Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties

Patrick Brosnan*[1] and Timothy Y. Chow†[2]

[1]University of Maryland, College Park
[2]Center for Communications Research, Princeton

February 5, 2018

Abstract

Motivated by a 1993 conjecture of Stanley and Stembridge, Shareshian and Wachs conjectured that the characteristic map takes the character of the dot action of the symmetric group on the cohomology of a regular semisimple Hessenberg variety to $\omega X_G(t)$, where $X_G(t)$ is the chromatic quasisymmetric function of the incomparability graph G of the corresponding natural unit interval order, and ω is the usual involution on symmetric functions. We prove the Shareshian–Wachs conjecture.

Our proof uses the local invariant cycle theorem of Beilinson–Bernstein–Deligne to obtain a surjection, which we call the local invariant cycle map, from the cohomology of a regular Hessenberg variety of Jordan type λ to a space of local invariant cycles. As λ ranges over all partitions, the local invariant cycles collectively contain all the information about the dot action on a regular semisimple Hessenberg variety. We then prove a result showing that, under suitable hypotheses, the local invariant cycle map is an isomorphism if and only if the special fiber has palindromic cohomology. (This is a general theorem, which independent of the Hessenberg variety context.) Applying this result to the universal family of Hessenberg varieties, we show that, in our case, the surjections are actually isomorphisms, thus reducing the Shareshian–Wachs conjecture to computing the cohomology of a regular Hessenberg variety. But this cohomology has already been described combinatorially by Tymoczko, and, using a new reciprocity theorem for certain quasisymmetric functions, we show that Tymoczko's description coincides with the combinatorics of the chromatic quasisymmetric function.

*Department of Mathematics, 1301 Mathematics Building, University of Maryland, College Park, MD 20742, USA; pbrosnan@umd.edu. Partially supported by NSF grant DMS-1361159.

†Center for Communications Research, 805 Bunn Drive, Princeton, NJ 08540, USA; tchow@alum.mit.edu.

1 Introduction

Let G be the incomparability graph of a unit interval order (also known as an *indifference graph*), i.e., a finite graph whose vertices are closed unit intervals on the real line, and whose edges join overlapping unit intervals. It is a longstanding conjecture [49] related to various deep conjectures about immanants that if G is such a graph, then the so-called *chromatic symmetric function* X_G studied by Stanley [47] is e-positive, i.e., a nonnegative combination of elementary symmetric functions. (In fact, Stanley and Stembridge conjectured something seemingly more general, but Guay-Paquet [19] has reduced their conjecture to the one stated here.) Early on, Haiman [21] proved that the expansion of X_G in terms of Schur functions has nonnegative coefficients, and Gasharov [16] showed that these coefficients enumerate certain combinatorial objects known as *P-tableaux*. It is well known that if χ is a character of the symmetric group S_n, then the image of χ under the so-called characteristic map ch

$$\operatorname{ch}\chi := \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)\,p_{\mathrm{cycletype}(\sigma)} \tag{1}$$

(where p here denotes the power-sum symmetric function) is a nonnegative linear combination of Schur functions, with the coefficients giving the multiplicities of the corresponding irreducible characters of S_n. One may therefore suspect that X_G is the image under ch of the character of some naturally occurring representation of S_n, but until recently, there was no candidate, even conjecturally, for such a representation.

Meanwhile, independently and seemingly unrelatedly, De Mari, Procesi, and Shayman [10] inaugurated the study of *Hessenberg varieties*. Let $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$ be a weakly increasing sequence of positive integers satisfying $i \le m_i \le n$ for all i, and let $s : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. The (type A) Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$ is defined by

$$\mathscr{H}(\mathbf{m}, s) := \{\text{complete flags } F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for } 1 \le i < n\}. \tag{2}$$

The geometry of a Hessenberg variety depends on the Jordan form of s. If the Jordan blocks have distinct eigenvalues then we say that s is *regular*, and, by extension, we also say that $\mathscr{H}(\mathbf{m}, s)$ is *regular*. Similarly, if s is diagonalizable then we say that $\mathscr{H}(\mathbf{m}, s)$ is *semisimple*. We say that s has *Jordan type* λ if λ is the partition of n given by the sizes of the Jordan blocks on s. Hessenberg varieties have many interesting properties, but of particular interest to us is the fact that there is a representation, called the *dot action*, of S_n on the cohomology of regular semisimple Hessenberg varieties. This dot action was first defined by Tymoczko, who asked for a complete description of it [52]; e.g., a combinatorial formula for the multiplicities of the irreducible representations and/or for the character values.

