ON THE ALGEBRAICITY OF THE ZERO LOCUS OF AN ADMISSIBLE NORMAL FUNCTION

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Abstract. We show that the zero locus of an admissible normal function on a smooth complex algebraic variety is algebraic.

1. Introduction

Let $\mathcal{H}$ be a variation of pure, polarizable Hodge structure of weight -1 over a smooth complex manifold S with integral structure $\mathcal{H}_{\mathbb{Z}}$ and $J(\mathcal{H})$ be the associated bundle of intermediate Jacobians over S. In this paper, we prove the following conjecture of Phillip Griffiths and Mark Green:

Conjecture 1.1. *If S is smooth complex algebraic variety and $\nu : S \to J(\mathcal{H})$ is an admissible normal function then the zero locus $\mathcal{Z}$ of ν is an algebraic subvariety of S.*

To prove (1.1), let $\bar{S}$ be a smooth partial compactification such that $D = \bar{S} - S$ is a normal crossing divisor, and ν is represented by an extension

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0$$

in the category of admissible variations of mixed Hodge structure over S. For elementary reasons, $\mathcal{Z}$ is a complex analytic subvariety of S. Therefore, by GAGA, the algebraicity of $\mathcal{Z}$ is equivalent to the following result:

Theorem 1.2. *If $p \in D$ is an accumulation point of $\mathcal{Z}$ then there exists an polydisk $P \subset \bar{S}$ containing p and an analytic subvariety A of P such that $A \cap S = \mathcal{Z} \cap P$.*

In the remainder of this section, we reduce the proof of Theorem (1.2) to a pair of technical results regarding the asymptotic of period maps of admissible variations of mixed Hodge structure. To this end, let $P^* = P - P \cap D$, $r = \dim S$ and Δ denote the unit disk in $\mathbb{C}$. Then, there exists a system of local coordinates $(s_1, \dots, s_r)$ on $P \cong \Delta^r$ relative to which $P \cap D$ is a union of hypersurfaces of the form $s_j = 0$. Therefore, after relabeling coordinates, $P^* \cong \Delta^{*m} \times \Delta^{r-m}$ where $\Delta^* \subset \Delta$ is the punctured disk. The local monodromy of $\mathcal{V}$ about p is quasi-unipotent, and hence by passage to a finite cover $f : P \to P$ we can assume that $\mathcal{V}$ has unipotent monodromy. By the proper mapping theorem, if the closure of the zero locus $f^*(\nu)$

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is an analytic subvariety of P then the closure of zero locus of ν is an analytic subvariety of P. Accordingly, we may assume without loss of generality that $\mathcal{V}$ has unipotent local monodromy at p. Furthermore, we can assume that $m = r$ above by taking the local monodromy of $\mathcal{V}$ about the last $r - m$ punctured disks to be trivial.

Let $\varphi : P^* \to \Gamma\backslash\mathcal{M}$ be the period map of $\mathcal{V}$ over P^* [21]. Let $U^r \subset \mathbb{C}^r$ denote the product of upper half-planes with coordinates $(z_1, \dots, z_r)$ and $U^r \to P$ denote the covering map defined by $s_j = e^{2\pi i z_j}$. Then, φ has a lifting $F : U^r \to \mathcal{M}$ which makes the following diagram commute:

$$\begin{array}{ccc} U^r & \xrightarrow{F} & \mathcal{M} \\ \downarrow & & \downarrow \\ P^* & \xrightarrow{\varphi} & \Gamma\backslash\mathcal{M} \end{array}$$

If $s \in P$ then $z \in U^r$ is a point over s if z maps to s via the covering map $U^r \to P$.

Given an increasing filtration W of a finite dimensional vector space V over a field of characteristic zero, a grading of V is a semisimple endomorphism Y of V such that W_k is the direct sum of W_{k-1} and the k-eigenspace $E_k(Y)$ for each index k. By a theorem of Deligne [7], a mixed Hodge structure (F, W) induces a unique, functorial decomposition

$$V_{\mathbb{C}} = \bigoplus_{r,s} I^{r,s}$$

of the underlying complex vector space $V_{\mathbb{C}}$ such that

(a) $F^p = \oplus_{r \geq p} I^{r,s}$;
(b) $W_k = \oplus_{r+s \leq k} I^{r,s}$;
(c) $\bar{I}^{p,q} = I^{q,p} \mod \oplus_{r<q,s<p} I^{r,s}$.

In particular, a mixed Hodge structure (F, W) induces a grading $Y_{(F,W)}$ of $V_{\mathbb{C}}$ by the requirement that $Y_{(F,W)}$ acts as multiplication by $p + q$ on $I^{p,q}$.

Lemma 1.3. *A point $s \in P^*$ belongs to $\mathcal{Z}$ if and only if $Y_{(F(z),W)}$ is an integral grading of W for any point $z \in U^r$ over s.*

To continue, let I be a closed and bounded subinterval of the real numbers. Then, the vertical strip associated to I is the set of all points $(z_1, \dots, z_r) \in U^r$ such that the real part of $z_j = x_j + iy_j$ belongs to I for each j. Let $s(m)$ sequence of points in P^* which converge to $0 \in P$ as $m \to \infty$, and $z(m)$ be a sequence of points in U^r over $s(m)$ which is contained in a vertical strip associated to an interval of length 2π. The sequence $z(m)$ will be said to be sl_2-convergent if

(i) $y_1(m) \geq y_2(m) \geq \cdots \geq y_r(m)$ and $y_r(m) \to \infty$;
(ii) for each $j = 1, \dots, r$, the sequence of points $\lambda_j(m) = [y_j(m), y_{j+1}(m)]$ converges in $\mathbb{RP}^1$, where $y_{r+1}(m) = 1$;
(iii) the sequence of points $(x_1(m), \dots, x_r(m))$ converges.

A straightforward argument shows that (after reordering the variables) given any sequence of points $s(m)$ in P^* which converge to 0 there exists a subsequence $s(m_j)$ of $s(m)$ and a sequence of points $z(m_j)$ over $s(m_j)$ such that $z(m_j)$ is sl_2-convergent.

By (i), $\lambda_j(m)$ takes values in the affine chart of $\mathbb{RP}^1$ with coordinates $[1, a]$ and $0 \leq a \leq 1$. We will freely identify

$$\lambda_j = \lim_{m \to \infty} \lambda_j(m)$$

with its limiting value in this affine chart. Let

$$\lambda = (\lambda_1, \dots, \lambda_r)$$

and form a graph with vertices $\{1, \dots, r\}$ by connecting i to $i+1$ by an edge if $\lambda_i \neq 0$. Let $P(\lambda)$ be the corresponding partition of $\{1, \dots, r\}$ into connected components. For example, the partition associated to $\lambda = (0, 1, 1, 0)$ is $\{1\} \cup \{2, 3, 4\}$.

Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit $\theta = e^{\sum_j z_j N_j}.F$. (See [12, §5] for an explanation of nilpotent orbits.) Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a sequence of non-negative real numbers and as above (i.e. $\lambda_r = 0$), let $P(\lambda)$ be the corresponding partition of $\{1, \dots, r\}$ defined by the vanishing of the λ_j's. Given $\sigma \in P(\lambda)$ let

$$C(\sigma) = \{\sum_{j\in\sigma} a_j N_j \mid a_j > 0\}$$

and

$$N(\sigma) = \sum_{j\in\sigma} (\Pi_{k\in\sigma, k<j}\, \lambda_k) N_j$$

(where the empty product is equal to 1, so $N(\sigma) = N_{\min\sigma} + \cdots$). Let $\mathcal{N}(\sigma)$ denote an open neighborhood of $N(\sigma)$ in $C(\sigma)$. The elements σ of $P(\lambda)$ can be ordered by the value of $\min(\sigma)$. We denote this ordering $\sigma_1, \sigma_2, \dots$.

Let $C = C(\{1, \dots, r\})$ and M be the relative weight filtration of W and N for any $N \in C$. By a theorem of Kashiwara, M is well defined independent of the choice of N. Furthermore, (F, M) is a mixed Hodge structure relative to which each N_j is $(-1, -1)$-morphism.

More generally, let J be a subset of $\{1, \dots, r\}$ and $C(J)$ be the open facet of the closure of C defined by positive linear combinations $\sum_{j\in J} a_j N_j$. Then, the relative weight filtration $M(C(J), W)$ is well defined, and

$$M(C(I), M(C(J), W)) = M(C(I \cup J), W)$$

Furthermore, if J' denotes the complement of J in $\{1, \dots, r\}$ then

$$\theta_J = (\exp(\sum_{j\in J'} z_j N_j).F, M(C(J), W))$$

is an admissible nilpotent orbit.

Remark 1.4. Let (F, W) denote a mixed Hodge structure with underlying vector space V. Cattani, Kaplan and Schmid associate to (F, W) a canonically defined Hodge structure $(\hat{F}, W)$ which is split over $\mathbb{R}$. The filtration $\hat{F}$ is related to the SL_2-orbit theorem and is denoted by the symbol $\tilde{F}_0$ in [6, (3.31)]. It is called the *canonical splitting* in [12, §1.2], but we call it the sl_2-splitting in this paper. There is a distinguished nilpotent element $\xi \in \operatorname{End} V_{\mathbb{C}}$ such that $\hat{F} = e^{-\xi} \cdot F$. In [12], this ξ is denoted by the symbol ϵ or $\epsilon(W, F)$. Occasionally in this paper, we will use the notation that if (F, W) is a mixed Hodge structure then $\hat{Y}_{(F,W)} = Y_{(\hat{F},W)}$ where $(\hat{F}, W)$ is the sl_2-splitting of (F, W).

Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit as above and let $W^0, \dots, W^r$ be the sequence of increasing filtrations defined by the requirement that $W^0 = W$ and $W^j = M(N_j, W^{j-1})$. Then, by a theorem of Deligne [1, 8, 20], the data $(N_1,\dots,N_r,Y_{(F,W^r)})$ defines a sequence of mutually commuting gradings (in the notation of equation (3.3) of [1])

$$Y^r = Y_{(F,M)},\quad Y^{r-1} = Y(N_r,Y^r),\quad \dots \tag{1.5}$$

such that Y^k grades W^k. Furthermore, if (F,W^r) is split over $\mathbb{R}$ this construction gives the corresponding gradings of the SL_2-orbit theorem. More precisely, let $(\hat{F},W^r)$ denote the sl_2-splitting of (F,W^r), and $\{\hat{Y}^j\}$ be the corresponding system of gradings. Let $\hat{H}_j = \hat{Y}^j - \hat{Y}^{j-1}$ and $\hat{N}_j$ denote the component of N_j with eigenvalue zero with respect to $\operatorname{ad}\hat{\mathrm{Y}}^{\mathrm{j}-1}$ for $j=1,\dots,r$. Then, each pair $(\hat{N}_j,\hat{H}_j)$ is an sl_2-pair which commutes with $(\hat{N}_k,\hat{H}_k)$.

In the above discussion, the weight filtration W was arbitrary. We now restrict to the case where W is of the type arising form a normal function.

We defer the proof of the next theorem until § 2.

Theorem 1.6. *Let $(iy_1(m),\dots,iy_r(m))$ be an sl_2-convergent sequence. Let $P(\lambda)$ denote the corresponding partition of $\{1,\dots,r\}$. Then,*

$$Y^* = \lim_{m\to\infty} Y_{(\theta(iy_1(m),\dots,iy_r(m)),W)} = Y(N(\sigma_1),Y(N(\sigma_2),\cdots,Y_{(\hat{F},M)})) \tag{1.7}$$

Let $\mathcal{V}$ be an admissible variation of mixed Hodge structure over Δ^{*r} with unipotent monodromy, with arbitrary weight filtration W. We assume that $\mathcal{V}$ is polarizable and fix polarizations on each of the graded pieces $\mathrm{Gr}^W_i\mathcal{V}$. Let φ be the associated period map and $F: U^r\to\mathcal{M}$ be a lifting of φ to the r-fold product of the upper half-plane relative to the covering map $s_j = e^{2\pi i z_j}$, $j=1,\dots,r$. Let V be any fiber of the variation $\mathcal{V}$ and let $\mathfrak{g}$ denote the Lie subalgebra of $\operatorname{End}V$ consisting of all elements which preserve W and act by infinitesimal isometries on $\mathrm{Gr}^W_i V$. Then, the limit mixed Hodge structure (F_∞,M) defines a distinguished vector space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{q}\oplus\mathfrak{g}_{\mathbb{C}}^{F_\infty} \tag{1.8}$$

where

$$\mathfrak{q} = \bigoplus_{r<0,s}\mathfrak{g}^{r,s}_{(F_\infty,M)}.$$

Relative to this decomposition, we can write (cf. (6.11) [15])

$$F(z_1,\dots,z_r) = e^{\sum_j z_jN_j}e^{\Gamma(s_1,\dots,s_r)}.F_\infty \tag{1.9}$$

where $\Gamma(s)$ is a $\mathfrak{q}$-valued holomorphic function which vanishes at the origin.

