

Problem 6. Suppose $\mathcal{A}$ is an additive category as defined in Weibel's book, and suppose that $f, g \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ are two morphisms. Show that $f + g$ is defined by the composition

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{(f,g)} Y \oplus Y \xrightarrow{\nabla} Y$$

where Δ is a the diagonal and $\nabla : Y \oplus Y \to Y$ is the map inducing the identity on both factors.

Problem 7. Suppose $T : \mathcal{A} \to \mathcal{B}$ is a functor between additive categories. Let X and Y denote objects in $\mathcal{A}$ and let s_X, p_X, s_Y, p_Y be as in Problem 5. Define $s : T(X) \oplus T(Y) \to T(X \oplus Y)$ by $s = T(s_X) \oplus T(s_Y)$. Define $p : T(X \oplus Y) \to T(X) \oplus T(Y)$ by $p = T(p_X) \times T(p_Y)$. Show that $p \circ s$ is the identity on $TX) \oplus T(Y)$. Then show that, if T is an additive functor, $s \circ p$ is the identity on $T(X \oplus Y)$. Conclude that additive functors preserve coproducts.

Problem 8. Suppose $T : \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories such that $T(0) = 0$. From the previous problem it follows that we have

$$T(X \oplus Y) = T(X) \oplus T(Y) \oplus T_2(X, Y)$$

where T_2 is the kernel of the $s \circ p$. What is T_2 when $T = \wedge^2 : \mathrm{Vect}_F \to \mathrm{Vect}_F$ where Vect_F is the category of vector spaces over a field?