HW6, due Wednesday, October 25
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections II.9, IV.1 and IV.5 of Aluffi's book. Work the following problems, but don't turn them in for a grade.

- II.6: 6, 7, 8, 9
- II.7: 11, 14
- II.8: 13
- II.9: 9, 11
- IV.1: 1, 4

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. Let $G = \mathbf{GL}_2(\mathbb{R})$, the group of invertible real 2×2-matrices. Show that the center of G is the subgroup consisting of diagonal matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

with $a \neq 0$.

2. Let G be the group of Problem 1 and let T denote the subgroup of diagonal matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $ab \neq 0$. Determine the normalizer $N_G(T)$ of T as well as the group $N_G(T)/T$ (up to isomorphism). What is $[N_G(T) : T]$?

3. Suppose G is a group. Show that products and equalizers exist in the category of G-sets. In fact, do the following.

(a) if X and Y are G-sets, show that the product of X and Y in the category of G-sets is just the cartesian product $X \times Y$ equipped with the G-action given by $g(x,y) = (gx, gy)$ (along with the projection maps onto X and Y).

(b) Show that if $\phi : X \to Y$ and $\psi : X \to Y$ are morphisms of G-sets, then the equalizer of E of ϕ and ψ in the category of G-sets, is just the set $E = \{x \in X : \phi(x) = \psi(x)\}$, which is a sub-$G$-set of X.

4. Suppose X is a G-set. We say that an equivalence relation R on X is *G-invariant* if, for $(x_1, x_2) \in R$ and $g \in G$, we have $(gx_1, gx_2) \in R$. Let R be a G-invariant equivalence relation on X, and let $Y = X/R$, the quotient of the set X by the equivalence relation R (in the category of sets). Let $\pi : X \to Y$ denote the quotient map sending $x \in X$ to its equivalence class $[x]$. Show that there is a unique G-action on Y making the map $\pi : X \to X/R$ into a morphism of G-sets.

5. Suppose the G-set X in Problem 4 is just G with the left action. If R is a G-invariant equivalence relations on G, show that $H(R) := \{g \in G : (g, 1) \in R\}$ is a subgroup of G. Then show that the map $R \mapsto H(R)$ sets up a one-one correspondence between G-invariant equivalence relations on G and subgroups of G.