HW1, due Wednesday, February 5
Math 403, Spring 2014
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1. Use the principle of mathematical induction to show that $1+2+\cdots+n = n(n+1)/2$ for any positive integer n.

2. Suppose n is an integer strictly greater than 1. Using the Fundamental Theorem of Arithmetic write

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

where the p_i are distinct primes and the a_i are positive integers. For each prime number p define $\nu_p(n)$ to be a_i if $p = p_i$ for some i. Define $\nu_p(n) = 0$ otherwise.

Suppose m is another integer strictly greater than 1. Show that, for all primes p, we have

$$\nu_p(nm) = \nu_p(n) + \nu_p(m).$$

3. For each prime p, set $\nu_p(1) = 0$. If n is a negative integer, set $\nu_p(n) = \nu_p(-n)$. Suppose a and b are non-zero integers. Show that $a|b$ if and only, for all primes p, $\nu_p(a) \leq \nu_p(b)$.

4. Suppose n and m are two positive integers and let $S = \{p_1, \dots, p_k\}$ be a finite set of primes containing all of the prime factors of n and all the prime factors of m. Using the Fundamental Theorem of Arithmetic write

$$n = \prod_{i=1}^{k} p_i^{a_i},$$
$$m = \prod_{i=1}^{k} p_i^{b_i}.$$

Set

$$(n,m) = \prod_{i=1}^{k} p_i^{\min(a_i,b_i)}$$
$$[n,m] = \prod_{i=1}^{k} p_i^{\max(a_i,b_i)}.$$

(1) Suppose that x is an integer such that $x|n$ and $x|m$. Show that $x|(n,m)$.
(2) Suppose y is an integer such that $n|y$ and $m|y$. Show that $[n,m]|y$.
(3) Show that $(n,m)[n,m] = nm$.

5. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. Prove the following:

(1) If f and g are one-one, then so is $g \circ f$.
(2) If f and g are onto, so is $g \circ f$.
(3) If $g \circ f$ is one-one, then so is f.
(4) If $g \circ f$ is onto, then so is g.