HW10, due Friday, December 2
Math 403, Fall 2011
Patrick Brosnan, Instructor

Reading Assignment

Begin reading about rings in Chapter 3.1-2.

Problem 1. (15 points) Let $K = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$ and let $A = \{a + b\sqrt{2} \in K : a, b \in \mathbf{Z}\}$.
(a) Show that K is a subring of $\mathbf{R}$.
(b) Show that A is a subring of K.
(c) Show that K is a field.

Problem 2. (10 points) Show that a subring of an integral domain is an integral domain.

Problem 3. (10 points) Suppose D is a division ring and I is a left ideal in D. Show that either $I = \{0\}$ or $I = D$. Then draw the following conclusion: If $\rho : D \to R$ is a homomorphism of rings where D is a division ring, then either $R = 0$ or ρ is one-to-one.

Problem 4. (10 points) Suppose R is a commutative ring.

(a) Suppose A is a subring of $R \times R$. Assume that A is also an equivalence relation on R. Show that $I_A := \{r \in R : (r, 0) \in A\}$ is an ideal in R.
(b) Suppose I is an ideal in R. Set $A_I := \{(r, s) \in R \times R : r - s \in I\}$. Show that A_I is a subring of $R \times R$ which is also an equivalence relation on R.
(c) (5 point bonus) Show that $I_{A_I} = I$ for any ideal $I \subset R$, and that $A_{I_A} = A$ for any equivalence relation $A \subset R \times R$.

Problem 5. (25 points) Suppose R is a ring.
(a) Show that, if A and B are subrings of R, then so is $A \cap B$.
(b) Generalize (a) in the following way. Suppose $\{A_i\}_{i \in I}$ is a set of subrings of R. Show that $A = \cap_{i \in I} A_i$ is a subgring of R.
(c) Suppose S is a subset of R. Let A denote the intersection of all subgrings of R containing S. Show that A is the smallest subring of R containing S. It is called the subring of R generated by S.
(d) Keeping a notation of (c), define a sequence A_n of subsets of R inductively as follows: $A_0 = \{0, 1\} \cup S$. $A_n = \{x - y, xy : x, y \in A_{n-1}\}$. Show that $A = \cup_{n=0}^{\infty} A_n$. (In other words, A is the union of the sets A_n).
(e) Now suppose that B is a subring of R and S is a subset of R. Let $B[S]$ denote the subring of R generated by $B \cup S$. Now set $R = \mathbf{R}$ (the ring of real numbers), $B = \mathbf{Q}$ and $S = \{\sqrt{2}\}$. Set $\mathbf{Q}[\sqrt{2}] := B[S]$. Show that $\mathbf{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$.

Problem 6. (30 points) Let

$$H = \{ \begin{pmatrix} t + xi & -y - zi \\ y - zi & t - xi \end{pmatrix} : t, x, y, z \in \mathbf{R}\}$$

contained in the ring $M_2(\mathbf{C})$ of 2×2 matrices with complex coefficients. To ease the notation, define

$$\vec{a} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \vec{b} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \vec{c} := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

These are elements of H and the general matrix in H can then be written (uniquely and more compactly) as $t + x\vec{a} + y\vec{b} + z\vec{c}$.

(a) Show that $\vec{a}^2 = \vec{b}^2 = \vec{c}^2 = -1$ and $\vec{a}\vec{b} = \vec{c}, \vec{b}\vec{c} = \vec{a}, \vec{c}\vec{a} = \vec{b}$.
(b) Show that H is a subring of $M_2(\mathbf{C})$.
(c) Show that the determinant of any element of H is a non-negative real number. Show further that $\det X = 0$ iff $X = 0$ for $X \in H$.
(d) For $X = t + x\vec{a} + y\vec{b} + z\vec{c} \in H$, define $X^* := t - x\vec{a} - y\vec{b} - z\vec{c}$. Show that $XX^* = \det X$.
(e) Show that, for $X \neq 0$, $X^{-1} = (\det X)^{-1} X^*$. Then conclude that H is a division ring.
(f) Show that $\vec{b}\vec{a} = -\vec{c}$ and conclude that H is not a field.