HW5, due Wednesday, October 4
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections II.4-6 of Aluffi's book.

Terminology: Recall from class that a submagma of a magma M is a subset of M which is closed under the binary operation of M. A submonoid of a monoid M is a submagma containing the idenity element of M.

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (24 points) Show that the intersection of a nonempty collection of submagmas (resp. submonoids, resp. subgroups) is a submagma (resp. submonoid, resp. subgroup). In other words,

(a) If M is a magma, I is a nonempty set, and $\{M_i\}_{i\in I}$ are submagmas of M, show that $\cap_{i\in I} M_i$ is a submagma.
(b) If M is a monoid, I is a nonempty set, and $\{M_i\}_{i\in I}$ are submonoids of M, show that $\cap_{i\in I} M_i$ is a submonoid.
(c) If M is a group, I is a nonempty set, and $\{M_i\}_{i\in I}$ are subgroups of M, show that $\cap_{i\in I} M_i$ is a subgroup.

Do not assume that I is finite.

2. (30 points) Suppose M is a magma (resp. monoid, resp. group) and S is a subset of M. The submagma (resp. submonoid, resp. subgroup) of M generated by S is the intersection of all submagmas (resp. submonoids, resp. subgroups) containing S.

Note that, by the previous problem, the submagma (resp. submonoid, resp. subgroup) of M generated by S is actually a submagma (resp. submonoid, resp. subgroup) of M.

(a) Suppose M is a monoid and $S \subseteq M$. Show that the submonoid $\langle S\rangle_m$ of M generated by S consists of the identity element of M and all elements of M, which can be written in the form $s_1 s_2 \cdots s_n$ with n a positive integer and $s_i \in S$.

(b) Suppose G is a group and $S \subseteq G$. Show that the subgroups $\langle S \rangle$ of G generated by S consists of the identity element of G and all elements of G, which can be written in the form $g_1 g_2 \cdots g_n$ with n a positive integer and either $g_i \in S$ or $g_i^{-1} \in S$.

3. (24 points) Let $\mathbf{GL}_2(\mathbb{R})$ denote the group of 2×2-matrices

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det T \neq 0$ and with a, b, c and d real. (The group structure is matrix multiplication, and, for T as above $\det T = ad - bc$.)

For a matrix T as above, write

$$T^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

for the transpose of T.

(a) Show that $\mathbf{SL}_2(\mathbb{R}) := \{T \in \mathbf{GL}_2(\mathbb{R}) : \det T = 1\}$ is a subgroup of $\mathbf{GL}_2(\mathbb{R})$. Traditionally, $\mathbf{GL}_2(\mathbb{R})$ is called the group of *general linear transformations*, and $\mathbf{SL}_2(\mathbb{R})$ is called the subgroup of *special linear transformations*.

(b) Write $\mathbf{O}_2(\mathbb{R}) = \{T \in \mathbf{GL}_2(\mathbb{R}) : TT^* = I\}$. Show that $\mathbf{O}_2(\mathbb{R}) \leq \mathbf{GL}_2(\mathbb{R})$ as well. Traditionally $\mathbf{O}_2(\mathbb{R})$ is called the group of *orthogonal transformations*. The group $\mathrm{SO}_2(\mathbb{R}) := \mathbf{O}_2(\mathbb{R}) \cap \mathbf{SL}_2(\mathbb{R})$ is called the *special orthogonal group*.

(c) For each $\theta \in \mathbb{R}$, set

$$\rho(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that $\rho(\theta) \in \mathrm{SO}_2(\mathbb{R})$ for each θ, and that the map $\rho : \mathbb{R} \to \mathrm{SO}_2(\mathbb{R})$ is onto.

(d) Show that $\rho : \mathbb{R} \to \mathrm{SO}_2(\mathbb{R})$ is a group homomorphism. (Here the binary operation on $\mathbb{R}$ is usual addition.)

(e) Let

$$\tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Show that $\tau^2 = I$ and $\tau\rho(\theta)\tau = \rho(-\theta)$.

(f) Show that $\mathbf{O}_2(\mathbb{R})$ is generated by τ and $\mathrm{SO}_2(\mathbb{R})$. In other words, $\mathbf{O}_2(\mathbb{R}) = \langle \tau \cup \mathrm{SO}_2(\mathbb{R}) \rangle$.

4. (22 points) Let n be a positive integer and let G_n denote the subgroup of $\mathbf{GL}_2(\mathbb{R})$ generated by the matrices τ and $r := \rho(2\pi/n)$ above. Show that every element of G_n can be written uniquely as

$\tau^i r^j$ where $i \in \{0,1\}$ and $j \in \{0,1,\ldots,n-1\}$. Conclude that G_n has $2n$ elements.