THE ZERO LOCUS OF AN ADMISSIBLE NORMAL FUNCTION

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Abstract. We prove that the zero locus of an admissible normal function over an algebraic parameter space S is algebraic in the case where S is a curve.

1. Introduction

Let S be a smooth, complex projective variety. Following Morihiko Saito [8], we define an admissible normal function on S to be an admissible variation of graded-polarized mixed Hodge structure $\mathcal{V}$ on the complement of a normal crossing divisor $D \subseteq S$ which is an extension of the trivial variation $\mathbb{Z}(0)$ by a variation of pure, polarized Hodge structure $\mathcal{H}$ of weight $w < 0$. That is an admissible normal function is an element $\nu \in \mathrm{Ext}^1_{\mathrm{AV}}(\mathbb{Z}(0), \mathcal{H})$ where AV denotes the abelian category of admissible variations of mixed Hodge structure on S which are smooth on $S - D$.

Henceforth, we assume that $w = -1$. In this case, an admissible normal function corresponds to the usual notion of a horizontal normal function on S with moderate growth along D together with existence of a suitable relative weight filtration along each irreducible component of D. In this article (Theorem 4.5), we settle the following conjecture communicated to us by M. Green and P. Griffiths in the case where S is a curve.

Conjecture 1.1. *Let ν be an admissible normal function on S. Then the zero locus $\mathcal{Z}$ of ν is an algebraic subvariety of S.*

In analogy with [1], an unconditional proof of this conjecture provides indirect evidence in support of standard conjectures on higher regulators and filtrations on Chow groups.

A rough outline of our proof is a follows: Let $\mathcal{U}$ be an open subset of S in the analytic topology which does not intersect D. Then the zero locus of ν on $\mathcal{U}$ is complex analytic since the restriction of ν to $\mathcal{U}$ is a holomorphic section of associated bundle of intermediate Jacobians.

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Thus, in order to prove that the zero locus of ν is algebraic, it is sufficient to show that:

(*) For each point $p \in D$ there exists an analytic open neighborhood $\mathcal{U}_p \subset S$ of p on which the zero locus of ν has only finitely many components.

In the case where S is a curve, we verify $(*)$ using the orbit theorems of the second author and results of P. Deligne.

The canonical real grading $Y(s)$ (described below) of the Hodge structure $\mathcal{V}_s$ at a point $s \in S - D$ will play a crucial role in our proof. The central idea is that ν is 0 at s if and only if $Y(s)$ is integral. It is therefore crucial to understand the asymptotics of $Y(s)$ as s tends to a point $s_0 \in D$. In Theorem 3.5, we use the SL_2-orbit theorem of [7] to show that $Y^{\ddagger} := \lim_{s\to s_0} Y(s)$ exists for $s_0 \in D$. Now, it is clear that ν can only vanish in a neighborhood of s_0 if $Y^{\ddagger}$ is non-integral. Knowing that the limit exists allows us to concentrate on the case where $Y^{\ddagger}$ is not integral. This case can then be handle by a rather explicit computation of the zero locus in the neighborhood of s_0.

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2. The Zero locus at a smooth point

As a preliminary step in our analysis of the zero locus of ν at infinity, we derive the local defining equations of $\mathcal{Z}$ at an interior point of S. To this end, we begin with a review of mixed Hodge structures and their gradings.

Gradings. Let V be a finite dimensional vector space over a field k. A grading for V is a direct sum decomposition $V = \oplus V_k$ of V into subspace V_k indexed by integers. An n-grading of V is a direct sum decomposition $V = \oplus V_w$ indexed by n-tuples of integers. It is well-known (and easy to see) that gradings are in one-one correspondence with linear actions of the multiplicative group $\mathbb{G}_m$ on V, and n-gradings are in one-one correspondence with linear actions of the n-torus $\mathbb{G}_m^n$ on V: To an n-grading $V = \oplus V_w$ one associates the action where $(t_1, \ldots, t_n)v = t_1^{w_1} \cdots t_n^{w_n} v$. Conversely, to an action of $\mathbb{G}_m^n$ on V one obtains a grading by writing V as a direct sum of its isotypical subspaces.

If k is a field of characteristic 0, then gradings are in one-one correspondence with semi-simple endomorphisms Y of V with integral eigenvalues. The correspondence is the one that takes a $\mathbb{G}_m$ action on V to its derivative at $1 \in \mathbb{G}_m$ viewed as an endomorphism of V. Conversely, it takes an endomorphism Y to the direct sum decomposition $V = \oplus_{k\in\mathbb{Z}} V_{Y,k}$ where $V_{Y,k} = \{v \in V | Yv = kv\}$. Similarly, n-gradings are in one-one correspondence with n-tuples of commuting semi-simple endomorphisms $Y_1, \dots, Y_n$ with integral eigenvalues.