A connection between these two apparently unrelated topics has been conjectured by Shareshian and Wachs [42, 43]. Motivated by the e-positivity conjecture, they have generalized X_G to something they call the *chromatic qua-*

sisymmetric function $X_G(t)$ of a graph, which is a polynomial in t with power series coefficients that reduces to X_G when $t = 1$. They also noted that, if we are given a sequence $\mathbf{m}$ as above, and we let $G(\mathbf{m})$ be the undirected graph on the vertex set $\{1, 2, \ldots, n\}$ such that i and j are adjacent if $i < j \leq m_i$, then $G(\mathbf{m})$ is an indifference graph, and moreover that every indifference graph is isomorphic to some $G(\mathbf{m})$. They then made the following conjecture. Let ω denote the usual involution on symmetric functions [48, Section 7.6].

Conjecture 3. *Let y be a regular semi-simple $n\times n$-matrix and let $\chi_{\mathbf{m},d}$ denote the character of the dot action on $\mathrm{H}^{2d}(\mathscr{H}(\mathbf{m}, y))$. Then $\operatorname{ch}\chi_{\mathbf{m},d}$ equals the coefficient of t^d in $\omega X_{G(\mathbf{m})}(t)$.*

This conjecture is intriguing because not only would it answer Tymoczko's question, but it would also open up the possibility of proving the e-positivity conjecture by geometric techniques.

The main result of the present paper is a proof of Conjecture 3 (Theorem 129). The linchpin of our proof is the following result (which is stated more formally later as Theorem 127).

Theorem 4. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of n. Let s be a regular element with Jordan type λ, and let $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ be the corresponding Young subgroup of the symmetric group S_n. Consider the restriction of $\chi_{\mathbf{m},d}$ to S_λ. Then the dimension of the subspace fixed by S_λ equals the Betti number β_{2d} of $\mathscr{H}(\mathbf{m}, s)$.*

What Theorem 4 does is to reduce the problem of computing the dot action on a regular semisimple Hessenberg variety to computing the cohomology of regular (but not necessarily semisimple) Hessenberg varieties. Fortunately, this latter task has already been largely carried out by Tymoczko [51], who has given a combinatorial description of the Betti numbers β_{2d} for all Hessenberg varieties in type A. So with Theorem 4 in hand, all that remains to prove Conjecture 3 is to give a bijection between Tymoczko's combinatorial description and the combinatorics of $\omega X_{G(\mathbf{m})}(t)$. More precisely, let m_λ denote the monomial symmetric function associated to the partition λ. (See [48, Section 7.3] or §2.2 below for monomial symmetric functions.) It is then a standard fact, proved explicitly in Proposition 10 below, that the dimension of the subspace fixed by S_λ in a representation χ is the coefficient of m_λ in the monomial symmetric function expansion of $\operatorname{ch}\chi$. So the first step of our proof is to compute the coefficients $c_{d,\lambda}(\mathbf{m})$ of $t^d m_\lambda$ in the monomial symmetric function expansion of $\omega X_{G(\mathbf{m})}(t)$. We do this with a generalization of a combinatorial reciprocity theorem of Chow (Theorem 29). This yields a description of $c_{d,\lambda}(\mathbf{m})$ that is almost, but not quite, identical to Tymoczko's description of β_{2d}; we show that the descriptions are equivalent by describing an explicit bijection between the two (Theorem 35). As a corollary (Corollary 36), we derive the fact that the Betti numbers of regular Hessenberg varieties form a palindromic sequence (even though the varieties are not smooth), because Shareshian and Wachs have proved that $\omega X_{G(\mathbf{m})}(t)$ is palindromic.