Given an admissible nilpotent orbit with monodromy logarithms $N_1,\dots,N_r$ and a point $x=(x_1,\dots,x_r)$ define

$$\mu(x) = \sum_j x_jN_j$$

Theorem 1.10. *Let $z(m) = x(m)+iy(m)$ be an sl_2-convergent sequence of points in U^r and $F: U^r\to\mathcal{M}$ denote the lifting of the period map of an admissible normal function over Δ^{*r} as above. Let $P(\lambda)$ be the corresponding partition. Then, after passage to a subsequence if necessary,*

$$Y^* = \lim_{m\to\infty} e^{-\mu(x(m))}.Y_{(F(z(m)),W)} = Y(N(\sigma_1),Y(N(\sigma_2),\cdots,Y_{(\hat{F}_\infty,M)})) \tag{1.11}$$

Remark 1.12. This result has been obtained independently by Kato, Nakayama and Usui [14] in their study of classifying spaces of degenerations of mixed Hodge structure. In particular, as part of their study of log intermediate Jacobians [12], they are able [14] to obtain an independent proof of Conjecture (1.1).

Recall that the zero locus $\mathcal{Z}$ of ν coincides with the set of points in Δ^{*r} where $Y_{(F(z),W)}$ is integral for some ($\implies$ any) lifting of $s \in \Delta^{*r}$ to U^r. In particular, in order for the origin to be an accumulation point of $\mathcal{Z}$,

$$Y_{\mathbb{Z}} = e^{\mu}.Y^*$$

must be an integral grading of W, for any sl_2-convergent sequence of points on $\mathcal{Z}$, where

$$\mu = \lim_{m\to\infty} \mu(x(m)) \tag{1.13}$$

Corollary 1.14. *There exist only finitely many integral gradings of the form $Y_{(F(z),W)}$ where $z \in U^r$ is a point in the vertical strip associated to an interval I of finite length.*

Proof. Otherwise, we can find a sequence of points $z(m)$ in the vertical strip such that each $Y_{(F(z(m)),W)}$ is integral and distinct from $Y_{(F(z(m'),W)}$ for $m \neq m'$. After reordering the variables, we can then pass to an sl_2-convergent sequence to get a convergent limit, which is a contradiction since the integral gradings are discrete. □

Remark 1.15. Theorem (1.10) implies that $Y_{(F(z),W)}$ remains bounded on any vertical strip. Indeed, if this is false then we can find a sequence of points $z(m)$ in the vertical strip such that

$$||Y_{(F(z(m+1),W)} - Y_{(F(z(m)),W)}|| > 1$$

with respect to a fixed norm on $W_{-1}\mathrm{gl}(V)$. Passage to an sl_2-convergent subsequence then gives a contradiction.

In order to derive the local equations for $\mathcal{Z}$ near the origin, we now record the following property of Deligne systems (1.5):

Lemma 1.16. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit. Then,*

$$Y = Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F,M)})))$$

preserves F, where M is the relative weight filtration of W and $N_1 + \dots + N_r$. More generally, Y preserves the Deligne $I^{p,q}$'s of (F, M).

The proof of Lemma (1.16) is given in section 2. Granting this lemma and Theorem (1.10), we now establish Conjecture (1.1) by verifying Theorem (1.2).

Proof of Theorem (1.2). We derive the local equations for $\mathcal{Z}$. Let $s(m)$ be a sequence of points in $\mathcal{Z}$ which accumulate to the origin. After passage to a subsequence as above, let $\lambda = (\lambda_1, \dots, \lambda_r)$ be the corresponding partition and $Y_{\mathbb{Z}}$ be the associated integral grading of W appearing in (1.10). Let

$$\Lambda^{-1,-1} = \bigoplus_{r,s<0} \mathfrak{g}^{r,s} \subset \mathfrak{g}_{\mathbb{C}}$$

with the respect to the bigrading of $\mathfrak{g}_{\mathbb{C}}$ induced by the limit mixed Hodge structure (F_∞, M). Then, applying the above results to the nilpotent orbit defined by the data $(N(\sigma_1), N(\sigma_2), \dots; F_\infty, W)$ we see that

$$Y_{\mathbb{Z}} = e^{-\tilde{\xi}}.Y_\infty, \qquad \tilde{\xi} \in \Lambda^{-1,-1}_{(F,M)} \cap (\cap_{j=1}^r \ker(\mathrm{ad}\,\mathrm{N_j}))$$

where $Y_\infty = Y(N(\sigma_1), Y(N(\sigma_2), \dots, Y_{(F_\infty,M)}))$ and $\tilde{\xi} = \xi - \mu$. On the other hand, by Lemma (1.16) Y_∞ preserves F_∞, and hence there exists a section $f(z)$ of $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ such that

$$Y_{(F(z),W)} = e^{\sum_j z_j N_j} e^{\Gamma(s)}.(Y_\infty + f(z)) \tag{1.17}$$

By equation (1.17), it then follows that the local defining equation for the branch of $\mathcal{Z}$ corresponding to $Y_{\mathbb{Z}}$ (on the given vertical strip) is

$$Y_{\mathbb{Z}} = e^{-\tilde{\xi}}.Y_\infty = e^{\sum_j z_j N_j} e^{\Gamma(s)}.(Y_\infty + f(z))$$

and hence

$$e^{-\Gamma(s)} e^{-\sum_j z_j N_j} e^{-\tilde{\xi}}.Y_\infty = Y_\infty + f(z) \tag{1.18}$$

In particular, since $\Gamma(s)$, $\tilde{\xi}$ and $N_1, \dots, N_r$ belong to $\mathfrak{q}$ whereas $f(z)$ takes values $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ and Y_∞ preserves the $I^{p,q}$'s of (F, M) it follows that we can write (1.18) as

$$e^{-\sum_j z_j N_j - \tilde{\xi}}.Y_\infty = e^{\Gamma(s)}.Y_\infty \tag{1.19}$$

since $\tilde{\xi}$ commutes with $N_1, \dots, N_r$.

We remark that equation (1.19) is independent of the choice of vertical strip since the factor

$$e^{-\sum_j z_j N_j - \tilde{\xi}} = e^{\mu - \sum_j z_j N_j} e^{-\xi}$$

is invariant under shifting I by $\omega \in R$ by virtue of equation (1.13).

Consider now the linear map $L : \mathbb{Q}^r \to W_{-1}\mathfrak{g}_{\mathbb{Q}}$ defined by the rule

$$L(u_1, \dots, u_r) = [\sum_j u_j N_j, Y_{\mathbb{Z}}]$$

Then, we can find a matrix A in reduced row echelon form such that the rows of A are a basis of $\ker(L)$. Let $\Omega \subset \{1, \dots, r\}$ index the non-pivot columns of A. Then, there exist unique, rational, linear forms β_j indexed by Ω such that

$$[\sum_j z_j N_j, Y_{\mathbb{Z}}] = [\sum_{j\in\Omega} \beta_j(z_1, \dots, z_r) N_j, Y_{\mathbb{Z}}].$$

Therefore, we can rewrite (1.19) as

$$e^{-\sum_{j\in\Omega} \beta_j(z_1,\dots,z_r)N_j - \tilde{\xi}}.Y_\infty = e^{\Gamma(s)}.Y_\infty \tag{1.20}$$

Note that the coefficient of z_j in β_j is 1.

To continue, let $\alpha = \alpha_0 + \alpha_{-1}$ denote the decomposition of $\alpha \in \mathfrak{g}_{\mathbb{C}}$ according to the eigenvalues of $\mathrm{ad}\,\mathrm{Y}_\infty$ and define

$$\tilde{L}(u_1, \dots, u_r) = [\sum_j u_j N_j, Y_\infty]$$

Then, since $Y_\infty = e^{\tilde{\xi}}.Y_{\mathbb{Z}}$ and $N_1, \dots, N_r$ commute with $\tilde{\xi}$, it follows that $\ker(\tilde{L}) = \ker(L)$. In particular, the set $\{(N_j)_{-1} \mid j \in \Omega\}$ is linearly independent.

To continue, we rewrite equation (1.20) as

$$e^{-\sum_{j\in\Omega} \beta_j(z_1,\dots,z_r)N_j - \tilde{\xi}} e^{\sum_{j\in\Omega} \beta_j(z_1,\dots,z_r)(N_j)_0 + (\tilde{\xi})_0}.Y_\infty = e^{\Gamma(s)}.Y_\infty \tag{1.21}$$

and observe that

$$e^{\alpha} = e^{-\sum_{j\in\Omega}\beta_j(z_1,\dots,z_r)N_j-\tilde{\xi}}e^{\sum_{j\in\Omega}\beta_j(z_1,\dots,z_r)(N_j)_0+(\tilde{\xi})_0} \in \exp(W_{-1}\mathfrak{g}_{\mathbb{C}}) \qquad (1.22)$$

Recall that $\exp(W_{-1}\mathfrak{g}_{\mathbb{C}})$ act simply transitively on the gradings of W. Therefore, by the Campbell–Baker–Hausdorff formula, Lemma (1.16), and the fact that $N_1,\dots,N_r$ and ζ belong to $\Lambda^{-1,-1}$, it then follows that

$$\lim_{m\to\infty} \tilde{\xi}_{-1}^{-1,-1} + \sum_{j\in\Omega}\beta_j(z_1(m),\dots,z_r(m))(N_j)_{-1} = 0 \qquad (1.23)$$

since $\lim_{m\to\infty}\Gamma(s(m)) = 0$, where $\tilde{\xi}_{-1}^{-1,-1}$ is the component of $\tilde{\xi}_{-1}$ in $\mathfrak{g}^{-1,-1}_{(F_\infty,M)}$. Indeed, the left hand side of (1.23) is exactly the projection α to $\mathfrak{g}^{-1,-1}_{(F_\infty,M)}$. Consequently, there exist complex numbers η_j indexed by $j\in\Omega$ such that

$$\tilde{\xi}_{-1}^{-1,-1} = \sum_{j\in\Omega}\eta_j(N_j)_{-1}$$

Accordingly, the branch of $\mathcal{Z}$ corresponding to $Y_{\mathbb{Z}}$ we must have

$$\beta_j(z_1,\dots,z_r)+\eta_j \to 0 \qquad (1.24)$$

Let $\beta_j(z_1,\dots,z_r) = \sum_k b_{jk}z_k$. Then, after multiplying equation (1.24) by $2\pi i$ and taking the exponential, it follows that

$$\Pi_k\, s_k^{b_{jk}} \sim e^{-2\pi i\eta_j} \qquad (1.25)$$

for each $j\in\Omega$. By passage to a finite ramified cover, we can make all of the b_{jk} integral. Therefore, the local defining equation for the branch of $\mathcal{Z}$ corresponding to $Y_{\mathbb{Z}}$ is

$$e^{-\sum_{j\in\Omega}\frac{1}{2\pi i}\log(\Pi_k\, s_k^{b_{jk}})N_j-\tilde{\xi}}.Y_\infty = e^{\Gamma(s)}.Y_\infty \qquad (1.26)$$

where each of the logarithmic terms is a finite, single valued holomorphic function by virtue of (1.25).

