HW6v2, due Friday, March 31
Math 403, Spring 2017
Patrick Brosnan, Instructor

Graded Problems: Work the following problems for a grade.

1. Suppose G is a group. The set $\operatorname{Aut} G$ of group automorphism of G is the set of all group homomorphisms $f: G \to G$ which are one-one and onto. Show that $\operatorname{Aut} G$ is a subgroup of the group $A(G)$ of all automorphisms of the set of elements of G.

2. Suppose that G is a group and $\varphi \in \operatorname{Aut} G$.
(a) If $g \in G$ and $k \in \mathbb{Z}$, show that $\varphi(g^k) = \varphi(g)^k$.
(b) If $g \in G$, show that $|g| = |\varphi(g)|$.
(c) If $G = \langle g \rangle$, show that $G = \langle \varphi(g) \rangle$ as well.
(d) If $G = \langle g \rangle = \langle h \rangle$ show that there exists a unique automorphism $\varphi \in \operatorname{Aut} G$ such that $\varphi(g) = h$.

3. Suppose G is a group. For each $g \in G$, define a map $\varphi_g : G \to G$ by setting $\varphi_g(h) = ghg^{-1}$.
(a) Show that $\varphi_g \in \operatorname{Aut} G$.
(b) Show that the map $\varphi : G \to \operatorname{Aut} G$ given by $g \mapsto \varphi_g$ is a group homomorphism.
(c) Show that the kernel of φ is the subgroup $Z(G)$ consisting of all $g \in G$ such that $gh = hg$ for all $h \in G$. This subgroup is called the *center* of G.

4. Suppose $f : M \to N$ is a morphism of magmas. Recall that this means that, for $m_1, m_2 \in M$, $f(m_1 m_2) = f(m_1) f(m_2)$. Show that the difference kernel $K(f) = \{(m_1, m_2) \in M \times M : f(m_1) = f(m_2)\}$ is a submagma of $M \times M$. Similarly, if $f : M \to N$ are monoids, show that the difference kernel $K(f)$ is a submonoid of $M \times M$.

5. Suppose G is a group and $H \leq G$. Set $R = \{(x, y) \in G \times G : x^{-1} y \in H\}$, $L = \{(x, y) \in G \times G : xy^{-1} \in H\}$.
(a) Show that both L and R are equivalence relations.
(b) Show that R is the difference kernel of the map $q_R : G \to G/H$ given by $g \mapsto gH$. Similarly, show that L is the difference kernel of the map $q_L : G \to H \backslash G$ given by $g \mapsto Hg$.
(c) Show that H is normal in G if and only if R is a subgroup of $G \times G$.

6 (20 point bonus)**.** Suppose M is a magma, N is a set and $q : M \to N$ is a surjective mapping with difference kernel $K(q) = \{(m_1, m_2) \in M \times M : q(m_1) = q(m_2)\}$. Suppose that $K(q)$ is a submagma of $M \times M$. Show that there is a unique binary operation operation on N such that $q : M \to N$ is a magma homomorphism. (**Hint:** Imitate the proof given in class of the corresponding result for groups.)