The idea behind the proof of Theorem 4 is to show that Tymoczko's dot action coincides with the monodromy action for the family $\mathscr{H}^{\mathrm{rs}}(\mathbf{m}) \to \mathfrak{g}^{\mathrm{rs}}$ of Hessenberg varieties over the space of regular semisimple $n \times n$ matrices (Theorem 125). This allows us to apply results from the theory of local systems and perverse sheaves to questions involving the dot action. In particular, the local invariant cycle theorem of Beilinson–Bernstein–Deligne, which is stated in our context as Theorem 54, implies that there is a surjective map from the cohomology of a regular Hessenberg variety to the space of local invariants of the monodromy action near a regular element s in the space $\mathfrak{g}$ of all $n \times n$-matrices.

In Theorem 57, we show that the local invariant cycle map is an isomorphism if and only if the Betti numbers of the special fiber are palindromic in a suitable sense. This is a general result in that it holds for any projective morphism of smooth, complex, quasi-projective schemes. Then, in Theorem 126, we show that the local invariant cycles near a regular element s with Jordan type λ coincide with the S_λ invariants of the dot action on the regular semisimple Hessenberg variety. The latter fact is proved by a monodromy argument that uses the Kostant section.

Here is a brief description of the contents by section. Section 2 mainly fixes notation and gives preliminary results. Section 3 proves the combinatorial reciprocity theorem, Theorem 29, mentioned above. Section 4 proves Theorem 35 on the Betti numbers of regular Hessenberg varieties, and derives palindromicity as a corollary of a theorem of Shareshian and Wachs. Section 5 reviews the concept of local monodromy and the related notion of a good fundamental system of neighborhoods to a point in topological space. Section 6 proves Theorem 57 on palindromicity and the local invariant cycle map. Along the way we review the proof of the local invariant cycle map from [4] and prove a slightly stronger version of it (Theorem 84) using the Kashiwara conjecture [25] (which, by the work of several authors, is now a theorem). We also prove Theorem 102, a more general version of our theorem on palindromicity and the local invariant cycle map. Section 7 proves Proposition 106 on the local monodromy of a Galois cover, which is applied later (in Lemma 112) to compute the local monodromy near a matrix s of type λ. Section 8 introduces the family $\mathscr{H}(\mathbf{m}) \to \mathfrak{g}$ of Hessenberg varieties. Finally, Section 9 shows that the monodromy action coincides with Tymoczko's dot action, and uses this fact to prove Theorem 129, which is a restatement of Conjecture 3.

1.1 Previous work

Prior to our work, Conjecture 3 was already known for some graphs G: a complete graph (trivial), a complete graph minus an edge [50], a complete graph minus a path of length three (Tymoczko, unpublished), and a path (by piecing together known results as explained in [43]). In a different direction, Abe, Harada, Horiguchi and Masuda (AHHM) proved that the multiplicity of the trivial representation is indeed as predicted by Conjecture 3. Hearing about this development and reading the last paragraph of the research announcement [2], which explains how to compute the multiplicity of the trivial representation in terms of the regular nilpotent Hessenberg variety, partially inspired our own proof. (Full details of the work of AHHM appeared on the arXiv in [1] shortly after the first draft [6] of this paper.) AHHM also computed the ring structure on regular semisimple Hessenberg varieties of type $(m_1, n, \dots, n)$, and deduced Conjecture 3 in that case from the computation.

In addition to the above work, very shortly after posting the first version of this paper on the arXiv, we learned of the series of papers by Chen, Vilonen and Xue (CVX) studying the the motives of certain generalized Hessenberg varieties as well as the action of monodromy as they vary in families. (See, for example, [7].) The context of this work is different from ours because, roughly speaking, generalized Hessenberg varieties are much further from combinatorics than the Hessenberg varieties which appear in our work. However, to the best of our knowledge, CVX were the first to exploit the idea of studying a universal family of Hessenberg varieties using its monodromy.

1.2 Later work

Since the first version of this paper appeared on the arXiv there have been a few related developments that may be helpful for understanding this work. Firstly, Guay-Paquet posted a proof of Conjecture 3 which is completely independent of and, in many ways, complementary to our proof [20].

On the other hand, using her generalization [36] of Tymoczko's computation of the Betti numbers of Hessenberg varieties in type A, Precup generalized our palindromicity results (Corollaries 36 and 37) to regular Hessenberg varieties for arbitrary complex semi-simple Lie algebras [35]. As it happens, our proof of palindromicity is rather indirect, relying in an essential way on the chromatic quasisymmetric function and a palindromicity theorem for that function proved by Shareshian and Wachs ([43, Corollary 4.6]). So, even in the type A case dealt with in this paper, Precup's direct proof is an important contribution.