Likewise, upon writing $\Gamma(s) = \Gamma(s)_0 + \Gamma(s)_{-1}$ with respect to $\operatorname{ad}\mathrm{Y}_\infty$ we have

$$e^{-\sum_{j\in\Omega}\frac{1}{2\pi i}\log(\Pi_k\, s_k^{b_{jk}})N_j-\tilde{\xi}}e^{\sum_{j\in\Omega}\frac{1}{2\pi i}\log(\Pi_k\, s_k^{b_{jk}})(N_j)_0+(\tilde{\xi})_0}.Y_\infty = e^{\Gamma(s)}e^{-\Gamma(s)_0}.Y_\infty \qquad (1.27)$$

Let $e^{\chi} = e^{\Gamma(s)}e^{-\Gamma(s)_0}$ and $\chi^{-1,-1}$ be the component of $\chi\in\mathfrak{g}^{-1,-1}_{(F_\infty,M)}$. Then, by the Campbell–Baker–Hausdorff formula, it follows from equation (1.27) that the zero locus is contained in the complex analytic subvariety

$$\mathcal{A} = \{\, s\in\Delta^r \mid \chi^{-1,-1}(s) \in \oplus_{j\in\Omega}\mathbb{C}(N_j)_{-1}\,\}$$

For a point $s\in\mathcal{A}$, let

$$\chi^{-1,-1}(s) + \tilde{\xi}_{-1}^{-1,-1} = \sum_j \tau_j(s)(N_j)_{-1}$$

Then, again only looking at $(-1,-1)$ components, we see that on $\mathcal{A}$, we must have

$$-\frac{1}{2\pi i}\log(\Pi_k\, s_k^{b_{jk}}) = \tau_j(s)$$

Therefore, over Δ^{*r}, the branch of $\mathcal{Z}$ corresponding to $Y_{\mathbb{Z}}$ is given by the system of complex analytic equations:

(a) $s\in\mathcal{A}$;
(b) $\Pi_k\ s_k^{b_{jk}} = \exp(-2\pi i\tau_j(s))$ for each $j\in\Omega$;

(c) $e^{-\sum_{j\in\Omega}\tau_j(s)N_j-\tilde{\xi}}.Y_\infty = e^{\Gamma(s)}.Y_\infty$

□

2. Deligne systems

We now reduce the proof of Lemma (1.16) to a corollary of the following sequence of lemmata:

Lemma 2.1. [8] *If $(N,\hat{F},W)$ defines an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$ then $(e^{iN}.\hat{F},W)$ is a mixed Hodge structure with* sl_2*-splitting*

$$(e^{i\hat{N}}.\hat{F},W)$$

in the notation of (1.5)*.*

Suppose now that $(N_1,\dots,N_r;\hat{F}_r,W)$ defines an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$. Following the notation of (1.5), let $W^0,\dots,W^r$ be the associated system of weight filtrations. Recall that by [6] and [11] that

$$(z_1,\dots,z_{r-1})\mapsto(e^{\sum_{j<r} z_jN_j}e^{iN_r}.\hat{F}_r,W^0)$$

is an admissible nilpotent orbit, and hence $(e^{iN_r}.\hat{F}_r,W^{r-1})$ is a mixed Hodge structure. Accordingly,

$$(z_1,\dots,z_{r-1})\mapsto(e^{\sum_{k\le r-1} z_kN_k}.\hat{F}_{r-1},W^0)$$

is an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$, where $(\hat{F}_{r-1},W^{r-1})=(e^{i\hat{N}_r}.\hat{F}_r,W^{r-1})$ is the sl_2-splitting of $(e^{iN_r}.\hat{F}_r,W^{r-1})$. Iterating this construction, we obtain a sequence of mixed Hodge structures

$$(\hat{F}_{j-1},W^{j-1})=(e^{i\hat{N}_j}.\hat{F}_j,W^{j-1})$$

and associated nilpotent orbits $(z_1,\dots,z_j)\mapsto e^{\sum_{k\le j} z_kN_k}.F_j$.

Lemma 2.2. [8][1] *In the setting of Lemma* (2.1)*,*

$$\hat{Y}=Y(N,Y_{(\hat{F},M)})$$

equals $Y_{(e^{i\hat{N}}.\hat{F},W)}$, and preserves $\hat{F}$.

In particular, given the data $(N_1,\dots,N_r;\hat{F}_r,W)$ of an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$, the sequence of gradings $\hat{Y}^j$ constructed in (1.5) is given by $\hat{Y}^j=Y_{(\hat{F}_j,W^j)}$. Since $N_1,\dots,N_j$ are $(-1,-1)$-morphisms of $(\hat{F}_j,W^j)$, it follows that

$$[N_k,\hat{H}_j]=0 \tag{2.3}$$

for $j>k$ where as in (1.5), $\hat{H}_j=\hat{Y}^j-\hat{Y}^{j-1}$.

Lemma 2.4. *Let $(N_1,\dots,N_r;\hat{F}_r,W)$ define an admissible nilpotent orbit with limiting mixed Hodge structure $(\hat{F},M)$ split over $\mathbb{R}$. Then,*

$$\hat{Y}^0=Y(N_1,Y(N_2,\dots,Y(N_r,Y_{(\hat{F},M)})))$$

preserves $\hat{F}$.

Proof. To begin, we recall[8][20] that $(\hat{N}_1, \hat{H}_1), \dots, (\hat{N}_r, \hat{H}_r)$ form a commuting system of sl_2-representations. Consequently,

$$[\hat{Y}^j, \hat{N}_k] = 0 \tag{2.5}$$

for $j < k$. Indeed, this is true by definition for $j = k-1$. Suppose that $j \leq k-2$. Then,

$$\begin{aligned} [\hat{Y}^j, \hat{N}_k] &= -[(\hat{Y}^{j+1} - Y^j) + \cdots + (\hat{Y}^{k-1} - Y^{k-2}), \hat{N}_k] \\ &= -[\hat{H}^{j+1} + \cdots + \hat{H}^{k-1}, \hat{N}_k] = 0 \end{aligned}$$

By the prior paragraphs, $\theta(z) = (e^{zN_1}.\hat{F}_1, W)$ is an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$, and hence by Lemma (2.2),

$$\hat{Y}^0(\hat{F}_1^p) \subseteq \hat{F}_1^p$$

Using the identity $F_1 = e^{\sum_{j>1} i\hat{N}_j}.\hat{F}$ and the fact that $\hat{Y}^0$ commutes with all $\hat{N}_j$, it then follows from the previous equation that $\hat{Y}^0$ preserves $\hat{F}$. □

Lemma 2.6. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit with sl_2-splitting $(\hat{F}, W^r) = (e^{-\xi}.F, W^r)$. Then,*

$$Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(\hat{F},M)}))) = e^{-\xi}.Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F,M)})))$$

Proof. ξ commutes with $N_1, \dots, N_r$ since ξ is a universal Lie polynomial in the Hodge components of Deligne's δ-splitting $(e^{-i\delta}.F, M)$ of (F, M) and δ commutes with all $(-1,-1)$-morphisms of (F, M), and hence in particular with $N_1, \dots, N_r$. Furthermore, since

$$Y_{(e^{-\xi}.F,M)} = e^{-\xi}.Y_{(F,M)}$$

and ξ commutes with N_r, we have

$$Y(N_r, Y_{(\hat{F},M)}) = e^{-\xi}.Y(N_r, Y_{(F.M)})$$

Iterating this process, we obtain,

$$Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(\hat{F},M)}))) = e^{-\xi}.Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F,M)})))$$

□

Lemma (1.16) now becomes:

Corollary 2.7. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit. Then,*

$$Y = Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F,M)})))$$

preserves F. More generally, Y preserves the Deligne $I^{p,q}$'s of (F, M).

Proof. Let $\hat{Y}$ denote the analog of Y obtained by replacing (F, M) by the sl_2-splitting $(\hat{F}, M)$. Then, $\hat{Y}$ is real and preserves both $\hat{F}$ and M. Therefore, $\hat{Y}$ preserves

$$I^{p,q}_{(\hat{F},M)} = \hat{F}^p \cap \overline{\hat{F}^q} \cap M_{p+q}$$

By Lemma (2.6), it then follows that Y preserves $I^{p,q}_{(F,M)}$. □

Proof of Theorem 1.6. The theorem follows from the several variable SL_2-orbit theorem of via dependence on parameters (cf. Proposition (10.8) in [12]), together with with Deligne's letter to Cattani and Kaplan[8]. Namely, via the theory of polarized Hodge structures, it follows that any choice of monodromy logarithms $\tilde{N}(\sigma_1) \in \mathcal{N}(\sigma_1), \tilde{N}(\sigma_2) \in \mathcal{N}(\sigma_2), \dots$ will define a preadmissible nilpotent orbit

$$\tilde{\theta} = e^{\sum z_j \tilde{N}(\sigma_j)}.F$$

i.e. $\tilde{\theta}$ satisfies Griffiths horizontality and induces nilpotent orbits of pure Hodge structure on Gr^W. It then follows from a theorem of Kashiwara that $\tilde{\theta}$ is admissible. By the stated hypothesis on the sequence $(y_1(m), \dots, y_r(m))$,

$$\theta(iy_1(m), \dots, iy_r(m)) = \tilde{\theta}(iy_1^*(m), \dots, iy_r^*(m))$$

where $(iy_1^*(m), \dots, iy_r^*(m))$ denote the projection of $(y_1(m), \dots, y_r(m))$ which only keeps $y_j(m)$ if $j = \min(\sigma_k)$ for some k, and–for each m–$\tilde{\theta}$ is defined by an appropriate collection of monodromy logarithms $\tilde{N}(\sigma_j)$ which converge to $N(\sigma_j)$ as $m \to \infty$. The limit in question is then $\hat{Y}^0$ for the orbit $\tilde{\theta}$, and this is equal to $Y(N(\sigma_1), Y(N(\sigma_2), \dots, Y_{(F_\infty, M)}))$. □

In [12], Kato, Nakayama and Usui associate to any admissible nilpotent orbit with data $(N_1, \dots, N_r; F, W)$ an associated semisimple endomorphism $t(y)$. For use in section 4, we now derive a formula for $t(y)$ in terms of the gradings $\hat{Y}^j$ constructed above. To this end, let us assume for the moment that $(N_1, \dots, N_r; F, W)$ underlies a nilpotent orbit of pure Hodge structure of weight k. Let $(\hat{F}_r, W^r)$ denote the sl_2-splitting of (F, W^r), and recall that W^r in this case is the monodromy weight filtration $W(N)[-k]$ for any element N in the cone of positive linear combinations of $N_1, \dots, N_r$. In particular, since any such N is a $(-1,-1)$-morphism of $(\hat{F}_r, W^r)$ it follows that the pair $(N, \hat{Y}_{(r)})$ where

$$\hat{Y}_{(r)} = \hat{Y}^r - k\mathbb{1}$$

defines an sl_2-pair. As above, we can iteratively define $\hat{Y}_{(j)} = \hat{Y}^j - k\mathbb{1}$ using the nilpotent orbit $(N_1, \dots, N_j; \hat{F}_j)$. Define,

$$\tilde{t}(y) = \Pi_{j=1}^r t_j^{\frac{1}{2}\hat{Y}_{(j)}} = (\Pi_{j=1}^r t_j^{-\frac{1}{2}k\mathbb{1}})(\Pi_{j=1}^r t_j^{\frac{1}{2}\hat{Y}^j})$$

where $t_j = y_{j+1}/y_j$, and hence $t_1 \dots t_r = y_{r+1}/y_1 = 1/y_r$. Accordingly,

$$\tilde{t}(y) = y_1^{(\frac{1}{2}k)\mathbb{1}} \Pi_{j=1}^r t_j^{\frac{1}{2}\hat{Y}^j}.$$

Remark 2.8. Along any sequence $y(m)$, $t_j(m) = \lambda_j(m)$.

By Theorem (0.5) of [12], the mixed version of $t(y)$ is to be constructed as follows: If $(N_1, \dots, N_r; F, W)$ defines an admissible nilpotent orbit then

$$\hat{Y}_{(e^{\sum iy_jN_j}.F, W)} \to \hat{Y}^0$$

provided that $t_j \to 0$ for all j. Let $t_k(y)$ denote the semisimple endomorphism $\tilde{t}(y)$ attached by the previous paragraph to the induced nilpotent orbit of pure Hodge structure of weight k on Gr_k^W. Then, $t(y)$ is constructed by multiplying each $t_k(y)$ by $y_1^{-\frac{1}{2}k}$ and then lifting the resulting semisimple element to the ambient vector space via the grading $\hat{Y}^0$. Accordingly, since the gradings $\hat{Y}^0, \dots, \hat{Y}^r$ are mutually commuting, it follows that

$$t(y) = \Pi_{j=1}^r t_j^{\frac{1}{2}\hat{Y}^j} \tag{2.9}$$

The following result appears in Proposition (10.4) of [12] with slightly different notation:

Lemma 2.10. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit. Then,*

$$\mathrm{Ad}\,(\mathrm{t}^{-1}(\mathrm{y}))\mathrm{e}^{\sum_\mathrm{j} \mathrm{i}\mathrm{y}_\mathrm{j}\mathrm{N}_\mathrm{j}} = \mathrm{e}^\mathrm{P}$$

where P is a polynomial in non-negative half integral powers of $t_1, \dots, t_r$ with constant term $iN_1 + i\sum_{j>1} \hat{N}_j$.

