1. (20 points) Determine if the equation

$$3x^2 = 100 + 163y$$

has a solution with x and y integers.

2. (20 points) Use Euler's theorem to find the last digit of the decimal expansion of $17^{200,007}$.

3. (20 points) For every positive integer n, let $\pi(n)$ denote the number of prime number less than or equal to n. Using Wilson's theorem, show that

$$\pi(n) = \sum_{j=2}^{n} \left\lfloor \frac{(j-1)!+1}{j} - \left\lfloor \frac{(j-1)!}{j} \right\rfloor \right\rfloor.$$

4. (20 points) The point of this question is to work through an alternate proof of the fact that $\left(\frac{2}{p}\right) = -1$ for p a prime congruent to 3 or 5 modulo 8. This proof is the one written in Gauss's book.

(a) Show that $\left(\frac{2}{3}\right) = \left(\frac{2}{5}\right) = -1$ by direct computation.

(b) Suppose p is an odd prime with $\left(\frac{2}{p}\right) = 1$. Show that there is an odd integer x such that $3 \le x \le p-4$ and an integer k such that

$$x^2 - 2 = pk.$$

(c) Show that the integer k above is odd and satisfies $0 < k < p$.

(d) Now suppose that p is congruent to 3 or 5 modulo 8. Show that the integer k above has a prime factor q which is congruent to 3 or 5 modulo 8.

(e) Using induction, conclude that $\left(\frac{2}{p}\right) = -1$ for all primes congruent to 3 or 5 modulo 8.

5. (20 points) Find at least three solutions to the linear diophantine equation

$$63x + 163y = 13.$$