Finally, using some of the ideas in this paper, Harada and Precup have proved the e-positivity of the coefficients of $X_{G(\mathbf{m})}(t)$ for certain sequences $\mathbf{m}$ corresponding to abelian ideals in the Lie algebra of strictly upper-triangular matrices [22]. This generalizes Remark 4.4 to Theorem 4.3 of [49].

1.3 Acknowledgments

Brosnan would like to thank Jim Carrell for discussions about Hessenberg varieties and equivariant cohomology, Mark Goresky for mentioning Springer's Bourbaki article [45], which eventually convinced us to look for a description of Tymoczko's dot action in terms of monodromy, and Joseph Bernstein for a conversation in MPI in Bonn that helped convince us to reformulate the palindromicity theorem, Theorem 57, in terms of perverse sheaves on the base as we do in Theorem 92. He would also like to thank Najmuddin Fakhruddin and Nero Budur for advice on §5-6, Hiraku Abe for discussions over email about flatness which led to the formulation of Theorem 109 below, and Tom Haines for discussions about Grothendieck's simultaneous resolution. He thanks the

Institute for Advanced Study for hospitality in Princeton, NJ during the Fall of 2014 when this work started.

We both thank the referees for many thoughtful suggestions and corrections.

2 Preliminaries

We fix some notation that will be used throughout the paper.

2.1 General notation

We let $\mathbb{P}$ denote the positive integers. If $n \in \mathbb{P}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

The vector $\mathbf{m} = (m_1, \ldots, m_{n-1})$ will always denote a *Hessenberg function*, by which we mean a sequence of positive integers satisfying

1. $m_1 \le m_2 \le \cdots \le m_{n-1} \le n$, and
2. $m_i \ge i$ for all i.

We also define

$$|\mathbf{m}| := \sum_{i=1}^{n-1} (m_i - i). \tag{5}$$

Given $\mathbf{m}$, let $P(\mathbf{m})$ denote the poset on the vertex set $[n]$ whose order relation $\prec$ is given by

$$i \prec j \iff j \in \{m_i + 1, m_i + 2, \ldots, n\}.$$

Such a poset is called a *natural unit interval order*. The *incomparability graph* $G(\mathbf{m})$ is the undirected graph on the vertex set $[n]$ in which i and j are adjacent if and only if i and j are incomparable in $P(\mathbf{m})$. In other words, if $i < j$ then i and j are adjacent in $G(\mathbf{m})$ if and only if $j \le m_i$.

An *integer partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of a positive integer n is a weakly decreasing sequence of positive integers that sum to n. Each λ_i is a *part* of λ, and the number of parts of λ is denoted by $\ell(\lambda)$. The *Young diagram* of λ comprises ℓ rows of boxes, left-justified, with λ_i boxes in the ith row from the top. We write $\lambda \vdash n$ to indicate that λ is a partition of n.

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of a positive integer n is a (not necessarily monotonic) sequence of positive integers that sum to n. Each α_i is a *part* of α, and the number of parts of α is denoted by $\ell(\alpha)$. It can be useful to visualize a composition of n by drawing vertical bars in some subset of the $n-1$ spaces between consecutive objects in a horizontal line of n objects; the parts are then the numbers of objects between successive bars. Motivated by the equivalence between compositions and sets of bars, we define:

- $|\alpha|$ for the number of bars of α (equivalently, $|\alpha| = \ell(\alpha) - 1$; *CAUTION:* $|\alpha|$ is *not* the sum of the parts of α);

- $\overline{\alpha}$ for the composition that has bars in precisely the positions where α does *not* have bars;
- $\alpha \cup \beta$ for the composition whose bars comprise the union of the bars of α and the bars of β; and
- $\alpha \le \beta$ if the bars of α are a subset of the bars of β.

We write S_n for the symmetric group. If S_n acts in the usual way on a set of size n, and α is a composition of n, then the *Young subgroup* S_α is the subgroup

$$S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_\ell} \subseteq S_n \tag{6}$$

comprising all the permutations that permute the first α_1 elements among themselves, the next α_2 elements among themselves, and so on.

