HW5, due Monday, April 13
Math 404, Spring 2015
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1. Suppose f is a separable polynomial over a field F. Suppose E/F and Ω/F are splitting field of f. Explain why the groups $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(\Omega/F)$ are isomorphic. We call the group $\mathrm{Gal}(E/F)$ (taken up to isomorphism) the Galois group of the polynomial f.

2. Let $f(x) = x^4 - 2$, and set $L = \mathbb{Q}[i]/\mathbb{Q}$ and set $M = \mathbb{Q}(\zeta)$ with $\zeta = 2^{1/4}$. View L as a subfield of $\mathbb{C}$ and M as a subfield of $\mathbb{R}$.

(a) Show that $E := LM$ is splitting field of f.
(b) Show that $H := \mathrm{Gal}(E/M)$ is isomorphic to $\mathbb{Z}/2$.
(c) Show that there is a unique element $\tau \in N := \mathrm{Gal}(E/L)$ with the property that $\tau(\zeta) = i\zeta$.
(d) Show that N is the cyclic group of order 4 generated by τ.
(e) Show that $\sigma\tau\sigma^{-1} = \tau^{-1}$.
(f) Conclude that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the dihedral group with 8 elements.

3 (**Bonus** 25 points)**.** Do problem 3-3 on page 46 of Milne's Field Theory book.