HW6, due Wednesday, October 18
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections II.7-8 of Aluffi's book.

Terminology: A subgroup $N \leq G$ of a group G is *normal* if, for all $g \in G$ and all $n \in N$, $gng^{-1} \in N$. (See Definition 7.1 on page 88 of Aluffi.)

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (**21 points**) Suppose G is a group with identity element 1.

(a) If $R \leq G \times G$ is a congruence on G, show that $N = N(R) := \{x \in G : (x,1) \in R\}$ is a normal subgroup of G.
(b) In the situation of (a) above, show that $(x,y) \in R \Leftrightarrow xN = yN$.
(c) Show that the map $R \mapsto N(R)$ sets up a one-one correspondence between congruences on G and normal subgroups of G.

2. (**20 points**) Let $\mathbb{Z}$ denote the set of integers considered as a group under addition. For G a group, let

$$\mathrm{ev} : \mathrm{Hom}_{\mathrm{Groups}}(\mathbb{Z}, G) \to G$$

be the map given by $\mathrm{ev}(\phi) = \phi(1)$. Show that ev is one-one and onto.

3. (**20 points**) Show that a morphism $f : M \to N$ in the category of monoids is a monomorphism if and only if it is one-one. (**Hint:** Use the evaluation map of H04.)

4. (**20 points**) Suppose $\phi : G \to H$ and $\psi : G \to H$ are group homomorphisms. Set $K := \{g \in G : \phi(g) = \psi(g)\}$, and write $i : K \to G$ for the inclusion. Show that (K,i) is the equalizer of ϕ and ψ.

5. (**19 points**) Suppose G is a commutative group and n is an integer. Show that the map $\phi_n : G \to G$ given by $g \mapsto g^n$ is a group homomorphism.