Proof. By (2.9),

$$\mathrm{Ad}\,(\mathrm{t}^{-1}(\mathrm{y}))\mathrm{y}_\mathrm{k}\mathrm{N}_\mathrm{k} = (\Pi_{\mathrm{j}\leq \mathrm{k}-1}\, \mathrm{t}_\mathrm{j}^{-\frac{1}{2}\hat{\mathrm{Y}}^\mathrm{j}})(\Pi_{\mathrm{j}\geq\mathrm{k}}\mathrm{t}_\mathrm{j}^{-\frac{1}{2}\hat{\mathrm{Y}}^\mathrm{j}})\mathrm{y}_\mathrm{k}\mathrm{N}_\mathrm{k}$$

where N_k is $(-1,-1)$-morphism of $(\hat{F}_j, W^j)$ for $j = k, \dots, r$, and hence $[N_k, \hat{Y}^j] = -2N_k$. Consequently,

$$(\Pi_{j\geq k} t_j^{-\frac{1}{2}\hat{Y}^j}) y_k N_k = t_k \dots t_r y_k N_k = N_k$$

On the other hand, N_k preserves W^j for $j < k$. Therefore,

$$(\Pi_{j<k}\, t_j^{-\frac{1}{2}\hat{Y}^j}) N_k$$

is a polynomial in non-negative, half-integral powers of t_j for $j < k$. Taking the limit as $t_1, \dots, t_r \to 0$ it then follows that the constant term of P is $i\sum_k N_k^\sharp$ where $N_k^\sharp$ is the projection of N_k to $\cap_{0<j<k} \ker(\mathrm{ad}\,\hat{\mathrm{Y}}^\mathrm{j})$ with respect to the mutually commuting gradings $\hat{Y}^j$. Accordingly, $N_1^\sharp = N_1$, whereas for $k > 1$, we can first project onto $\ker(\mathrm{ad}\,\mathrm{N}_{\mathrm{k}-1})$ to obtain $\hat{N}_k$. By (2.5), $\hat{N}_k$ commutes with $\hat{Y}^j$ for $j < k$, and hence $N_k^\sharp = \hat{N}_k$. □

Remark 2.11. For nilpotent orbits of pure Hodge structure, this statement appears in Lemma (4.5) of [5]; note however that in [5], t_j is defined to be y_j/y_{j+1} which is reciprocal to our convention.

3. Surface case

Let $z(m) = (z_1(m), \dots, z_r(m))$ be an sl_2-convergent sequence with limiting ratios $\lambda = (0, \dots, 0)$. Then, for an index $j \leq r$, the sequence $z(m)$ is said to have non-polynomial growth with respect to $z_j(m)$ if for each positive integer $d > 1$ it follows that

$$\frac{y_{j+1}^d(m)}{y_j(m)} \to 0 \tag{3.1}$$

after passage to a suitable subsequence [which may depend on d]. In particular, by our convention that $y_{r+1}(m) = 1$, it follows that $z(m)$ always has non-polynomial growth with respect to $z_r(m)$, and hence there is a smallest integer ι with respect to which $z(m)$ has non-polynomial growth. Furthermore, negating the definition of non-polynomial growth, it follows that for each $j < \iota$ there exists an integer $d_j > 1$ such that

$$y_{j+1}^{d_j}(m) \geq y_j(m) \tag{3.2}$$

Remark 3.3. We only have to define the notion of non-polynomial growth in the case where $\lambda = (0, \dots, 0)$, since otherwise we can group the variables with $\lambda_j \neq 0$ together as in the proof of Theorem (1.6)

In this section, we prove Conjecture (1.1) in the case where S is a surface, i.e. we prove Theorem (1.10) for $r = 2$. Given an sl_2-convergent sequence $z(m)$ the possible limiting ratios are $\lambda = (\lambda_1, 0)$ with $\lambda_1 \neq 0$ and $\lambda = (0, 0)$.

Theorem 3.4. *Theorem* (1.10) *holds for* sl_2*-convergent sequences* $z(m) = (z_1(m), z_2(m))$ *with limiting ratios* $\lambda = (\lambda_1, 0)$ *with* $\lambda_1 \neq 0$.

Proof. Mutatis mutandis, this follows from the proof of Theorem (3.9) of [1] (see also [2]). More precisely, by the SL_2-orbit theorem of [16] if $(e^{zN}.F_\infty, W)$ is a nilpotent orbit arising from an admissible normal function then

$$e^{iyN}.F_\infty = g(y) y^{-\frac{1}{2}H}.\hat{F}_\infty$$

where the coefficients of the $G_{\mathbb{R}}$-valued function

$$g(y) = 1 + g_1 y^{-1} + g_2 y^{-2} + \cdots \tag{3.5}$$

are given by universal Lie polynomials in the Hodge components of the Deligne (or sl_2)-splitting of $(F_\infty, M(N, W))$ and $\mathrm{ad}\, \mathrm{N}_0^+$ where (N_0, H, N_0^+) is the associated sl_2-triple. Accordingly, given a two variable admissible nilpotent orbit $e^{z_1 N_1 + z_2 N_2}.F_\infty, W)$ with W as above, we have

$$e^{iy(N_1 + \tau N_2)}.F_\infty = g_\tau(y) e^{iy(N_1 + \tau N_2)}.\hat{F}_\infty$$

where the coefficients $g_k(\tau)$ of $g_\tau(y)$ are real analytic functions of τ. In particular, If $\lambda_1 \neq 0$ then

$$e^{iy_1 N_1 + iy_2 N_2}.F_\infty = e^{iy_1(N_1 + y_2/y_1 N_2)}.F_\infty$$

where $N_1 + y_2/y_1 N_2 \to N_1 + \lambda_1 N_2$. Using this observation, the proof of Theorem (1.10) now proceeds as in Theorem (3.9) of [1] using the local normal form of the period map (1.9) and the fact that $\lambda_1 \neq 0$ also implies that $y_j^n s_k \to 0$. □

Remark 3.6. In [16], the function $g(y)$ does not have leading coefficient 1 (cf. (3.5)) because the construction of [16] is done with respect to Deligne's δ-splitting of the limit mixed Hodge structure. As explained in [2], we can renormalize $g(y)$ to have leading coefficient 1 by basing the construction at the sl_2-splitting of the limit mixed Hodge structure. We make this renormalization throughout this article. For a comparison of the results of [16] and [12] see section 11 of [12].

Remark 3.7. The analytic dependence of the coefficients of $g_\tau(y)$ on τ also appears in 10.8 of [12].

Returning now to our sl_2-convergent sequence $z(m) = (z_1(m), z_2(m))$, the case $\lambda = (0, 0)$ can be subdivided according to the value of ι. Suppose therefore that $\iota = 2$. Then, $z(m)$ has polynomial growth with respect to $z_1(m)$ and hence there exists an integer ℓ such that

$$y_2(m) \leq y_1(m) \leq y_2^\ell(m) \tag{3.8}$$

Let $\mathcal{V} \to \Delta^{*2}$ be an admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy, and local normal form

$$F(z_1, z_2) = e^{z_1 N_1 + z_2 N_2} e^{\Gamma(s_1, s_2)}.F_\infty \tag{3.9}$$

Assume the weight filtration W has only two non-zero graded quotients which are adjacent. Define

$$\tilde{F}(z) = e^{-\mu(x)}.F(z) = e^{-x_1 N_1 - x_2 N_2}.F(z)$$

and note hat $(\tilde{F}(z), W)$ is split over $\mathbb{R}$ due to the short length of $W = W^0$.

Let $e^{z_1 N_1 + z_2 N_2}.F_\infty$ be the associated nilpotent orbit of $F(z)$. Following the conventions of section 2, let $(\hat{F}_2, W^2)$ be the sl_2-splitting of (F_∞, W^2) and $\mathbf{r} = \hat{F}_0$. Then, by the several variable SL_2-orbit theorem of [12]

$$e^{iy_1 N_1 + iy_2 N_2}.F_\infty = t(y)^e g(y).\mathbf{r}$$

where ${}^e g(y)$ is a $G_{\mathbb{R}}$-valued function which has a convergent series expansion in terms of the variables $t_j^{\frac{1}{2}}$ with leading coefficient 1, where $t_j = y_{j+1}/y_j$. Therefore,

$$\begin{aligned}
Y_{(\tilde{F}(z),W)} &= Y_{((\mathrm{Ad}\,(e^{iy_1N_1+iy_2N_2})e^{\Gamma(s_1,s_2)})e^{iy_1N_1+iy_2N_2}.F_\infty,W)} \\
&= Y_{((\mathrm{Ad}\,(e^{iy_1N_1+iy_2N_2})e^{\Gamma(s_1,s_2)})^e g(y)t(y).\mathbf{r},W)} \\
&= t(y)^e g(y).Y_{((\mathrm{Ad}\,({}^e g^{-1}(y)t^{-1}(y)e^{iy_1N_1+iy_2N_2})e^{\Gamma(s_1,s_2)}).\mathbf{r},W)} \\
&= t(y)^e g(y).Y_{(e^u.\mathbf{r},W)}
\end{aligned}$$

where

$$e^u = \mathrm{Ad}\,({}^e g^{-1}(y)t^{-1}(y)e^{iy_1N_1+iy_2N_2})e^{\Gamma(s_1,s_2)}$$

In particular, since the function $\Gamma(s_1, s_2)$ is holomorphic and vanishes at $(0,0)$, it follows by (3.8) that $u \to 0$ along such a sequence $(y_1(m), y_2(m))$. Indeed,

$$\mathrm{Ad}\,(t^{-1}(y)e^{iy_1N_1+iy_2N_2})\Gamma(s_1,s_2) \in I[y_1^{\frac{1}{2}}, y_1^{-\frac{1}{2}}, y_2^{\frac{1}{2}}, y_2^{-\frac{1}{2}}]$$

where I is the ideal of $\mathfrak{g}_{\mathbb{C}}$-valued functions which vanish at $(0,0)$. Therefore, by (3.8) and the fact that $q^j e^{-q} \to 0$ as $q \to \infty$, the previous statement implies that $u \to 0$, since ${}^e g(y)$ is bounded as $(y_1, y_2) \to \infty$ with $y_2/y_1 \to 0$. Accordingly,

$$Y_{(\tilde{F}(z),W)} = t(y)^e g(y) g_{\mathbb{R}}(y).Y_{(\mathbf{r},W)}$$

where $g_{\mathbb{R}}(y) = e^\gamma$ such that $|\gamma|$ can be bounded by the norm of an element of $I[y_1^{\frac{1}{2}}, y_1^{-\frac{1}{2}}, y_2^{\frac{1}{2}}, y_2^{-\frac{1}{2}}]$ with $|s_j| = e^{-2\pi y_j}$. By [12][8], $t(y)$ is at worst a polynomial in half integral powers of y_1 and y_2 and fixes $Y_{(\mathbf{r},W)}$, while ${}^e g(y) \sim 1$ for $y_2/y_1 \sim 0$. Therefore,

$$Y_{(\tilde{F}(z(m)),W)} \to Y_{(\mathbf{r},W)}$$

under (3.8). Again, comparing Deligne's construction [8] to [12], it follows that

$$Y_{(\mathbf{r},W)} = Y(N_1, Y(N_2, Y_{(\hat{F}_\infty,M)}))$$

where $(\hat{F}_\infty, M)$ is the sl_2-splitting of (F_∞, M).

Remark 3.10. Mutatis mutandis, the same argument works for any number of variables r provided that $\iota = r$.

Suppose now that $z(m) = (z_1(m), z_2(m))$ is an sl_2-convergent sequence with non-polynomial growth with respect to $z_1(m)$. Define

$$F_\infty(z_2) = e^{iy_2 N_2} e^{\Gamma(0,s_2)}.F_\infty \tag{3.11}$$

Then, for any fixed value of z_2,

$$\theta_{z_2}(z_1) = e^{z_1 N_1}.F_\infty(z_2) \tag{3.12}$$

pairs with the weight filtration W^0 of $\mathcal{V}$ to define an admissible nilpotent orbit. Furthermore,

$$z_2 \mapsto e^{x_2 N_2}.F_\infty(z_2) = e^{z_2 N_2} e^{\Gamma(0,s_2)}.F_\infty \tag{3.13}$$

pairs with $W^1 = M(N_2, W^0)$ to define the lifted period map of an admissible variation of mixed Hodge structure over Δ^*.

Lemma 3.14. *Let $\delta(z_2)$ denote the δ-splitting of $(F_\infty(z_2), W^1)$. Then, the Hodge components of $\delta(z_2)$ are bounded by polynomials in y_2.*

Proof. Let $z = x + iy$. Then, by application application of Corollary (12.8) of [12] to the period map $z \mapsto e^{xN_2}.F_\infty(z)$, it follows that

$$t(y)^{-1}.Y_{(F_\infty(z),W^1)} \to Y_{(e^{\epsilon_0}.\mathbf{r},W^1)}$$

(see [12] for notation). Therefore,

$$Y_{(F_\infty(z),W^1)} \sim t(y).Y_{(e^{\epsilon_0}.\mathbf{r},W^1)}$$

for sufficiently large values of y. □

Remark 3.15. Mutatis mutandis, this lemma and its proof remain valid in several variables. Also, the pair $(e^{\epsilon_0}.\mathbf{r}, W^1)$ is a mixed Hodge structure since $(\mathbf{r}, W^1)$ is a mixed Hodge structure and ϵ_0 is derived from the sl_2-splitting operation and hence acts trivially on Gr^{W^1}.