Throughout this paper, the vector spaces V which will occur will be over fields of characteristic 0. We will work with the first notion and the last notion of a grading interchangeably.

Suppose that V is equipped with a filtration

$$V = W_r V \supset W_{r-1} V \supset \cdots \supset W_{l-1} V = \{0\} \tag{2.1}$$

by subspaces W_i with $i \in \mathbb{Z}$. A *grading* of $W_\bullet$ is then a grading $V = \oplus_{i\in\mathbb{Z}} V_i$ of V such that $W_k = \oplus_{i\le k} V_i$.

Deligne's Grading. We now recall a fundamental result of Deligne concerning mixed Hodge structures. (See [3, Lemme 1.2.8] for the original theorem or [2, Theorem 2.13] where the result appears in the notation used below.)

A mixed Hodge structure (F, W) induces a functorial bigrading

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{2.2}$$

such that

(1) $F^p = \bigoplus_{a\ge p} I^{a,b}$;
(2) $W_k = \bigoplus_{a+b\le k} I^{a,b}$;
(3) $\bar{I}^{p,q} = I^{q,p} \mod \oplus_{r<q,s<p} I^{r,s}$.

In particular, a mixed Hodge structure (F, W) induces a grading of W via the semisimple endomorphism $Y : V_{\mathbb{C}} \to V_{\mathbb{C}}$ which acts as multiplication by $p + q$ on $I^{p,q}$. We will call this grading *Deligne's grading* $Y_{F,W}$.

Normal functions. Returning now to the normal function setting, let S be a complex manifold of dimension n and $\mathcal{V}$ be the variation of mixed Hodge structure associated to ν. Let $p \in \mathbb{Z}$ and $(s_1, \dots, s_n)$ be local holomorphic coordinates on a polydisk $\Delta^n \subseteq S$ which vanish at p. Then, since Δ^n is simply connected, we can parallel translate the data of $\mathcal{V}$ back the reference fiber $V = \mathcal{V}_p$. The Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ then corresponds to a holomorphic, horizontal decreasing filtration

$F(s)$ of $V_{\mathbb{C}}$. The weight filtration $\mathcal{W}$ of $\mathcal{V}$ corresponds to a constant filtration W of $V_{\mathbb{Z}}$ with weight graded-quotients

$$\mathrm{Gr}^W_0(V_{\mathbb{Z}}) = \mathbb{Z}(0), \qquad \mathrm{Gr}^W_{-1}(V_{\mathbb{Z}}) = H_{\mathbb{Z}}$$

and $\mathrm{Gr}^W_k = 0$ for $k \neq 0, -1$. Similarly, the graded-polarizations of $\mathcal{W}$ correspond to constant polarizations of Gr^W.

On account of the short length of W, the grading

$$Y(s) = Y_{(F(s),W)} \tag{2.3}$$

defined by (2.2) can be characterized as the unique real grading of W which preserves $F(s)$. If $Y_{\mathbb{Z}}$ is any integral grading of W then the image $1 \in \mathbb{Z}(0)$ under the map

$$Y(s) - Y_{\mathbb{Z}} : \mathbb{Z}(0) \to H_{\mathbb{R}}/H_{\mathbb{Z}}$$

gives the point in the Griffiths intermediate Jacobian corresponding to the extension (2.2) via the standard isomorphism

$$H_{\mathbb{R}}/H_{\mathbb{Z}} \cong \frac{H_{\mathbb{C}}}{F^0 H_{\mathbb{C}} + H_{\mathbb{Z}}}.$$

Accordingly, p belongs to the zero locus of ν if and only if $Y(p)$ is an integral grading of W. Consequently, since $Y(s)$ is real analytic[1] in s and the set of integral gradings of W is a discrete subset of the affine space of $\mathbb{R}$-gradings of W, there exists a neighborhood of p in which the zero locus of ν is given by the equation

$$Y(s) = Y(p).$$

The filtration $F(s)$ takes its values in a classifying space $\mathcal{M}$ of graded-polarized mixed Hodge structure [5]. Let $G_{\mathbb{C}}$ denote the Lie group consisting of all automorphisms of $V_{\mathbb{C}}$ which preserve W and act by complex isometries on Gr^W. Then, for each point $F \in \mathcal{M}$ there exists a neighborhood $U_{\mathbb{C}}$ of zero in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ such that the map

$$u \mapsto e^u.F \tag{2.4}$$

is a holomorphic submersion from $U_{\mathbb{C}}$ onto a neighborhood of F in $\mathcal{M}$. Accordingly, if

