HW10, due Wednesday, December 6
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections III.4 and III.5 of Aluffi's book. Then do the following problems:

III.4 2, 3, 4, 8, 15
III.5 1, 3, 5

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (20 points) Suppose R is a ring, and $S \subseteq R$. The *centralizer* $C(S)$ of S is the set of all $x \in R$ such that, for all $s \in S$, $xs = sx$. Show that $C(S)$ is a subring of R.

The *center* of R is, by definition, $C(R)$. So this shows that the center of R is a subring.

2. (20 points) A (not necessarily commutative) ring R is *simple* if there are exactly two two-sided ideals in R (necessarily the zero ideal and R itself). Show that the characteristic of a simple ring R is either 0 or a prime number p.

3. (20 points) Let R be a commutative ring and let $I, J \subseteq R$ be two ideals in R. Write IJ for the product ideal and write $P = \{ab : a \in I, b \in J\}$.

(a) Show that $IJ = P$ if I and J are principal.
(b) Set $R = \mathbb{R}[x,y]$ and $I = (x,y)$. Show that $\{ab : a, b \in I\} \subsetneq I^2$. (Here $I^2 = II$.)

4. (40 points) Let $R = M_2(\mathbb{R})$, the ring of 2×2 matrices with coefficients in $\mathbb{R}$.

(a) Suppose $v = (a,b) \in \mathbb{R}^2$, and set $I(v) := \{T \in R : Tv = 0\}$. Show that $I(v)$ is a non-zero left-ideal in R, which is proper as long as $v \neq 0$.
(b) Show that all nonzero left ideals in R are of the form $I(v)$ for some vector v.
(c) Show that R is simple.
(d) Show that the center of R is the set of matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

where $\lambda \in \mathbb{R}$.