HW1, due Friday, February 3
Math 403, Spring 2017
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Practice Problems: Do the following problems from Herstein for practice, but do not turn them in. The format below is that **H4.5** means "Chapter 4, Section 5 of Herstein."

H1.2: 2, 5
H1.3: 1, 3, 7, 14, 15
H1.4: 2, 4, 7

Graded Problems: Work the following problems for a grade.

1. Suppose $m, n \in \mathbb{Z}$ with $n > 0$. Show that $m + 2m^2 n \geq 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \sin x$.

(1) Is f one-one?
(2) Is f onto?
(3) What is $f(\mathbb{R})$?
(4) What is $f([0, 10])$?
(5) What is $f^{-1}([-1, 1])$?
(6) What is $f^{-1}([2, \infty))$?

3. Let $f \in S_4$ be the permutation given by the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

(1) Compute f^2, f^3, f^4 and f^5.
(2) Compute f^{-1}.
(3) (**10 point bonus**). Notice a pattern and say what f^k is for all $k \in \mathbb{Z}$.

4. Suppose S is a set. A subset $R \subset S \times S$ is called an *equivalence relation* on S if the following three properties hold.

(1) For all $s \in S$, $(s, s) \in R$.
(2) If $(s_1, s_2) \in R$ then $(s_2, s_1) \in R$.
(3) If $(s_1, s_2) \in R$ and $(s_2, s_3) \in R$, then $(s_1, s_3) \in R$.

Here (1) is known as the *reflexive property*, (2) is known as the *symmetric property* and (3) is known as the *transitive property*.

It is traditional to write $x \sim_R y$ if $(x, y) \in R$. And often we drop the R when R is understood and just write $x \sim y$. If we want to emphasize the R we sometimes also write xRy.

Which of the following are equivalence relations on $\mathbb{Z}$?

(1) The set $R = \{(x, y) \in \mathbb{Z}^2 : 2 \mid x - y\}$.
(2) The set $R = \{(x, y) \in \mathbb{Z}^2 : x - y \geq 0\}$.
(3) The set $R = \{(x, y) \in \mathbb{Z}^2 : \sin(x\pi/5) = \sin(y\pi/5)\}$.
(4) The set of all pairs $(x, y) \in \mathbb{Z}^2$ such that $x - y = z^3$ for some $z \in \mathbb{Z}$.

5. Suppose $f : X \to Z$ and $g : Y \to Z$ are mappings. The *fiber product* of f and g is the set

$$X \times_Z Y := \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

One particular example of this is when $X = Y$ and $f = g$. Then the product $X \times_Z X$ is called the *difference kernel* of f. Sometimes it is written as $K(f)$.

Show that, for any mapping $f : X \to Z$, the difference kernel $K(f)$ is an equivalence relation on X.