An *ordered (set) partition* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ of a finite set S is a sequence of pairwise disjoint non-empty subsets of S whose union is S.

A *sequencing* q of a finite set S of cardinality n is a bijective map $q : [n] \to S$. It is helpful to think of q as the sequence $q(1), \ldots, q(n)$ of elements of S.

By a *digraph* we mean a finite directed graph with no loops or multiple edges but that may have bidirected edges, i.e., it may contain both $u \to v$ and $v \to u$ simultaneously. If D is a digraph, we write $\overline{D}$ for the *complement* of D, i.e., the digraph with the same vertex set as D but with a directed edge $u \to v$ if and only if there does *not* exist a directed edge $u \to v$ in D.

2.2 Symmetric and quasisymmetric functions

We mostly follow the notation of Stanley [48] for symmetric and quasisymmetric functions. For convenience, we recall some of the notation here.

Let $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$ be a countable set of independent indeterminates. If $\kappa : [n] \to \mathbb{P}$ is a map then we write $\mathbf{x}_\kappa$ for the monomial $x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$. A formal power series in $\mathbb{Q}[[\mathbf{x}]] = \mathbb{Q}[[x_1, x_2, \ldots]]$ is a *symmetric function* if it is of bounded degree and invariant under any permutation of the variables $\mathbf{x}$. We write Λ for the subring of $\mathbb{Q}[[\mathbf{x}]]$ consisting of symmetric functions. Then $\Lambda = \oplus_{n\ge 0} \Lambda_n$ where Λ_n denotes the space of homogeneous symmetric functions of degree n.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an integer partition, then the *monomial symmetric function* m_λ is the symmetric function of minimal support that contains the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$. For example,

$$m_{2,1,1} = x_1^2 x_2 x_3 + x_2^2 x_1 x_3 + x_3^2 x_1 x_2 + x_1^2 x_3 x_4 + x_3^2 x_1 x_4 + x_4^2 x_1 x_3 + \cdots$$

There is an unfortunate conflict between our notation for monomial symmetric functions and our notation $\mathbf{m}$ for Hessenberg functions. It should be clear from context which is meant since the subscript of a monomial symmetric function is a partition, whereas the entries of $\mathbf{m}$ have integer subscripts.

Set $h_n := \sum_{\lambda \vdash n} m_\lambda$. Then, if λ is a partition, set $h_\lambda = \prod h_{\lambda_i}$. The h_λ are called the *complete homogeneous symmetric functions* [48].

Both $\{h_\lambda\}_{\lambda\vdash n}$ and $\{m_\lambda\}_{\lambda\vdash n}$ form bases of Λ_n. So we get a non-degenerate scalar product on Λ_n (and on Λ as well) by setting

$$\langle m_\lambda, h_\mu\rangle = \delta_{\lambda\mu} \text{ (Kronecker delta)} \tag{7}$$

as in [48, Equation 7.30]. This scalar product is symmetric [48, Proposition 7.9.1].

Write CF_n for the space of $\mathbb{Q}$-valued class functions on S_n, and set $\mathrm{CF} = \oplus_{n\geq 0}\mathrm{CF}_n$. The *characteristic map* $\mathrm{ch} : \mathrm{CF}_n \to \Lambda_n$ is a function that sends class functions χ on the symmetric group to symmetric functions via the formula

$$\mathrm{ch}\,\chi := \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)\,p_{\mathrm{cycletype}(\sigma)} \tag{8}$$

where cycletype(σ) is the integer partition consisting of the cycle sizes of σ, listed with multiplicity in weakly decreasing order, and p denotes the power-sum symmetric function. It turns out that ch is an isomorphism of $\mathbb{Q}$-vector spaces. Moreover, if we give CF_n the standard inner product $\langle\cdot,\cdot\rangle$ on class functions, then ch is an isometry [48, Proposition 7.18.1]:

$$\langle \mathrm{ch}\,f, \mathrm{ch}\,g\rangle = \langle f, g\rangle. \tag{9}$$

Note that all complex characters of finite dimensional representations of S_n are actually rational. In fact, even the representations themselves are realizable over $\mathbb{Q}$ [41, Example 1, page 103]. So it makes sense to work with the $\mathbb{Q}$-valued class functions even if we are interested in complex representations of S_n.