Corollary 3.16. *Let $(\hat{F}_\infty(z_2), W^1) = (e^{-\epsilon(z_2)}.F_\infty(z_2), W^1)$ denote the sl_2-splitting of $(F_\infty(z_2), W^1)$. Then, the Hodge components of $\epsilon(z_2)$ are bounded by polynomials in y_2 with respect to any fixed basis of $\mathfrak{g}_\mathbb{C}$ as $y_2 \to \infty$ (and x_2 restricted to an interval of finite length). Likewise, the components of the grading $Y_{(\hat{F}_\infty(z_2),W^1)}$ with respect to any basis of $\mathrm{gl}(V_\mathbb{C})$ are bounded by polynomials in y_2.*

In addition to [12], we have an another proof of the 1-variable SL_2-orbit theorem [16]. Following [16], fix z_2 and let

$$\theta_{z_2}(iy_1) = g_{z_2}(y_1) e^{iy_1 N_1}.\hat{F}_\infty(z_2) \tag{3.17}$$

be the asymptotic SL_2-orbit expansion of $\theta_{z_2}(iy)$ for $y_1 > a(z_2)$, normalized so that

$$g_{z_2}(y_1) = 1 + g_1(z_2) y_1^{-1} + g_2(z_2) y_1^{-2} + \cdots \tag{3.18}$$

i.e. $(\hat{F}_\infty(z_2), W^1)$ is the sl_2-splitting of $(F_\infty(z_2), W^1)$ and not the δ-splitting which appears in [16].

To continue, recall that by a theorem of Deligne [8][10], the sl_2-representation attached to the nilpotent orbit $\theta_{z_2}(z_1)$ is constructed as follows: Let

$$H_1(z_2) = Y_{(\hat{F}_\infty(z_2),W^1)} - Y(N_1, Y_{(\hat{F}_\infty(z_2),W^1)}) \tag{3.19}$$

where $Y(N_1, Y_{(\hat{F}_\infty(z_2),W^1)}) = Y_{(\mathbf{r},W^0)}$ Then, $(N_1, H_1(z_2))$ is an sl_2-pair. Moreover, due to the short length of W^0, in this case we have

$$Y(N_1, Y_{(\hat{F}_\infty(z_2),W^1)}) = Y_{(F_o(z_2),W^0)}$$

where $F_o(z_2) = e^{iN_1}.\hat{F}_\infty(z_2)$.

Corollary 3.20. *Let $(N_1, H_1(z_2), N_1^+(z_2))$ be the sl_2-triple associated to $(N_1, H_1(z_2))$. Then $H_1(z_2)$ and $N_1^+(z_2)$ are bounded by polynomials in y_2 as $y_2 \to \infty$ with x_2 restricted to an interval of finite length.*

By [16], $g_j(z_2)$ is given by universal Lie polynomials in the Hodge components of Deligne's δ-splitting of $(F_\infty(z_2), W^1)$ and $\operatorname{ad}\mathrm{N}_1^+(\mathrm{z}_2)$. By, Lemma (3.14) and Corollary (3.20), both of these ingredients are bounded by polynomials of y_2.

To continue, observe that since $z(m)$ has non-polynomial growth with respect to $z_1(m)$, given a positive integer ℓ, it follows that after passage to a subsequence, we can assume that

$$y_1(m) \geq y_2^\ell(m) \tag{3.21}$$

for all m sufficiently large. The arguments below will produce many quantities which are of polynomial growth with respect to $y_2(m)$. Therefore, by taking ℓ sufficiently large and passage to a subsequence, we will be able to ensure that all of these quantities vanish when paired against $y_1(m)^{-1}$ as $m \to \infty$.

Theorem 3.22. *Let $z(m) = (z_1(m), z_2(m))$ be an sl_2-convergent sequence with non-polynomial growth with respect to $z_1(m)$. Then, upon passage to a subsequence, for m sufficiently large, $y_1(m) > a(z_2(m))$ and*

$$\lim_{m\to\infty} g_{z_2(m)}(y_1(m)) = 1$$

Proof. Recall the proof of the 1-variable SL_2-orbit theorem in [16]. On page 62 there is a formula for a quantity

$$C_{\ell+1} = i \sum_{p,q\geq 1, p+q\geq \ell+1} b_{p-1,q-1}^{\ell-1}\eta^{-p,-q}$$

in terms of the Hodge components of a quantity η and the constants $b_{r,s}^t$ defined by

$$(1-x)^r(1+x)^s = \sum_t b_{r,s}^t x^t$$

Let

$$C_{\ell+1} = \frac{(-i)^\ell}{\ell!}(\operatorname{ad}\mathrm{N})^\ell \mathrm{B}_{\ell+1} \tag{3.23}$$

Then, by Corollary (8.30) of [16], the coefficient g_k appearing in the SL_2-orbit expansion can be expressed as a universal non-commutative polynomial of degree k in $B_2, \dots, B_{k+1}$ where B_j is assigned degree $j-1$. By [16], η is bounded by a polynomial of degree d in y_2 since $\delta(z_2)$ is bounded by a polynomial in y_2. Therefore, B_j is also bounded by a polynomial of degree d in y_2, and hence g_k will be bounded by a polynomial of degree kd in y_2. Therefore, $g_k y_1^{-k}$ will be bounded by a polynomial of degree k in $(y_2^d)/y_1$. By condition (3.21), $y_2^d/y_1 \to 0$ along our sequence, and hence we can make (y_2^d/y_1) as small as we please. Accordingly, by a comparison test, we obtain the convergence of $g_{z_2}(y_1) \to 1$ along our sequence. □

Remark 3.24. By equation (8.29) of [16] paper, if we decompose $B_{\ell+1}$ into isotypical components with respect to the associated sl_2-representation, we see that the factor $\operatorname{ad}(\mathrm{N})^\ell$ appearing in equation (3.23) does not kill any components of $B_{\ell+1}$. Therefore, the degree of $B_{\ell+1}$ in y_2 is equal to the degree of $C_{\ell+1}$ in y_2.

Returning to (3.9) let $\mu(x) = x_1N_1 + x_2N_2$. Then,

$$\begin{aligned} Y_{(F(z_1,z_2),W^0)} &= e^{\mu(x)}.Y_{(e^{iy_1N_1+iy_2N_2}e^{\Gamma(s_1,s_2)}.F_\infty,W^0)} \\ &= e^{\mu(x)}.Y_{((\mathrm{Ad}\,(\mathrm{e}^{\mathrm{i}y_1\mathrm{N}_1+\mathrm{i}y_2\mathrm{N}_2})(\mathrm{e}^{\Gamma(s_1,s_2)}\mathrm{e}^{-\Gamma(0,s_2)})).\theta_{\mathrm{z}_2}(\mathrm{i}y_1),\mathrm{W}^0)} \\ &= e^{\mu(x)}.Y_{((\mathrm{Ad}\,(\mathrm{e}^{\mathrm{i}y_1\mathrm{N}_1+\mathrm{i}y_2\mathrm{N}_2})(\mathrm{e}^{\tilde{\Gamma}(s_1,s_2)}).\theta_{\mathrm{z}_2}(\mathrm{i}y_1),\mathrm{W}^0)} \end{aligned} \tag{3.25}$$

where

$$e^{\tilde{\Gamma}(s_1,s_2)} = e^{\Gamma(s_1,s_2)}e^{-\Gamma(0,s_2)}$$

Observe that since $\Gamma(s_1,s_2)$ is a holomorphic functions of s_1 and s_2 which vanishes at $(0,0)$, it follows from the defining equation of $\tilde{\Gamma}$ and the Baker–Campbell–Hausdorff formula that $s_1|\tilde{\Gamma}$ in $\mathcal{O}(\Delta^2)$. Furthermore, if we temporarily view s_1, s_2, y_1, y_2 as independent variables, then since $\mathrm{ad}\,\mathrm{N}_1$ and $\mathrm{ad}\,\mathrm{N}_2$ are nilpotent and preserve the subalgebra $\mathfrak{q}$ in which $\Gamma(s_1,s_2)$ assumes its values, we have

$$\mathrm{Ad}\,(\mathrm{e}^{\mathrm{i}y_1\mathrm{N}_1+\mathrm{i}y_2\mathrm{N}_2})\mathrm{e}^{\tilde{\Gamma}(s_1,s_2)} \in \mathrm{I}[y_1,y_2]\otimes\mathfrak{q}$$

where I is the ideal of holomorphic functions on Δ^2 which are divisible by s_1.

Let $F_o(z_2) = e^{iN_1}.\hat{F}_\infty(z_2)$ and $H_1(z_2)$ be the semisimple element (3.19). Then, via the SL_2-orbit theorem, we have

$$Y_{(F(z_1,z_2),W^0)} = e^{\mu(x)}.Y_{(e^v g_{z_2}(y_1)y^{-\frac{1}{2}H_1(z_2)}.F_0(z_2),W^0)} \tag{3.26}$$

where

$$e^v = (\mathrm{Ad}\,(\mathrm{e}^{\mathrm{i}y_1\mathrm{N}_1+\mathrm{i}y_2\mathrm{N}_2})(\mathrm{e}^{\tilde{\Gamma}(s_1,s_2)})$$

To continue, observe that since $y_1 \geq y_2$ implies

$$\begin{aligned} |y_1^{\frac{1}{2}j}y_2^{\frac{1}{2}k}s_1| &= (y_1^{\frac{1}{2}j}|s_1|^{\frac{1}{2}})(y_2^{\frac{1}{2}k}|s_1|^{\frac{1}{2}}) \\ &\leq (y_1^{\frac{1}{2}j}|s_1|^{\frac{1}{2}})(y_2^{\frac{1}{2}k}|s_2|^{\frac{1}{2}}) = (y_1^{\frac{1}{2}j}e^{-\pi y_1})(y_2^{\frac{1}{2}k}e^{-\pi y_2}) \end{aligned} \tag{3.27}$$

for any non-negative integers j and k. Taking the limit as $\ell \to \infty$, it then follows that

$$|y_1^{\frac{1}{2}j}(m)y_2^{\frac{1}{2}k}(m)s_1(m)| = 0 \tag{3.28}$$

Returning to (3.26), we have

$$\begin{aligned} Y_{(F(z_1,z_2),W^0)} &= e^{\mu(x)}.Y_{(g_{z_2}(y_1)g_{z_2}^{-1}(y_1)e^v g_{z_2}(y_1)y_1^{-\frac{1}{2}H_1(z_2)}.F_0(z_2),W^0)} \\ &= e^{\mu(x)}g_{z_2}(y_1).Y_{((\mathrm{Ad}\,(\mathrm{g}_{\mathrm{z}_2}^{-1}(\mathrm{y}_1))\mathrm{e}^{\mathrm{v}})\mathrm{y}^{-\frac{1}{2}\mathrm{H}_1(\mathrm{z}_2)}.\mathrm{F}_0(\mathrm{z}_2),\mathrm{W}^0)} \\ &= e^{\mu(x)}g_{z_2}(y_1)y_1^{-\frac{1}{2}H_1(z_2)}.Y_{((\mathrm{Ad}\,(\mathrm{y}_1^{\frac{1}{2}\mathrm{H}_1(\mathrm{z}_2)}\mathrm{g}_{\mathrm{z}_2}^{-1}(\mathrm{y}_1))(\mathrm{e}^{\mathrm{v}}).\mathrm{F}_0(\mathrm{z}_2),\mathrm{W}^0)} \end{aligned}$$

By Corollary (3.20), $H_1(z_2)$ is bounded by polynomials in y_2. Consequently, since $s_1|\tilde{\Gamma}(s_1,s_2)$, it follows from equations (3.27), (3.28) and Theorem (3.22) that

$$e^u = (\mathrm{Ad}\,(\mathrm{y}_1^{\frac{1}{2}\mathrm{H}_1(\mathrm{z}_2)}\mathrm{g}_{\mathrm{z}_2}^{-1}(\mathrm{y}_1))\mathrm{e}^{\mathrm{v}} \to 1 \tag{3.29}$$

and more strongly, by the boundedness of $g_{z_2}(y_1)$, the left hand side of (3.29) is bounded by a polynomial in half integral powers of y_1 and y_2 – arising from the $y_1^{H_1(z_2)}$ and $e^{iy_1N_1+iy_2N_2}$ – and an element of I coming from $\tilde{\Gamma}(s_1,s_2)$. In particular, since (Deligne [10]), $H_1(z_2)$ preserves $Y_{(F_0(z_2),W^0)}$ it follows that

$$Y_{(F(z_1,z_2),W^0)} = e^{\mu(x)}e^v g_{z_2}(y_1)\mathrm{Ad}\,(\mathrm{y}_1^{-\frac{1}{2}\mathrm{H}_1(\mathrm{z}_2)}\mathrm{e}^{-\varphi(\mathrm{u})}).\mathrm{Y}_{(\mathrm{F}_0(\mathrm{z}_2),\mathrm{W}^0)} \tag{3.30}$$

where

$$e^u = g_{\mathbb{R}}(u)e^{\varphi(u)}$$

with $g_{\mathbb{R}}(u) \in G_{\mathbb{R}}$ and $\varphi(u) \in Lie(G_{\mathbb{C}}^{F_0(z_2)})$ such that

$$\bar{\varphi}(u)^{0,0} = -\varphi(u)^{0,0}$$

[Hodge components with respect to $(F_0(z_2),W^0)$]. Again, because of the form of $F_o(z_2)$, the projection operators onto Hodge components with respect to $F_o(z_2)$ are bounded by polynomials in y_2. Therefore, since $\varphi(u)$ is given by universal Lie series in the Hodge components of u, $\bar{u}$, etc., the fact u is bounded by polynomials in half integral powers of y_1, y_2 times elements of I forces

$$\mathrm{Ad}\,(\mathrm{y}_1^{-\frac{1}{2}\mathrm{H}_1(\mathrm{z}_1)})\mathrm{e}^{-\varphi(\mathrm{u})} \to 1$$

along a sequence $(z_1(m), z_2(m))$ which satisfies condition (3.21). Likewise, along such a sequence,

$$(\mathrm{Ad}\,(\mathrm{e}^{\mathrm{iy}_1\mathrm{N}_1+\mathrm{iy}_2\mathrm{N}_2})(\mathrm{e}^{\tilde{\Gamma}(\mathrm{s}_1,\mathrm{s}_2)})) \to 1$$

by (3.27), while $g_{z_2}(y_1) \to 1$ by (3.22).

