HW8, due Monday, April 17
Math 403, Spring 2017
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Reminders about Semi-direct Products. In class, I covered semi-direct products. But they are not covered in the text. So here are some reminders about them.

Suppose N and H are groups and $\varphi : H \to \operatorname{Aut} N$ is a group homomorphism. We can put a binary operation on the Cartesian product $N \times H$ by setting

$$(n_1, h_1)(n_2, h_2) := (n_1\varphi(h_1)(n_2), h_1h_2).$$

Note that this is equal to the usual external direct product operation exactly in the case that $\varphi : H \to \operatorname{Aut} N$ is the trivial homomorphism. In general, we call it the *semi-direct product* binary operation.

In class, I showed that the semi-direct product binary operation gives a group structure on the Cartesian product. We write the resulting group as $N \rtimes H$ or $N \rtimes_\varphi H$ if we want to be specific about φ. It is called the *semi-direct product* of N with H. I wrote $i : N \to N \rtimes H$, $j : H \to N \rtimes H$ and $p : N \rtimes H \to H$ for the maps give as $i(n) = (n, e)$, $j(h) = (e, h)$ and $p(n, h) = h$. I showed that i and j are injective group homomorphisms, while p is surjective with kernel $i(N)$. Then I showed that, for all $n \in N$ and $h \in H$

$$j(h)i(n)j(h)^{-1} = i(\varphi(h)(n)).$$

If G is a group with normal subgroup N, then we get a natural homomorphism $\varphi_G : G \to \operatorname{Aut} N$ given by $\varphi_G(g)(n) = gng^{-1}$. Similarly, if H is a subgroup of G we get a homomorphism $\varphi : H \to \operatorname{Aut} N$ given by restricting the homomorphism φ_G to H. Suppose $G = NH$ and $N \cap H = e$. Then I showed that the map

$$f : N \rtimes_\varphi H \to G$$

given by $f(n, h) = nh$ is an isomorphism of groups.

Practice Problems: Do the following problems from Herstein for practice, but do not turn them in. The format below is that **H4.5** means "Chapter 4, Section 5 of Herstein."

H4.1: 1, 2, 8, 31
H4.2: 1, 4, 5, 6

Graded Problems: Work the following problems for a grade.

1. Define $\varphi : \mathbb{R}^\times \to \operatorname{Aut} \mathbb{R}$ by $\varphi(\alpha)(\beta) = \alpha^2\beta$. Then define

$$B = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \in \mathbf{GL}_2(\mathbb{R}) : x \in \mathbb{R}^\times, y \in \mathbb{R} \right\}.$$

Show that $B \cong \mathbb{R} \rtimes_\varphi \mathbb{R}^\times$.

2. Suppose G is a non-abelian group of order 8 containing at least 2 elements of order 2. Show that $G \cong D_4$.

3. Suppose G is a group and H is an abelian group. Define a binary operation $+$ on $\operatorname{Hom}(G, H)$ by writing $(\varphi + \psi)(g) = \varphi(g) + \psi(g)$.

(a) Show that, with this binary operation, $\operatorname{Hom}(G, H)$ is an abelian group.
(b) Suppose $f : G_1 \to G_2$ is a homomorphism of groups. Show that the map $f^* : \operatorname{Hom}(G_2, H) \to \operatorname{Hom}(G_1, H)$ given by $\varphi \mapsto \varphi \circ f$ is also a group homomorphism.

(c) (**10 point bonus**) Show that $\mathrm{Hom}(C_n, \mathbb{Q}/\mathbb{Z}) \cong C_n$.

4. A map $f : A \to B$ between two rings is a *ring homomorphism* if

(a) For all $x, y \in A$, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.
(b) $f(1_A) = 1_B$.

Suppose A is a ring. Show that there is exactly one ring homomorphism $f : \mathbb{Z} \to A$.