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p+q\leq 0} \mathfrak{g}^{p,q} \tag{2.5}$$

[1]The grading $Y(s)$ is holomorphic when viewed as a section of the bundle of intermediate Jacobians

denotes the Deligne bigrading of the induced mixed Hodge structure $(F\mathfrak{g}_{\mathbb{C}}, W\mathfrak{g}_{\mathbb{C}})$ then map (2.3) restricts to a biholomorphism from a neighborhood of zero in subalgebra

$$q_F = \bigoplus_{p<0,p+q\leq 0} \mathfrak{g}^{p,q}$$

onto a neighborhood of F in $\mathcal{M}$.

Letting $F = F(p)$, the constructions of the previous paragraph show that near p we can write

$$F(s) = e^{\Gamma(s)}.F$$

relative to a unique holomorphic function $\Gamma(s)$ with values in q_F which vanishes at p. Let $Y = Y(p)$ and

$$\Gamma(s) = \Gamma_0(s) + \Gamma_{-1}(s)$$

denote the decomposition of $\Gamma(s)$ according to the eigenvalues of Y. By the Campbell–Baker–Hausdorff formula, there exists a universal power series $\Psi(t)$ such that

$$e^{u+v}e^{-u} = e^{\Psi(\operatorname{ad} u)v}.$$

In particular,

$$\begin{aligned} e^{\Gamma(s)}.Y &= e^{\Gamma_0(s)+\Gamma_{-1}(s)}e^{-\Gamma_0(s)}e^{\Gamma_0(s)}.Y \\ &= e^{\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)}.Y = Y + \Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) \end{aligned}$$

is a holomorphic grading of the weight filtration (over $\mathbb{C}$) which preserves $F(s)$. Therefore, there exists a real analytic section $\zeta(s)$ of $\mathfrak{g}_{\mathbb{C}}^{F(s)} \cap W_{-1}\mathfrak{g}_{\mathbb{C}}$ such that

$$Y(s) = Y + \Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s)$$

and hence the equation $Y(s) = Y(p)$ is equivalent to

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s) = 0. \tag{2.6}$$

Equation (2.6) implies that, near p on the zero locus of ν,

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) \in \mathfrak{g}_{\mathbb{C}}^{F(s)} \cap W_{-1}\mathfrak{g}_{\mathbb{C}} \tag{2.7}$$

on the zero locus of ν. Conversely, whenever equation (2.7) holds, $Y = Y(p)$ is a real grading of W which preserves $F(s)$. Because these two properties specify $Y(s)$ uniquely, it then follows that whenever equation (2.7) holds, $Y(s) = Y(p)$. Thus, on a neighborhood of p, the zero locus of ν is given by equation (2.7).

Applying $e^{-\operatorname{ad}\Gamma(s)}$ to both sides of (2.7), it then follows that the equation for the zero locus near p is

$$e^{-\operatorname{ad}\Gamma(s)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) \in \mathfrak{g}_{\mathbb{C}}^{F} \cap W_{-1}\mathfrak{g}_{\mathbb{C}}. \tag{2.8}$$

However, q_F is a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which is a vector space complement to $\mathfrak{g}_{\mathbb{C}}^F$ in $\mathfrak{g}_{\mathbb{C}}$. Furthermore, q_F is closed under $\operatorname{ad} Y$. Therefore, $\Gamma(s)$, $\Gamma_0(s)$ and $\Gamma_{-1}(s)$ take values in q_F, and $e^{-\operatorname{ad}\Gamma(s)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)$ takes values in q_F. Consequently, equation (2.8) is equivalent to

$$e^{-\operatorname{ad}\Gamma(s)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0. \tag{2.9}$$

Thus, since $e^{-\operatorname{ad}\Gamma(s)}$ is invertible, equation (2.9) implies the following result.

