HW2, due Tuesday, September 24
Math 403, Fall 2013
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Note. Problems 3 and 7 are worth 5 points. All other problems are worth 15 points.

1. Suppose a, b, c are non-zero integers. Show that $(a, bc) = 1$ if and only if $(a, b) = (a, c) = 1$.

2. Suppose $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Show that $(a, b) = (a, b + ca)$.

3. Suppose X and Y are finite sets. Let n denote the number of elements of X and let m denote the number of elements of Y. Write $\mathrm{Fun}(X, Y)$ for the set of functions from X to Y. In class, we noted that $\mathrm{Fun}(X, Y)$ has m^n elements. How many one-one functions are there in $\mathrm{Fun}(X, Y)$? You do not have to rigorously prove your answer, but you should give a convincing argument.

4. Let $M_2(\mathbb{R})$ denote the set of 2×2-matrices with coefficients in the real numbers, and let

$$* : M_2(\mathbb{R}) \times M_2(\mathbb{R})$$

denote the binary operation $X * Y = XY - YX$ where XY denote the matrix multiplication of X and Y. Show that $*$ is not associative. The operation $*$ is known as the *Lie bracket* operation. Usually $X * Y$ is written as $[X, Y]$.

5. Find the inverse of the element $[31]$ of the group $U(54)$. Write your answer as $[n]$ for some integer $0 < n < 54$.

6. Suppose G is a group with identity element e. If $g^2 = e$ for all $g \in G$, show that G is abelian.

7. Suppose M is a monoid with binary operation $*$ and identity element e. We say that an element $m \in M$ is *central* if, for all $n \in M$, $m * n = n * m$. The center of M is the set $Z(M)$ of all central elements of M. Show that $Z(M)$ is a submonoid of M. That is, show that $e \in Z(M)$ and that, if $m, n \in Z(M)$ then $m * n \in Z(M)$.

8. Let M denote the monoid $M_2(\mathbb{R})$. What is $Z(M)$? Prove your answer.