Corollary 3.31. *Let $z(m) = (z_1(m), z_2(m))$ be an* sl_2*-convergent sequence with non-polynomial growth with respect to $z_1(m)$. Then,*

$$Y_{(F(z(m)),W)} \to e^{\mu}.\hat{Y}^0$$

as required to finish the proof Theorem (1.10) *in dimension two provided that*

$$Y_{(F_o(z_2),W)} = Y(N_1, Y_{(\hat{F}(z_2),W^1)}) \to \hat{Y}^0$$

Equivalently,

$$Y_{(\hat{F}(z_2),W^1)} \to \hat{Y}^1 = Y(N_2, \hat{Y}_{(\hat{F}_\infty,W^2)}) \tag{3.32}$$

For the remainder of this section, we drop the assumption that our weight filtration has only two non-zero graded-quotients and prove the following result, which contains (3.32) as a special case.

Theorem 3.33. *Let $F : U \to \mathcal{M}$ be the lifting of the period map of an admissible variation of mixed Hodge structure $\mathcal{V} \to \Delta^*$ with unipotent monodromy to the upper half-plane. Let $z(m) = x(m) + iy(m)$ be an* sl_2*-convergent sequence. Then, after passage to a subsequence,*

$$\lim_{m\to\infty} \hat{Y}_{(F(z),W)} = \lim_{m\to\infty} Y_{(\widehat{F(z),W})} = e^{\mu}.Y_{(\mathbf{r},W)} \tag{3.34}$$

where $\mu = \lim_{m\to\infty}, x(m)N$.

By way of preliminary discussions, let (F, W) be a mixed Hodge structure and recall that Deligne's δ-splitting [6] is defined to be the unique real element δ of $\Lambda^{-1,-1}_{(F,W)}$ such that

$$\bar{Y}_{(F,W)} = e^{-2i\delta}.Y_{(F,W)}$$

and hence $(e^{-i\delta}.F, W)$ is split over $\mathbb{R}$.

Lemma 3.35. *Let (F, W) be a mixed Hodge structure and $\lambda \in \Lambda^{-1,-1}_{(F,W)}$. Let $\delta(\lambda)$ be Deligne's δ-splitting for $(e^{\lambda}.F, W)$. Then,*

$$e^{-2i\delta(\lambda)} = e^{\bar{\lambda}}e^{-2i\delta(0)}e^{-\lambda}$$

Proof. On the one hand,

$$\bar{Y}_{(e^{\lambda}.F,W)} = \overline{e^{\lambda}.Y_{(F,W)}} = e^{\bar{\lambda}}.\bar{Y}_{(F,W)} = e^{\bar{\lambda}}e^{-2i\delta(0)}.Y_{(F,W)}$$

On the other hand,

$$\bar{Y}_{(e^{\lambda}.F,W)} = e^{-2i\delta(\lambda)}.Y_{(e^{\lambda}.F,W)} = e^{-2i\delta(\lambda)}e^{\lambda}.Y_{(F,W)}$$

Comparing these two equations and taking note of the fact that $\Lambda^{-1,-1}_{(F,W)} \subset W_{-1}\mathfrak{g}_{\mathbb{C}}$, it follows that

$$e^{-2i\delta(\lambda)} = e^{\bar{\lambda}}e^{-2i\delta(0)}e^{-\lambda}$$

as required. □

We also recall (see [12]) that Deligne's δ-splitting $e^{-i\delta}.F$ and the sl_2-splitting $e^{-\epsilon}.F$ of (F,W) are related by formula

$$\delta = \frac{i}{2}H(\epsilon, -\bar{\epsilon}) \tag{3.36}$$

where $e^{H(a,b)} = e^a e^b$ is the Baker–Campbell–Hausdorff formula.

Let

$$F(z) = e^{zN}e^{\Gamma(s)}.F_\infty$$

be the local normal form of the period map appearing in Theorem (3.33).

Let $\tilde{F}(z) = e^{-xN}.F(z)$. Recall that by [12]

$$e^{iyN}.F_\infty = t(y)^e g(y)e^{\epsilon(y)}.\mathbf{r}$$

Accordingly,

$$\begin{aligned}
\hat{Y}_{(\tilde{F}(z),W)} &= \hat{Y}_{(e^{iyN}e^{\Gamma(s)}.F_\infty,W)} \\
&= \hat{Y}_{((\mathrm{Ad}\,(e^{iyN})e^{\Gamma(s)})e^{iyN}.F_\infty,W)} \\
&= \hat{Y}_{((\mathrm{Ad}\,(e^{iyN})e^{\Gamma(s)})t(y)^e g(y)e^{\epsilon(y)}.\mathbf{r},W^1)} \\
&= t(y)^e g(y).\hat{Y}_{((\mathrm{Ad}\,({}^e g^{-1}(y)t^{-1}(y)e^{iyN})e^{\Gamma(s)})e^{\epsilon(y)}.\mathbf{r},W)}
\end{aligned} \tag{3.37}$$

Let I denotes the ideal of holomorphic functions of s which vanish at $s = 0$ and

$$e^u = \mathrm{Ad}\,({}^e g^{-1}(y)t^{-1}(y)e^{iyN})e^{\Gamma(s)}$$

For the moment, let us view y and s as independent variables. Then, $u(s,y)$ is a real analytic function of s and $y^{-\frac{1}{2}}$, and polynomial in $y^{\frac{1}{2}}$. Furthermore, there exists an element $j(s,y) \in I[y^{\frac{1}{2}}, y^{-\frac{1}{2}}]$ such that $|u(s,y)| < |j(s,y)|$ for all s sufficiently close to $0 \in \Delta$ and y sufficiently large. Indeed,

$$\mathrm{Ad}\,(t^{-1}(y)e^{iyN})e^{\Gamma} = \exp(\alpha)$$

for some $\alpha \in \mathfrak{g}_{\mathbb{C}} \otimes I[y^{\frac{1}{2}}, y^{-\frac{1}{2}}]$. Therefore, since ${}^e g(y)$ is real-analytic in $y^{-\frac{1}{2}}$, it is a bounded operator as $y \to \infty$, and hence we can find an element $j(s,y) \in I[y^{\frac{1}{2}}, y^{-\frac{1}{2}}]$ which bounds $u(s,y)$ as $y \to \infty$ and $s \to 0$. In particular, if $s \to 0$ and $y \to \infty$ along a sequence such that $|s| = e^{-2\pi y}$ then $u(s,y) \to 0$ at a rate which is given by a constant times $y^{\frac{1}{2}n}e^{-2\pi y}$ for some integer n.

Next, we recall that if (F,W) is a graded-polarized mixed Hodge structure and $u \in \mathfrak{g}_{\mathbb{C}}$ is sufficiently small, then there is a distinguished decomposition

$$e^u = g_{\mathbb{R}}(u)e^{\lambda(u)}f(u) \tag{3.38}$$

where $g_{\mathbb{R}}(u) \in G_{\mathbb{R}}$, $\lambda(u) \in \Lambda^{-1,-1}_{(F,W)}$ and $f(u) \in G^F_{\mathbb{C}}$ are given by universal Lie series in u, $\bar{u}$, and their Hodge components with respect to (F,W). More generally, if F depends real-analytically on a real parameter $t \sim 0$ then the decomposition

$$e^u = g_{\mathbb{R}}(u,t)e^{\lambda(u,t)}f(u,t)$$

with respect to $(F(t),W)$ will be given by Lie universal series in u, $\bar{u}$ and their Hodge components with respect to $(F(t),W)$. Accordingly, since we can express the Hodge components with respect to $(F(t),W)$ as real analytic functions of t in the Hodge components with respect to $(F(0),W)$ it follows that $g_{\mathbb{R}}(u,t)$ etc. will be given by Lie series in u, $\bar{u}$ and their Hodge components with respect to $(F(0),W)$ with real-analytic coefficients. The norms of these real-analytic functions will be determined by the coefficients of the universal series for $g_{\mathbb{R}}(u)$ etc. determined by (3.38) and real-analytic functions which give projection onto Hodge components with respect to $(F(t), W)$ in terms of projection onto Hodge components with respect to $(F(0), W)$. In particular, since $\epsilon(y)$ is real-analytic in $y^{-\frac{1}{2}}$ the operations of taking Hodge components with respect to $e^{\epsilon(y)}.\mathbf{r}$ are also real-analytic in $y^{-\frac{1}{2}}$. Therefore, if we let $g_{\mathbb{R}}(u, y)$ etc. denote the decomposition of e^u with respect to $e^{\epsilon(y)}.\mathbf{r}$ then $g_{\mathbb{R}}(u, y)$ etc. will be given by Lie series in u, $\bar{u}$ and their Hodge components with respect to $(\mathbf{r}, W)$ with real-analytic functions of $y^{-\frac{1}{2}}$ as coefficients.

Returning now to (3.37), we have

$$\begin{aligned} \hat{Y}_{(\tilde{F}(z),W)} &= t(y)^e g(y).\hat{Y}_{(e^{u(s,y)}e^{\epsilon(y)}.\mathbf{r},W)} \\ &= t(y)^e g(y) g_{\mathbb{R}}(u(s,y),y)..\hat{Y}_{(e^{\lambda(u(s,y),y)}e^{\epsilon(y)}.\mathbf{r},W)} \end{aligned} \tag{3.39}$$

Accordingly, by Lemma (3.35) and the Baker–Campbell–Hausdorff formula, if $\delta(y)$ is the δ-splitting of $(e^{\epsilon(y)}.\mathbf{r}, W)$ and $\delta(\lambda, y)$ is the δ-splitting of $(e^{\lambda}e^{\epsilon(y)}.\mathbf{r}, W)$ where $\lambda = \lambda(u(s, y), y)$ then

$$\delta(\lambda, y) = \delta(y) + \beta(\lambda, y)$$

where $\beta(\lambda, y)$ is given by a universal Lie series in λ, $\bar{\lambda}$ and $\delta(y)$ such that every term of $\beta(\lambda, y)$ contains either λ or $\bar{\lambda}$. By equation (3.36), it then follows that

$$\epsilon(\lambda, y) = \epsilon(y) + \gamma(\lambda, y)$$

[the sl_2-splitting of $(e^{\lambda}e^{\epsilon(y)}.\mathbf{r}, W)$] where $\gamma(\lambda, y)$ is a universal series in λ, $\bar{\lambda}$, $\epsilon(y)$, $\bar{\epsilon}(y)$ and their Hodge components with respect to $(\mathbf{r}, W)$ such that every term of $\gamma(\lambda, y)$ contains either a Hodge component of λ or $\bar{\lambda}$. [Recall:

$$\Lambda^{-1,-1}_{(e^{\lambda}e^{\epsilon}.\mathbf{r},W)} = \Lambda^{-1,-1}_{(e^{\epsilon}.\mathbf{r},W)}$$

since $\lambda \in \Lambda^{-1,-1}_{(e^{\epsilon}.\mathbf{r},W)}$, and that in solving for ϵ in terms of δ using (3.36), we only use the bigraded structure $\oplus_{r,s<0}\,\mathfrak{g}^{r,s}$ of $\Lambda^{-1,-1}$]. Consequently,

$$\begin{aligned} (\widehat{e^{\lambda}e^{\epsilon(y)}}.\mathbf{r}, W) &= (e^{-\epsilon-\gamma(\lambda,y)}e^{\lambda}e^{\epsilon(y)}.\mathbf{r}, W) \\ &= (e^{-\epsilon(y)-\gamma(\lambda,y)}e^{\epsilon(y)}e^{-\epsilon(y)}e^{\lambda}e^{\epsilon(y)}.\mathbf{r}, W) \\ &= (e^{\sigma(\lambda,\epsilon)}.\mathbf{r}, W) \end{aligned} \tag{3.40}$$

where all of the Hodge components of $\sigma(\lambda, \epsilon)$ are given by universal series which contain at least one Hodge component of λ or $\bar{\lambda}$.