Theorem 2.10. *Near p, the zero locus of ν is given by the equation $\Gamma_{-1}(s) = 0$.*

Proof. Applying $e^{\operatorname{ad}\Gamma(s)}$ to (2.9) implies that the zero locus is given by the equation

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0 \tag{2.11}$$

By the Campbell–Baker–Hausdorff formula,

$$\Psi(u)v = v + \sum_{j>0} c_j(\operatorname{ad} u)^j v$$

and hence

$$\Psi(\operatorname{ad}\Gamma_0)\Gamma_{-1} = \Gamma_{-1} + \sum_{j>0} c_j(\operatorname{ad}\Gamma_0)^j\Gamma_{-1}. \tag{2.12}$$

Consequently, if

$$\Gamma_0 = \sum_{k>0}\Gamma^{-k,k}, \qquad \Gamma_{-1} = \sum_{\ell>0}\Gamma^{-\ell,\ell-1}$$

denote the decomposition of Γ_0 and Γ_{-1} into Hodge components with respect to the bigrading (2.5) then

$$\Psi(\operatorname{ad}\Gamma_0)\Gamma_{-1} = \Gamma^{-1,0} \mod \bigoplus_{r\geq 2}\mathfrak{g}^{-r,r-1}.$$

As such, the equation (2.11) then implies that $\Gamma^{-1,0} = 0$. Proceeding by induction, assume that $\Gamma^{-\ell,1-\ell} = 0$ for $\ell < n$. Then,

$$\Psi(\operatorname{ad}\Gamma_0)\Gamma_{-1} = \Gamma^{-n,n-1} \mod \bigoplus_{r\geq n+1}\mathfrak{g}^{-r,r-1}$$

and hence the equation (2.11), $\Gamma^{-n,n-1} = 0$. Thus, $\Gamma_{-1} = 0$ is the local defining equation for $\mathcal{Z}$. □

3. Limiting Grading

In this section, we prove that when S is a curve, the grading (2.2) has a well defined limit $Y^{\ddagger}$ as s approaches a puncture $p \in S$. Simple examples show that in higher dimensions, the limiting value of (2.2) depends not only on the point in the boundary divisor but also the direction of approach.

Let $\Delta \subset S$ be a disk containing the puncture p. By passing to a finite cover if necessary, we can assume that the local monodromy of the restriction of $\mathcal{V}$ to the punctured disk $\Delta^* = \Delta - \{p\}$ is unipotent. Let s be a local coordinate on Δ which vanishes at p, let A be an angular sector of Δ^* and s_o be a point in A. Then, we can parallel translate the Hodge filtration of $\mathcal{V}$ back to a single valued filtration $F(s)$ on $V = \mathcal{V}_{s_o}$. Analytic continuation of $F(s)$ to all of Δ^* then gives the multivalued period map

$$\varphi : \Delta^* \to \Gamma\backslash\mathcal{M}$$

of $\mathcal{V}$. By local liftablity, there exists a holomorphic, horizontal lifting of φ to a map $\tilde{F}$ from the upper half-plane U into $\mathcal{M}$ which makes the following diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{\tilde{F}} & \mathcal{M} \\ {\scriptstyle e^{2\pi i z}}\downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\varphi} & \Gamma\backslash\mathcal{M} \end{array}$$

Furthermore, upon picking a branch of $\log(s)$ on A and letting

$$z = x + iy = \frac{1}{2\pi i}\log(s)$$

there is unique lifting $\tilde{F}(z)$ such that, for $s \in A$,

$$\tilde{F}(z) = F(s).$$

By unipotent monodromy, we $\tilde{F}(z+1) = e^N.\tilde{F}(z)$ and hence

$$\tilde{\varphi}(z) = e^{-zN}.\tilde{F}(z)$$

drops to a map $\tilde{\varphi}$ from Δ^* into the "compact dual" of $\mathcal{M}$. The admissibility of $\mathcal{V}$ then asserts that

(a) $F_\infty = \lim_{s\to 0}\tilde{\varphi}(s)$ exists;
(b) The relative weight filtration M of W and N exists.

From these properties, together with Schmid's nilpotent orbit theorem, Deligne then deduces [9] that the pair (F_∞, M) is a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism.

The mixed Hodge structure (F_∞, M) induces a mixed Hodge structure on $\mathfrak{g}_{\mathbb{C}}$ with Deligne bigrading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r,s} \mathfrak{g}^{r,s}. \tag{3.1}$$

Note that in equation (3.1), it is possible to have $r + s > 0$ since elements of $\mathfrak{g}_{\mathbb{C}}$ only preserve W and not M. Nonetheless, the nilpotent subalgebra

$$q_\infty = \bigoplus_{r<0} \mathfrak{g}^{r,s}$$

is a vector space complement to $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$. Reasoning as in §2 (cf. [P1]), it then follows that near the puncture $s = 0$ we can write $\tilde{\varphi}(s) = e^{\Gamma(s)}.F_\infty$ relative to a unique holomorphic function $\Gamma(s)$ which takes values in q_∞ and vanishes at $s = 0$. Untwisting the definition of $\tilde{\varphi}$, it then follows that

$$F(s) = e^{\frac{1}{2\pi i}\log(s)N} e^{\Gamma(s)}.F_\infty \tag{3.2}$$

over the angular sector A.

