HW9, due Wednesday, November 15
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections III.2 and III.3 of Aluffi's book.

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (20 points) Suppose R is a set with two binary operations $+$ and $\cdot$ such that $(R,+)$ is a group, $(R,\cdot)$ is a monoid and the distributive laws

$$x\cdot(y+z) = x\cdot y + x\cdot z$$
$$(x+y)\cdot z = x\cdot y + y\cdot z$$

hold. In other words, suppose $(R,+,\cdot)$ satisfies all the ring axioms except the axiom that addition is commutative. Show that, in fact, addition is forced to be commutative; so $(R,+,\cdot)$ is a ring.

2. (20 points) A *subring* of a ring $R = (R,+,\cdot)$ is a subset S of R, which is both a subgroup of $(R,+)$ and a submonoid of $(R,\cdot)$. Let R denote the set of complex numbers of the form $\frac{m}{2} + \frac{n}{2}\sqrt{-3}$, where m and n are integers and $m+n$ is even. Show that R is a subring of $\mathbb{C}$.

3. (30 points) An element of x in a ring R is *nilpotent* if $x^n = 0$ for some positive integer n. Say what the nilpotent elements are in the following rings.

(a) $\mathbb{Z}$.
(b) $\mathbb{Z}/10$.
(c) $\mathbb{Z}/12$.

Here $\mathbb{Z}/10$ and $\mathbb{Z}/12$ have the usual ring structure of addition and multiplication described in class.

4. (30 points) A *ring homomorphism* from a ring A to a ring B is a map $\varphi : A \to B$ which is simultaneously a group homomorphism from $(A,+)$ to $(B,+)$ and a monoid homomorphism from $(A,\cdot)$ to $(B,\cdot)$. From this we get a category Rings whose objects are rings and whose morphism are ring homomorphisms.

(a) Show that $\operatorname{End}_{\text{Rings}} \mathbb{Z} = \operatorname{End}_{\text{Rings}} \mathbb{Q} = \{\text{id}\}$. So, there are no nontrivial endomorphisms of $\mathbb{Z}$ or $\mathbb{Q}$ in the category of rings. It follows, of course, that $\operatorname{Aut}_{\text{Rings}} \mathbb{Z} = \operatorname{Aut}_{\text{Rings}} \mathbb{Q} = \{\text{id}\}$.

(b) Show that $\operatorname{End}_{\text{Rings}} \mathbb{R} = \operatorname{Aut}_{\text{Rings}} \mathbb{R} = \{\text{id}\}$ as well. To do this, the trick is to note that any endomorphism of $\mathbb{R}$ has to take squares to squares. Then use this to show that any ring endomorphism σ of $\mathbb{R}$ has to be continuous.

(c) Show that complex conjugation is a ring automorphism of $\mathbb{C}$, and use that to conclude that $\operatorname{Aut}_{\text{Rings}} \mathbb{C}$ contains a subgroup isomorphic to $\mathbb{Z}/2$.