HW1, due Wednesday, February 18
Math 404, Spring 2014
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1. Let p be a prime number. Recall that $\mathbb{N} = \{0,1,2,\dots\}$. For each non-zero integer a, let $\nu(a) := \max\{n \in \mathbb{N} : p^n | a\}$. For each non-zero rational number a/b, set $\nu(a/b) = \nu(a) - \nu(b)$. For a rational number x, define

$$|x|_p = \begin{cases} p^{-\nu_p(x)}, & x \neq 0; \\ 0, & \text{else.} \end{cases}$$

(a) Show that ν is well-defined. In other words, show that, if $a/b = a'/b'$ for integers a, a', b, b', then $\nu(a/b) = \nu(a'/b')$.
(b) Show that, for $x, y \in \mathbb{Q}^\times$, $\nu(xy) = \nu(x) + \nu(y)$. Conclude that $|xy|_p = |x|_p|y|_p$ for all $x, y \in \mathbb{Q}$.
(c) Suppose $|x|_p \leq 1$ for some rational number x. Show that $|1+x|_p \leq 1$ as well.
(d) Show that, for $x, y \in \mathbb{Q}$, $|x+y|_p \leq \max\{|x|_p, |y|_p\}$.

2. Let R denote the set of all a/b in $\mathbb{Q}$ where a and b are integers and b is not divisible by p.

(a) Show that $R = \{x \in \mathbb{Q} : |x|_p \leq 1\}$.
(b) Show that R is a subring of $\mathbb{Q}$. (**Hint:** You can do this by hand, but it's easier to use (b) and (d) from the last problem.)
(c) Show that $R^\times = \{x \in R : |x|_p = 1\}$.
(d) Show that every non-zero element of x of R is of the form $p^n u$ for some $u \in R^\times$ and $n \in \mathbb{N}$.
(e) Show that R is a Euclidean domain with Euclidean norm ν.
(f) Show that every ideal in R is either 0 or equal to $p^n R$ for some non-negative integer n.

3. Let $\mathbb{F}_3$ denote the field $\mathbb{Z}/3\mathbb{Z}$ with 3 elements. Show that the polynomial $f = x^3 - x + 1$ is irreducible.

4. Let $K = \mathbb{F}_3[x]/(x^3 - x + 1)$. Explain why the previous problem shows that K is a field. Then set $\alpha = 1 + x^2$. What is α^{-1}? Write it in the form

$$\alpha^{-1} = a + bx + cx^2$$

where $a, b, c \in \mathbb{F}_3$.