To determine the asymptotic behavior of the grading

$$Y(s) = Y_{(F(s),W)}$$

on A we shall use equation (3.2) together with the SL_2-orbit theorem of [7] and a result of Deligne which constructs a grading Y of the weight filtration W which is well adapted to both N and the limiting mixed Hodge structure (F_∞, M).

More precisely, suppose that Y_M is a grading of M which preserves W and satisfies

$$[Y_M, N] = -2N.$$

Then, Deligne (see [4, Appendix, Theorem 1]) shows that there exists a unique, functorial grading

$$Y' = Y'(N, Y_M) \tag{3.3}$$

such that Y' commutes with both N and Y_M[2]. Furthermore,

(a) If Y_M is defined over $\mathbb{R}$ the so is Y';
(b) If (F, M) is a mixed Hodge for which N is a $(-1,-1)$-morphism and induces sub mixed Hodge structures on W then the grading Y' produced from N and the grading of M by the $I^{p,q}$'s of (F, M) preserves F.

[2]The general statement [6] of Deligne's result for longer weight filtrations involves the interplay between the decomposition of N according to $\operatorname{ad} Y'$ and the graded representations of sl_2.

To show the existence of the limiting grading

$$Y^{\ddagger} = \lim_{s\to 0} Y(s)$$

we now invoke the SL_2-orbit theorem of [2, Proposition 2.20]: Let

$$(\hat{F}, M) = (e^{i\delta}.F_\infty, M)$$

denote Deligne's splitting of the limiting mixed Hodge structure of $\mathcal{V}$ and

$$\Lambda^{-1,-1} = \bigoplus_{r,s<0} \mathfrak{g}^{r,s}_{(\hat{F},M)}.$$

Define $G_{\mathbb{R}} = G_{\mathbb{C}} \cap \mathrm{GL}(V_{\mathbb{R}})$ and let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$. Then, there exists a distinguished, real analytic function

$$g : (a,\infty) : (a,\infty) \to G_{\mathbb{R}}$$

and element

$$\zeta \in \mathfrak{g}_{\mathbb{R}} \cap \ker(N) \cap \Lambda^{-1,-1}$$

such that

(a) $e^{iyN}.F_\infty = g(y)e^{iyN}.\hat{F}$;

(b) $g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form

$$g(y) = e^{\zeta}(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots)$$
$$g^{-1}(y) = (1 + f_1 y^{-1} + f_2 y^{-2} + \cdots)e^{-\zeta}$$

with $g_k,\ f_k \in \ker(\mathrm{ad}\, N)^{k+1}$;

(c) δ, ζ and the coefficients g_k are related by the formula

$$e^{i\delta} = e^{\zeta}\left(1 + \sum_{k>0} \frac{1}{k!}(-i)^k(\mathrm{ad}\ N_0)^k\, g_k\right).$$

Combining this result with equation (3.2), we obtain the following asymptotic formula for $Y(s)$ over the angular sector A:

$$\begin{aligned} F(s) = e^{zN}e^{\Gamma(s)}.F_\infty &= e^{xN}e^{\Gamma_1(s)}e^{iyN}.F_\infty \\ &= e^{xN}e^{\Gamma_1(s)}g(y)e^{iyN}.\hat{F} = e^{xN}g(y)e^{\Gamma_2(s)}e^{iyN}.\hat{F} \end{aligned}$$

where $\Gamma_1(s) = \mathrm{Ad}(e^{iyN})\Gamma(s)$ and $\Gamma_2(s) = \mathrm{Ad}(g^{-1}(y))\Gamma_1(s)$.

