HW11, due Monday, December 12
Math 403, Fall 2011
Patrick Brosnan, Instructor

Reading Assignment

Begin reading about rings in Chapter 3.1-2.

Problem 1. (40 points) Recall that, if X is a set, $A(X)$ denotes the group of all maps $\phi : X \to X$, which are one-one and onto. If G is a group and $g \in G$, define $L(g) : G \to G$ by $L(g)(h) = gh$.

(a) Show that, if $g_1, g_2 \in G$, then $L(g_1 g_2) = L(g_1) \circ L(g_2)$.
(b) Show that, if $g \in G$, $L(g)$ is one-one and onto. Conclude that $L : G \to A(G)$ given by $g \mapsto L(g)$ is a homomorpism of groups.
(c) Show that $L(g)$ is the identity in $A(G)$ iff g is the identity in G. Conclude that $L : G \to A(G)$ is one-to-one.
(d) Draw the following conclusion: If G is a group with n elements, then G is isomorphic to a subgroup of the symmetric group S_n. (This is called *Cayley's theorem.*)

Problem 2. (40 points) Let A be a ring.

(a) Show that there exists exactly one ring homomorphism $h : \mathbf{Z} \to A$. (**Hint**: If h is a ring homomorphism, we must have $h(1) = 1$, so $h(2) = 1 + 1, h(-2) = -(1 + 1)$, etc.)
(b) Let $h : \mathbf{Z} \to A$ be as in (a). Then $\ker h = n\mathbf{Z}$ for some (uniquely determined) $n \in \mathbf{N}$. Set $\operatorname{char} A = n$; this is called the *characteristic* of A. Show that, if A is an integral domain, then $\operatorname{char} A$ is either prime or 0.
(c) Show that any field of characteristic $p > 0$ contains a subfield isomorphic to $\mathbf{Z}/p$.
(d) Show that any field of characteristic 0 contains a subfield isomorphic to the rationals.

Problem 3. (20 points) Write $\mathbf{F}_2 := \mathbf{Z}/2$ for the field with 2 elements. Set $p = x^2 + x + 1 \in \mathbf{F}_2[x]$.

(a) Show that p is irreducible.
(b) Show that $K := \mathbf{F}_2[x]/p\mathbf{F}_2[x]$ is a field.
(c) Show K has 4 elements and write down the addition and multiplication table in K.