Proof of Theorem (3.33). Inserting (3.40) into (3.39), it follows that

$$\hat{Y}_{(\tilde{F}(z),W)} = t(y)^e g(y) g_{\mathbb{R}} e^{\sigma(\lambda,\epsilon)}.Y_{(\mathbf{r},W)} \tag{3.41}$$

where $g_{\mathbb{R}} = g_{\mathbb{R}}(u(s, y), y)$, $\lambda = \lambda(u(s, y), y)$ and $\epsilon = \epsilon(y)$. Moreover, $t(y)$ fixes $Y_{(\mathbf{r},W)}$ (by construction), $t(y)^e g(y) = g(y)t(y)$ and $\mathrm{Ad}\,(\mathrm{t(y)})\gamma_{\mathbb{R}}$, $\mathrm{Ad}\,(\mathrm{t(y)})\sigma(\lambda, \epsilon) \to 1$ along any sequence $(y(\ell), s(\ell))$ such that $|s(\ell)| = e^{-2\pi y}$, since $|u(s, y)|$ can be bounded by the norm of an element of $J[y^{\frac{1}{2}}, y^{-\frac{1}{2}}]$. Recall that $g(y) = u(y)\tilde{g}(y)$ where $u(y) \to 1$. Likewise, $\tilde{g}(y)Gr^W \to 1$ by construction [6] and hence $\tilde{g}(y) \to 1$. Inserting these limits into (3.41), yields

$$\lim_{m\to\infty} \hat{Y}_{(F(z),W)} = e^{\mu}.Y_{(\mathbf{r},W)} \tag{3.42}$$

□

4. Higher dimensional case

In this section we prove the following generalization of Theorem (1.10) by induction on dimension using the ideas developed in our study of the surface case:

Theorem 4.1. *Let $z(m)$ be an* sl_2*-convergent sequence of points in U^r and $F: U^r \to \mathcal{M}$ denote the lifting of the period map of an admissible variation of mixed Hodge structure over Δ^{*r} with unipotent monodromy and weight filtration $W = W^0$. Let $P(\lambda)$ be the corresponding partition. Then, after passage to a subsequence if necessary,*

$$Y^* = \lim_{m\to\infty} e^{-\mu(x(m))}.\hat{Y}_{(F(z(m),W)} = Y(N(\sigma_1), Y(N(\sigma_2), \cdots, Y_{(\hat{F}_\infty, M)})) \tag{4.2}$$

where $\hat{Y}_{(F(z(m),W)}$ is the grading of the sl_2*-splitting of $(F(z(m), W)$ and $(\hat{F}_\infty, M)$ is the* sl_2*-splitting of the limit mixed Hodge structure of $(F(z), W)$.*

For $r = 1$, this is Theorem (3.33). Furthermore, as in the proof of Theorem (1.6) and (3.4), by grouping variables together it is sufficient to consider sl_2-convergent sequences $z(m) = (z_1(m), \dots, z_r(m))$ with limiting ratios $\lambda = (0, \dots, 0)$. Accordingly, as in the surface case, we consider the smallest index ι with respect to which $z(m)$ has non-polynomial growth with respect to $z_\iota(m)$. Then, since by (3.2) the variables y_j for $j \le \iota$ share mutual polynomial bounds, it follows as in (3.30) that

$$||\hat{Y}_{(F(z(m),W)} - \hat{Y}_{(F_\iota(z(m)),W)}|| \to 0$$

for any fixed norm on $W_{-1}\mathfrak{q}$, where $F_\iota(z_1, \dots, z_r)$ is the nilpotent orbit in $z_1, \dots, z_\iota$ obtained by degenerating $z_1, \dots, z_\iota$ in $F(z)$, i.e.

$$F_\iota(z_1, \dots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma_{[\iota]}}.F_\infty \tag{4.3}$$

where $\Gamma_{[\iota]}$ is obtained from the local normal form (1.9) of $F(z)$ by taking $\Gamma(s)$ and setting $s_j = 0$ for $j \le \iota$. Consequently, it is sufficient to prove Theorem (4.1) for period maps of the form (4.3).

Heuristically, the proof of Theorem (4.1) now reduces to the inductive application of Theorem (0.5) of [12] to F_ι, viewed as a nilpotent orbit in $z_1, \dots, z_\iota$ with base point

$$F_\infty(z_{\iota+1}, \dots, z_r) = e^{\sum_{j>\iota} z_j N_j} e^{\Gamma_{[\iota]}}.F_\infty$$

However, since this base point need not be contained in a bounded set we can not directly apply [12]. We can however observe that the pair $(F_\infty(z_{\iota+1}, \dots, z_r), W^\iota)$ is an admissible variation of mixed Hodge structure with associated nilpotent orbit θ_ι defined by $(N_{\iota+1}, \dots, N_r; F_\infty, W^\iota)$. Let $t_\iota(y)$ be the associated semisimple operator appearing the several variable SL_2-orbit theorem of [12]. Then, by Theorem (12.8) of [12],

$$t_\iota^{-1}(y) F_\infty(z_{\iota+1}, \dots, z_r) \to F_\sharp := \exp(\epsilon_0).\hat{F}_\iota$$

Furthermore, specializing to the case where $F_\infty(z_{\iota+1}, \dots, z_r)$ is in fact a nilpotent orbit and comparing Proposition (10.4) of [12] with Lemma (2.10), it follows that

$$\exp(\epsilon_0) = e^{iN_{\iota+1}} e^{-i\hat{N}_{\iota+1}}, \qquad F_\sharp = e^{P_\iota(0)}.\hat{F}_\infty$$

where $\mathrm{Ad}\,(\mathrm{t}^{-1}(\mathrm{y}))\mathrm{e}^{\mathrm{i}\sum_{\mathrm{j}>\iota} \mathrm{y_j N_j}} = \mathrm{e}^{\mathrm{P}_\iota(\mathrm{t})}$ as in Lemma (2.10).

Likewise, via the method of Theorem (12.8) of [12],

$$\mathrm{Ad}\,(\mathrm{t}_\iota^{-1}(\mathrm{y}))\Gamma_{[\iota]} \to 1$$

it follows that as $y_{\iota+1},\dots,y_r\to\infty$ in such a way that $t_{\iota+1},\dots,t_r\to 0$. Similarly, let $(e^{\tilde\epsilon}.F_\infty, W^r)$ denote the sl_2-splitting of (F_∞, W^r). Then, since $\tilde\epsilon$ preserves each W^j and $\tilde\epsilon\in\Lambda^{-1,-1}_{(\hat F_\infty, W^r)}$ it follows that

$$\tilde\epsilon(t) = \mathrm{Ad}\,(\mathrm{t}_\iota^{-1}(\mathrm{y}))\tilde\epsilon$$

is a polynomial in $t_{\iota+1},\dots,t_r$ with constant term 0 where $t_j = y_{j+1}/y_j$.

To continue, note that

$$t_\iota(y) = \Pi_{j>\iota}\, y_j^{\frac12 \hat Y^j} = y_{\iota+1}^{-\frac12 \hat Y^{\iota+1}}\Pi_{j>\iota}\, y_j^{\hat H_j} \tag{4.4}$$

Using the above remarks, we can now rewrite (4.3) as

$$\begin{aligned} F_\iota(z_1,\dots,z_r) &= e^{\sum_j z_jN_j}e^{\Gamma_{[\iota]}}.F_\infty \\ &= e^{\mu(x)}e^{\sum_j iy_jN_j}e^{\Gamma_{[\iota]}}.F_\infty \\ &= e^{\mu(x)}e^{\sum_{j\le\iota} iy_jN_j}e^{\sum_{j>\iota} iy_jN_j}e^{\Gamma_{[\iota]}}.F_\infty \\ &= e^{\mu(x)}e^{\sum_{j\le\iota} iy_jN_j}t_\iota(y)t_\iota^{-1}(y)e^{\sum_{j>\iota} iy_jN_j}e^{\Gamma_{[\iota]}}.F_\infty \end{aligned} \tag{4.5}$$

In particular, since $[N_k,\hat H_j]=0$ for $j>k$ by (2.3) it follows by (4.4) that

$$\begin{aligned} \mathrm{Ad}\,(\mathrm{t}_\iota^{-1}(\mathrm{y}))\mathrm{e}^{\mathrm{i}\sum_{\mathrm{k}\le\iota}\mathrm{y_kN_k}} &= \mathrm{Ad}\,(\mathrm{y}_{\iota+1}^{\frac12\hat{\mathrm{Y}}^{\iota+1}})\Pi_{\mathrm{j}>\iota}\,\mathrm{Ad}\,(\mathrm{y_j^{-\hat H_j}})\mathrm{e}^{\mathrm{i}\sum_{\mathrm{k}\le\iota}\mathrm{y_kN_k}} \\ &= \mathrm{Ad}\,(\mathrm{y}_{\iota+1}^{\frac12\hat{\mathrm{Y}}^{\iota+1}})\mathrm{e}^{\mathrm{i}\sum_{\mathrm{k}\le\iota}\mathrm{y_kN_k}} \qquad (4.6)\\ &= e^{i\sum_{k\le\iota}(y_k/y_{\iota+1})N_k} \qquad (4.7) \end{aligned}$$

Note that properties of non-polynomial growth with respect to $z_\iota(m)$ remain unchanged by replacing y_k by $y_k/y_{\iota+1}$. Accordingly, (4.5) becomes

$$F_\iota(z_1,\dots,z_r) = e^{\mu(x)}t_\iota(y)e^{i\sum_{k\le\iota}(y_k/y_{\iota+1})N_k}t_\iota^{-1}(y)e^{\sum_{j>\iota} iy_jN_j}e^{\Gamma_{[\iota]}}.F_\infty \tag{4.8}$$

Similarly,

$$t_\iota^{-1}(y)e^{\sum_{j>\iota} iy_jN_j}e^{\Gamma_{[\iota]}}.F_\infty = (\mathrm{Ad}\,(\mathrm{t}_\iota^{-1})\mathrm{e}^{\sum_{\mathrm{j}>\iota}\mathrm{iy_jN_j}})(\mathrm{Ad}\,(\mathrm{t}_\iota^{-1})\mathrm{e}^{\Gamma_{[\iota]}})\mathrm{e}^{\tilde\epsilon(\mathrm{t})}.\hat{\mathrm{F}}_\infty \tag{4.9}$$

Lemma 4.10. *If $k\le j$ and $\alpha\in\ker(\mathrm{ad}\,\mathrm{N_k})$ then each eigencomponent of α with respect to $\mathrm{ad}\,\hat{\mathrm{Y}}^{\mathrm{j}}$ belongs to $\ker(\mathrm{ad}\,\mathrm{N_k})$.*

Proof. By the Jacobi identity,

$$[N_k,[\hat Y^j,\alpha]] = [[N_k,\hat Y^j],\alpha] + [\hat Y^j,[N_k,\alpha]] = [2N_k,\alpha] = 0$$

since $[N_k,\hat Y^j]=2N_k$. Consequently, each eigencomponent of α must also belong to $\ker(\mathrm{ad}\,\mathrm{N_k})$ since $\mathrm{ad}\,\mathrm{N_k}$ decreases eigenvalues with respect to $\mathrm{ad}\,\hat{\mathrm{Y}}^{\mathrm{j}}$ by 2. □

Corollary 4.11. *If α commutes with $N_1,\dots,N_\iota$ then so does $\mathrm{Ad}\,(\mathrm{t}_\iota^{-1}(\mathrm{y}))\alpha$.*

Proof. Decompose α with respect to $\hat Y^{\iota+1},\dots,\hat Y^r$ and apply the previous lemma. □

In particular, both $P_\iota(t)$ and $\tilde\epsilon(t)$ commute with $N_1,\dots,N_\iota$. Furthermore, as in Proposition (2.6) of [5], it follows via equation (6.10) of [15] that

$$\Gamma_{[\iota]}\in\ker(\mathrm{ad}\,\mathrm{N}_1)\cap\cdots\cap\ker(\mathrm{ad}\,\mathrm{N}_\iota)$$

and hence $\mathrm{Ad}\,(\mathrm{t}_\iota^{-1}(\mathrm{y}))\Gamma_{[\iota]}$ inherits this property as well. Define

$$e^\beta = e^{-P_\iota(0)}e^{P_\iota(t)}(\mathrm{Ad}\,(\mathrm{t}_\iota^{-1})\mathrm{e}^{\Gamma_{[\iota]}})\mathrm{e}^{\tilde\epsilon(\mathrm{t})}$$

and note that by the above β commutes with $N_1, \dots, N_\iota$ and goes to zero as $y_{\iota+1}, \dots, y_r \to \infty$ in such a way that $t_{\iota+1}, \dots, t_r \to 0$. Furthermore, $\beta \in \mathfrak{q}$ (cf. (1.8)). Accordingly, we can rewrite (4.8) as

$$F_\iota(z_1, \dots, z_r) = e^{\mu(x)} t_\iota(y) e^{i \sum_{k \le \iota} (y_k/y_{\iota+1}) N_k} (\mathrm{Ad}\,(\mathrm{e}^{\mathrm{P}_\iota(0)})\mathrm{e}^{\beta}).\mathrm{F}_\sharp \tag{4.12}$$