Let $\hat{Y}_M$ denote the grading of M defined by the $I^{p,q}$'s of $(\hat{F}, M)$ and $\hat{Y}$ be the grading of W defined by application of Deligne's construction to the pair $(N, \hat{Y}_M)$. Then [7],

$$H = \hat{Y}_M - \hat{Y}$$

belongs to $\mathfrak{g}_{\mathbb{R}}$ and satisfies $[H,N] = -2N$. Furthermore, since $\hat{Y}_M$ and $\hat{Y}$ preserve $\hat{F}$, so does H. Therefore,

$$e^{iyN}.\hat{F} = y^{-\frac{1}{2}H}.F_o$$

where $F_o = e^{iN}.\hat{F}$. By the SL_2-orbit theorem, F_o belongs to $\mathcal{M}$. Consequently,

$$F(s) = e^{xN}g(y)e^{\Gamma_2(s)}y^{-\frac{1}{2}H}.F_o = e^{xN}g(y)y^{-\frac{1}{2}H}e^{\Gamma_3(s)}.F_o$$

where

$$\Gamma_3(s) = \mathrm{Ad}(y^{\frac{1}{2}H})\Gamma_2(s) = \mathrm{Ad}(y^{\frac{1}{2}H}g(y)e^{iyN})\Gamma(s).$$

To continue, observe that since, $y = -\frac{1}{2\pi}\log|s|$ and H has only finitely many eigenvalues (all of which are integral), the action of $\mathrm{Ad}(y^{\frac{1}{2}H})$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by an integral power of $y^{\frac{1}{2}}$. Similarly, since $g(y)$ is bounded as $s \to 0$, so is the action of $\mathrm{Ad}(g(y))$. Likewise, since N is nilpotent, the action of $\mathrm{Ad}(e^{iyN})$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by a power of y. Therefore, since $\Gamma(s)$ is a holomorphic function of s which vanishes at $s = 0$, $\Gamma_3(s)$ is a real analytic function on A which satisfies the growth condition

$$\Gamma_3(s) = O((\log|s|)^b s)$$

for some half integral power b. In particular, near $s = 0$,

$$Y_{(e^{\Gamma_3(s)}.F_o,W)} = Y_{(F_o,W)} + \gamma_4(s)$$

for some real analytic function $\gamma_4(s)$ which is again of order $(\log|s|)^b s$. By Deligne [2],

$$Y_{(F_o,W)} = \hat{Y}.$$

Therefore,

$$\begin{aligned} Y(s) &= e^{xN}g(y)y^{-\frac{1}{2}H}.Y_{(e^{\Gamma_3(s)}.F_o,W)} \\ &= e^{xN}g(y)y^{-\frac{1}{2}H}.(Y + \gamma_4(s)) \\ &= e^{xN}g(y).(\hat{Y} + \gamma_5(s)) \end{aligned}$$

where $\gamma_5(s) = \mathrm{Ad}(y^{-\frac{1}{2}H})\gamma_4(s)$ is again of order $\log|s|^{b'}s$ for some half-integral power b'.

Define

$$\begin{aligned} \tilde{g}(s) &= e^{xN}g(y)e^{-xN} \\ &= e^{\zeta}\left(1 + \sum_{k>0}(\mathrm{Ad}(e^{xN}g_k))y^{-k}\right). \end{aligned}$$

Then, since $x = \frac{1}{2\pi}\mathrm{Arg}(s)$ is bounded on the angular sector A,

$$\lim_{s\to 0} g(s) = e^{\zeta}.$$

Consequently, because N commutes with $\hat{Y}$,

$$Y(s) = \tilde{g}(s).(\hat{Y} + Ad(e^{xN})\gamma_5(s)). \tag{3.4}$$

Therefore, since $\gamma_5(s)$ is order $(\log(s))^{b'}s$, we can take the limit of equation (3.4) to obtain:

Theorem 3.5.

$$Y^{\ddagger} = \lim_{s\to 0} Y(s) = e^{\zeta}.\hat{Y}. \tag{3.6}$$

Remark 3.7. Since the right hand side of (3.6) depends only on the triple (F_∞, W, N), $Y^{\ddagger}$ is independent of choice of angular sector A. Likewise, a change of local coordinate s changes F_∞ to $e^{\lambda N}.F_\infty$. Therefore, due to the functorial nature of Deligne's construction of the grading Y' and the fact that $[Y', N] = 0$, the right hand side of (3.6) is independent of the choice of coordinate s. Likewise, since the right hand side of (3.6) commutes with N, it is well defined independent of the choice of reference fiber. Consequently, in the geometric setting, $Y^{\ddagger}$ should have a direct geometric meaning.

4. Zero Locus at Infinity

To verify the conjecture in the case where S is a curve, we now note that the finiteness condition $(*)$ is preserved under passage to finite covers. Therefore, we may assume as in §3 that the associated variation of mixed Hodge structure $\mathcal{V}$ has unipotent monodromy about each point $p \in D$. The requirement that the zero locus of ν has only finitely many irreducible components on a neighborhood of $p \in D$ is then equivalent to the existence of a disk $\Delta \subset S$ such that $\Delta \cap D = \{p\}$ on which the zero locus of ν is either

(a) The empty set;
(b) All of Δ, in which case $\mathcal{V}$ is the trivial extension of $\mathbb{Z}(0)$ by $\mathcal{H}$.

Applying Deligne's construction (3.3) to the limiting mixed Hodge structure (F_∞, M), we get a grading Y_∞ of W which preserves F_∞. Therefore,

$$Y_\infty(s) = e^{\frac{1}{2\pi i}\log(s)N} e^{\Gamma(s)}.Y_\infty$$

is a (complex) grading of W which preserves the Hodge filtration of $F(s)$ near $s = 0$ over the angular sector A. Decomposing $\Gamma(s)$ as

$$\Gamma(s) = \Gamma_0(s) + \Gamma_{-1}(s)$$

according to the eigenvalues of $\operatorname{ad} Y_\infty$, it then follows that

$$\begin{aligned} Y_\infty(s) &= e^{\frac{1}{2\pi i}\log(s)N} e^{\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)}.Y_\infty \\ &= e^{\frac{1}{2\pi i}\log(s)N}.(Y_\infty + \Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)) \\ &= Y_\infty + e^{\frac{1}{2\pi i}\log(s)\operatorname{ad}N}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s). \end{aligned} \tag{4.1}$$

As in the derivation of equation (2.11), we then have

$$Y(s) = Y_\infty(s) + \zeta(s) \tag{4.2}$$

for some section $\zeta(s)$ of $W_{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^{F(s)}$.

Now, unlike a normal function over Δ^n considered earlier, the the function $\zeta(s)$ defined by equation (4.2) may in principle have singularities at $s = 0$. To show that this is not the case, observe that since $\Gamma(s)$ is holomorphic and vanishes at $s = 0$ and N is nilpotent,

$$\lim_{s\to 0} e^{\frac{1}{2\pi i}\log(s)\operatorname{ad}N}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0. \tag{4.3}$$

Therefore, since the limit

$$Y^{\ddagger} = \lim_{s\to 0} Y(s)$$

exists by Theorem (3.5), equations (4.2) and (4.3) imply that $\zeta(s)$ also has a continuous extension to 0 in the angular sector A.

In particular, if $Y^{\ddagger}$ is not an integral grading of W then there is a neighborhood of zero in angular sector A on which $Y(s)$ is not integral, and hence ν has no zeros on this neighborhood. Thus, it remains to consider the case where $Y^{\ddagger}$ is integral. By [7],

$$\hat{Y} = e^{-i\delta}.Y_\infty$$

and hence

$$Y^{\ddagger} = e^{\zeta}.\hat{Y} = e^{\zeta}e^{-i\delta}.Y_\infty.$$

We can write

$$e^{\zeta}e^{-i\delta} = e^{\xi}$$

for some (unique)

$$\xi \in \ker(\operatorname{ad}N) \cap \Lambda^{-1,-1}_{(\hat{F},M)}$$

since both ζ and δ belong to the subalgebra $\ker(N) \cap \Lambda^{-1,-1}_{(\hat{F},M)}$.

To continue, note that

$$\mathfrak{g}^{r,s}_{(F_\infty,W)} = e^{i\operatorname{ad}\delta}(\mathfrak{g}^{r,s}_{(\hat{F},M)})$$

and hence

$$\Lambda^{-1,-1}_{(F_\infty,W)} = e^{i\operatorname{ad}\delta}\Lambda^{-1,-1}_{(\hat{F},M)} = \Lambda^{-1,-1}_{(\hat{F},M)}$$

since $\Lambda^{-1,-1}_{(\hat{F},M)}$ is closed under $\operatorname{ad}\delta$. As such

$$\xi \in \ker(\operatorname{ad}N)\Lambda^{-1,-1}_{(\hat{F},M)} = \ker(\operatorname{ad}N) \cap \Lambda^{-1,-1}_{(F_\infty,M)}.$$