To continue, we note that $(N_1, \dots, N_\iota; F_\sharp, W)$ defines an admissible nilpotent orbit. Returning to (1.8), note that both (F_∞, W^r) and $(\hat{F}_\infty, W^r)$ define the same subalgebra $\mathfrak{q}$ since

$$\tilde{\epsilon} \in \Lambda^{-1,-1}_{(F_\infty, W^r)} \subseteq \mathfrak{q}$$

Applying applying $\mathrm{Ad}\,(\mathrm{e}^{\mathrm{P}_\iota(0)})$ to both sides of (1.8) for $(\hat{F}_\infty, W^r)$ it follows that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{F_\sharp} \oplus \mathfrak{q}_\sharp$$

where[1] $\mathfrak{q}_\sharp = \mathrm{Ad}\,(\mathrm{e}^{\mathrm{P}_\iota(0)})\mathfrak{q}$. Let

$$\mathfrak{v} = \mathfrak{q}_\sharp \cap \ker(\mathrm{ad}\,\mathrm{N}_1) \cap \cdots \ker(\mathrm{ad}\,\mathrm{N}_\iota)$$

Then, for any element $v \in \mathfrak{v}$, the map

$$(z_1, \dots, z_\iota) \mapsto e^{\sum_{j \le \iota} z_j N_j} e^v.F_\sharp \tag{4.13}$$

is horizontal. It therefore follows from the theory of polarized mixed Hodge structures (cf. Theorem (2.3), [5] and [4][15]) and the results of Kashiwara [11] that there is a neighborhood $\mathfrak{v}_o$ of $0 \in \mathfrak{v}$ such that $v \in \mathfrak{v}_o$ implies that (4.13) is an admissible nilpotent orbit. In particular,

$$e^v = \mathrm{Ad}\,(\mathrm{e}^{\mathrm{P}_\iota(0)})\mathrm{e}^{\beta} \tag{4.14}$$

will satisfy $v \in \mathfrak{v}_o$ as $y_{\iota+1}, \dots, y_r \to \infty$ in such a way that $t_{\iota+1}, \dots, t_r \to 0$.

Setting aside (4.14) for the moment, given $v \in \mathfrak{v}_o$ define

$$\hat{Y}(v) = Y(N_1, \dots, Y(N_\iota, \hat{Y}_{(e^v.F_\sharp, W^\iota)}))$$

to be the limiting grading of

$$\hat{Y}_{(e^{\sum_{j \le \iota} i y_j N_j} e^v.F_\sharp.W)}$$

as $y_1, \dots, y_\iota \to \infty$ in such a way that $t_1, \dots, t_\iota \to 0$. Then, by Theorem (0.5) of [12],

$$\hat{Y}_{(e^{\sum_{j \le \iota} i y_j N_j} e^v.F_\sharp.W)} = \exp(u(\tau; v)).\hat{Y}(v) \tag{4.15}$$

where $u(\tau; v)$ has a convergent series expansion

$$u(\tau; v) = \sum_m u_m(v) \Pi_{j=1}^r \tau_j^{m(j)}$$

in $\tau_1 = y_2/y_1, \dots, \tau_\iota = 1/y_\iota$ with constant term 0. Furthermore, by Theorem (10.8) of [12], the coefficients $u_m(v)$ are analytic functions of $v \in \mathfrak{v}_o$. For future reference, note that $\tau_j = t_j$ for $j < \iota$.

Combining the above, we obtain

$$\hat{Y}_{(F_\iota(z_1, \dots, z_r), W)} = e^{\mu(x)} t_\iota(y) e^{u(t_1, \dots, t_\iota; v)}.\hat{Y}(v) \tag{4.16}$$

where we have now reimposed condition (4.14). In particular, if $\iota = 1$ it follows from the definition of non-polynomial growth that

$$\mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))\mathrm{e}^{\mathrm{u}(\mathrm{t}_1; \mathrm{v})} \to 0$$

[1] In fact $\mathfrak{q}^\sharp = \mathfrak{q}$ since $N_{\iota+1}, \dots, N_r$ are $(-1,-1)$-morphisms of $\hat{F}_\infty, W^r)$ whereas $\hat{Y}^{\iota+1}, \dots, \hat{Y}^r$ are of type $(0,0)$ with respect to $(\hat{F}_\infty, W^r)$ by Lemma (1.16).

along an appropriate subsequence of $z(m)$ since the action of $\mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))$ on $\mathfrak{g}_\mathbb{C}$ is bounded by a polynomial in y_2 and we can arrange for

$$y_2^{d+1}(m)/y_1(m) = t_1(m)y_2^d(m) \to 0$$

It therefore remains to consider (in this case)

$$t(y).\hat{Y}(v) = t(y).Y(N_1, \dots, Y(N_\iota, \hat{Y}_{(e^v.F_\sharp, W^\iota)})) \tag{4.17}$$

By definition [8], the right hand side of equation (4.17) is invariant under rescaling $N_k \mapsto \alpha N_k$ for $k = 1, \dots, \iota$. It therefore follows from (4.4) and (4.17) that

$$\begin{aligned} t(y).\hat{Y}(v) &= Y(N_1, \dots, Y(N_\iota, \hat{Y}_{(t(y)e^v.F_\sharp, W^\iota)})) \\ &= Y(N_1, \dots, Y(N_\iota, \hat{Y}_{(e^{\sum_{j>\iota} iy_jN_j} e^{\Gamma_{[\iota]}}, W^\iota)})) \end{aligned}$$

Using our induction hypothesis, we now obtain Theorem (4.1).

Suppose now that $\iota > 1$ and let us again temporarily set aside (4.14). Let

$$u_1 = u(t_1, \dots, t_\iota; v) - u(t_1, \dots, t_{\iota-1}, 0; v), \qquad u_2 = u(t_1, \dots, t_{\iota-1}, 0; v)$$

Then, $\exp(u(t_1, \dots, t_\iota; v)) = \exp(u_1 + u_2)$ where u_1 is divisible by t_ι in the ring of real-analytic functions of $t_1, \dots, t_\iota$. Therefore,

$$\mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))\exp(\mathrm{u}(\mathrm{t}_1, \dots, \mathrm{t}_\iota; \mathrm{v})) = \mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))(\mathrm{e}^{\mathrm{u}_1+\mathrm{u}_2}\mathrm{e}^{-\mathrm{u}_2})\mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))\mathrm{e}^{\mathrm{u}_2} \tag{4.18}$$

where $e^{u_1+u_2}e^{-u_2} = e^{u_3}$ with u_3 again divisible by t_ι in the ring of real-analytic functions in $t_1, \dots, t_\iota$. Consequently, as above, it follows that

$$\mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))(\mathrm{e}^{\mathrm{u}_1+\mathrm{u}_2}\mathrm{e}^{-\mathrm{u}_2}) \to 1 \tag{4.19}$$

along some subsequence of $z(m)$ [i.e., any subsequence along which t_ι dominates the action of $\mathrm{Ad}\,(\mathrm{t}_\iota(\mathrm{y}))$ on $W_{-1}\mathfrak{g}_\mathbb{C}$].

Accordingly, by the previous paragraph it follows that

$$t_\iota(y)e^{u(t_1,\dots,t_\iota;v)}.\hat{Y}(v) \to t_\iota(y)e^{u_2}.\hat{Y}(v)$$

along a suitable subsequence of $z(m)$ once we reimpose (4.14) provided that $t_\iota(y)e^{u_2}.\hat{Y}(v)$ is convergent along this sequence. To establish this, we need a formula for $t_\iota(y)e^{u_2}.\hat{Y}(v)$. For this purpose, we once again drop (4.14) and fix $v \in \mathfrak{v}_o$ and $\tau_1, \dots, \tau_{\iota-1} > 0$. Then, by equation (4.15) it follows that

$$\begin{aligned} e^{u_2}.\hat{Y}(v) &= \lim_{y\to\infty} \hat{Y}_{(e^{(\sum_{j\le\iota} \alpha_jN_j)y}e^v.F_\sharp, W)} \\ &= Y(\sum_{j\le\iota} \alpha_jN_j, \hat{Y}_{(e^v.F_\sharp, W^\iota)}) \end{aligned}$$

where $\alpha_j = y_j/y_\iota = \Pi_{j\le\iota-1}\, t_j^{-1}$. Reimposing (4.14), it then follows that

$$t_\iota(y)e^{u_2}.\hat{Y}(v) = Y(\sum_{j\le\iota} \alpha_jN_j, \hat{Y}_{(e^{\sum_{j>\iota} iy_jN_j} e^{\Gamma_{[\iota]}(s)}.F_\infty, W^\iota)}) \tag{4.20}$$

By our induction hypothesis,

$$\hat{Y}_{(e^{\sum_{j>\iota} iy_jN_j} e^{\Gamma_{[\iota]}(s)}.F_\infty, W^\iota)} \to \hat{Y}^\iota = Y_{(\hat{F}_\iota, W^\iota)} \tag{4.21}$$

(after passage to a to a subsequence of $z(m)$), and hence there exists a unique function $W_{-1}^\iota\mathrm{gl}(V)$-valued function γ such that

$$\hat{Y}_{(e^{\sum_{j>\iota} iy_jN_j} e^{\Gamma_{[\iota]}(s)}.F_\infty, W^\iota)} = e^{\gamma(z_{\iota+1},\dots,z_r)}.\hat{Y}^\iota$$

with $e^{\gamma} \to 1$ along any subsequence for which (4.21) holds. Furthermore, because $e^{\gamma}.\hat{Y}^{\iota}$ arises from the sl_2-splitting of the limit mixed Hodge structure of a nilpotent orbit with monodromy logarithms $N_1, \dots, N_\iota$ and weight filtration $W = W^0$ it follows that:

(a) $[e^{\gamma}.\hat{Y}^{\iota}] = -2N_j$ for $j \leq \iota$;
(b) $e^{\gamma}.\hat{Y}^{\iota}$ preserves W^0.

By the same reasoning, conditions (a) and (b) also hold for $\hat{Y}^{\iota}$ in place of $e^{\gamma}.\hat{Y}^{\iota}$. In particular, by virtue of the fact that $W^{\iota}_{-1}\mathrm{gl}(V)$ is a nilpotent graded ideal of $\mathrm{gl}(V)$ which acts simply transitively on the gradings of W^{ι}, it follows from property (a) for $e^{\gamma}.\hat{Y}^{\iota}$ and $\hat{Y}^{\iota}$ that $[\gamma, N_j] = 0$. Likewise, it follows from property (b) for $e^{\gamma}.\hat{Y}^{\iota}$ and $\hat{Y}^{\iota}$ that γ preserves W^0. Invoking the functoriality of Deligne's construction, it then follows from (4.20) that

$$\begin{aligned} t_\iota(y)e^{u_2}.\hat{Y}(v) &= e^{\gamma}.Y(\sum_{j\leq\iota}\alpha_j N_j, \hat{Y}^{\iota}) \\ &= e^{\gamma}.\hat{Y}_{(e^{\sum_{j\leq\iota}\alpha_j N_j}.\hat{F}_\iota, W^0)} \end{aligned} \tag{4.22}$$

In particular, since the limit mixed Hodge structure $(\hat{F}_\iota, W^{\iota})$ of the nilpotent orbit defined by $(\sum_{j\leq\iota}\alpha_j N_j; \hat{F}_\iota, W^0)$ is split over $\mathbb{R}$,

$$\begin{aligned} Y_{(e^{\sum_{j\leq\iota}\alpha_j N_j}.\hat{F}_\iota, W^0)} &= Y_{(e^{\sum_{j\leq\iota} y_\iota \alpha_j N_j}.\hat{F}_\iota, W^0)} \\ &= Y_{(e^{\sum_{j\leq\iota} y_j N_j}.\hat{F}_\iota, W^0)} \end{aligned}$$

By Theorem (1.7) and the results of section 2,

$$Y_{(e^{\sum_{j\leq\iota} y_j N_j}.\hat{F}_\iota, W^0)} \to \hat{Y}^0$$

along a subsequence of $z(m)$. Combining this observation with (4.22) and the fact that $e^{\gamma} \to 1$, it then follows that $t_\iota(y)e^{u_2}.\hat{Y}(v) \to \hat{Y}^0$.

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