Consequently,

$$Y^{\ddagger} = e^{\xi}.Y_\infty = Y_\infty + \Psi(\operatorname{ad}\xi_0)\xi_{-1}$$

where ξ_0 and ξ_{-1} both belong to $\ker(\operatorname{ad}N)\cap\Lambda^{-1,-1}_{(F_\infty,M)}$ since Y_∞ commutes with N and preserves each summand $\mathfrak{g}^{r,s}_{(F_\infty,M)}$ of $\Lambda^{-1,-1}_{(F_\infty,M)}$. As such,

$$\Psi(\operatorname{ad}\xi_0)\xi_{-1} \in \ker(\operatorname{ad}N) \cap \Lambda^{-1,-1}_{(F_\infty,M)}.$$

Returning now to equation (4.2), it then follows that

$$Y(s) = Y^{\ddagger} - \Psi(\operatorname{ad}\xi_0)\xi_{-1} + e^{\frac{1}{2\pi i}\log(s)\operatorname{ad}N}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s)$$

where $\zeta(s)$ is a real analytic section of $\mathfrak{g}_{\mathbb{C}}^{F(s)} \cap W_{-1}\mathfrak{g}_{\mathbb{C}}$. In particular, since

$$\lim_{s\to 0} Y(s) = Y^{\ddagger}$$

is integral, it then follows from the continuity of $Y(s)$ that near $s=0$ the zeros of ν occur where

$$-\Psi(\operatorname{ad}\xi_0)\xi_{-1} + e^{\frac{1}{2\pi i}\log(s)\operatorname{ad}N}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s) = 0.$$

Equivalently,

$$\begin{aligned} &\operatorname{Ad}(e^{\frac{1}{2\pi i}\log(s)N}e^{\Gamma(s)})^{-1}\Big(\Psi(\operatorname{ad}\xi_0)\xi_{-1} - e^{\frac{1}{2\pi i}\log(s)\operatorname{ad}N}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)\Big) \\ &\qquad = \operatorname{Ad}(e^{\frac{1}{2\pi i}\log(s)N}e^{\Gamma(s)})^{-1}\zeta(s). \end{aligned} \tag{4.4}$$

Thus, since the right hand side of equation (4.4) takes values in $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ and the left hand side of (4.4) takes values in q_∞, it then follows that the zeros of ν occur exactly where

$$e^{\frac{1}{2\pi i}\log(s)\operatorname{ad}N}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = \Psi(\operatorname{ad}\xi_0)\xi_{-1}.$$

Since $\Psi(\operatorname{ad}\xi_0)\xi_{-1} \in \ker(\operatorname{ad}N)$, the equation can be further reduced to just

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = -\Psi(\operatorname{ad}\xi_0)\xi_{-1}.$$

Because $\Gamma(s)$ is a holomorphic function which vanishes at zero, the above equation only has solutions near $s=0$ only if

$$\Psi(\operatorname{ad}\xi_0)\xi_{-1} = 0$$

(i.e. $Y^{\ddagger} = Y_\infty$). In this case, the equation is just

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0.$$

Again, because $\Gamma(s)$ is holomorphic at $s=0$, the solutions to the above equation are either isolated or all of A.

Thus, we have obtained the following.

Theorem 4.5. *Let ν be an admissible normal function on a complex, projective curve S smooth outside of a finite set $D \subset S$. Then the zero locus $\mathcal{Z}$ of ν is an algebraic subset of $S - D$.*

References

[1] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan, *On the locus of Hodge classes*, J. Amer. Math. Soc. **8** (1995), 483–506. MR **1273413 (95e:**14003)

[2] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), 457–535. MR **840721 (88a:**32029)

[3] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), 5–57. MR 0498551 (58 #16653a) (French)

[4] Aroldo Kaplan and Gregory Pearlstein, *Singularities of variations of mixed Hodge structure*, Asian J. Math. **7** (2003), 307–336. MR 2129326

[5] Gregory J. Pearlstein, *Variations of mixed Hodge structure, Higgs fields, and quantum cohomology*, Manuscripta Math. **102** (2000), 269–310. MR **1777521 (2001m:**32033)

[6] ———, *Degenerations of mixed Hodge structure*, Duke Math. J. **110** (2001), 217–251. MR **1865240 (2002h:**14014)

[7] ———, SL_2*-orbits and degenerations of mixed Hodge structure* (2003), arXiv:math.AG/0309439, to appear in J. Differential Geom.

[8] Morihiko Saito, *Admissible normal functions*, J. Algebraic Geom. **5** (1996), 235–276. MR **1374710 (97c:**14044)

[9] Joseph Steenbrink and Steven Zucker, *Variation of mixed Hodge structure. I*, Invent. Math. **80** (1985), 489–542. MR **791673 (87h:**32050a)

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