As we explained in the introduction, the following standard fact is an important ingredient in our proof.

Proposition 10. *Let ρ be a finite-dimensional complex representation of S_n, and let χ be its character. Let $\mathrm{ch}\,\chi = \sum_\lambda c_\lambda m_\lambda$ be the monomial symmetric function expansion of $\mathrm{ch}\,\chi$. Then c_λ equals the dimension of the subspace fixed by any Young subgroup $S_\lambda \subseteq S_n$. In particular, knowing c_λ for all λ uniquely determines χ.*

Proof. Let $\chi{\downarrow}^{S_n}_{S_\lambda}$ denote the restriction of χ to S_λ, and let d_λ be the dimension of the subspace fixed by S_λ. Then d_λ equals the multiplicity of the trivial representation $\mathbf{1}$ in $\chi{\downarrow}^{S_n}_{S_\lambda}$, i.e., $d_\lambda = \langle \mathbf{1}, \chi{\downarrow}^{S_n}_{S_\lambda}\rangle$. By Frobenius reciprocity [39, Theorem 1.12.6],

$$\langle \mathbf{1}, \chi{\downarrow}^{S_n}_{S_\lambda}\rangle = \langle \mathbf{1}{\uparrow}^{S_n}_{S_\lambda}, \chi\rangle, \tag{11}$$

where $\mathbf{1}{\uparrow}^{S_n}_{S_\lambda}$ is the induction of $\mathbf{1}$ from S_λ up to S_n. But $\mathrm{ch}\,\mathbf{1}{\uparrow}^{S_n}_{S_\lambda}$ is just the homogeneous symmetric function h_λ [48, Corollary 7.18.3]. The monomial symmetric functions and the complete homogeneous symmetric functions are dual bases, so $d_\lambda = \langle h_\lambda, \mathrm{ch}\,\chi\rangle = c_\lambda$. □

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of n. The *monomial quasisymmetric function* M_α is the formal power series defined by

$$M_\alpha := \sum_{i_1<\cdots<i_\ell} x_{i_1}^{\alpha_1}\cdots x_{i_\ell}^{\alpha_\ell}, \tag{12}$$

where the sum is over all strictly increasing sequences $(i_1, \ldots, i_\ell)$ of positive integers. In addition, we define a degree-zero monomial quasisymmetric function by $M_\varnothing := 1$. A formal power series is a *quasisymmetric function* if it is a finite rational linear combination of monomial quasisymmetric functions. We write $\mathscr{Q}$ for the algebra of quasisymmetric functions and $\mathscr{Q}_n$ for the space of homogeneous quasisymmetric functions of degree n [48, Section 7.19]. Clearly, we have $\mathscr{Q} = \oplus \mathscr{Q}_n$, and clearly Λ is a subalgebra of $\mathscr{Q}$. Note that it is a proper subalgebra. (For example, $M_{2,1} \in \mathscr{Q} \setminus \Lambda$.)

The *fundamental quasisymmetric function* F_α of Gessel [17] is defined by

$$F_\alpha := \sum_{\beta \geq \alpha} M_\beta, \tag{13}$$

and again we set $F_\varnothing := 1$. By inclusion-exclusion,

$$M_\alpha = \sum_{\beta \geq \alpha} (-1)^{|\beta| - |\alpha|} F_\beta. \tag{14}$$

2.3 Hessenberg varieties

As mentioned in the introduction, if $\mathbf{m}$ is a Hessenberg function and $s : \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation, then we define the *Hessenberg variety* (of type A, which is the only type that we consider in this paper) by

$$\mathscr{H}(\mathbf{m}, s) := \{\text{complete flags } F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for } 1 \leq i < n\}.$$

If the Jordan blocks of s have distinct eigenvalues then we say that $\mathscr{H}(\mathbf{m}, s)$ is *regular*, if s is diagonalizable then we say that $\mathscr{H}(\mathbf{m}, s)$ is *semisimple*, and if s is nilpotent then we say that $\mathscr{H}(\mathbf{m}, s)$ is *nilpotent*. Since $\mathscr{H}(\mathbf{m}, s)$ can equal $\mathscr{H}(\mathbf{m}, s')$ for $s \neq s'$ (e.g., if $s' - s$ is a constant), this is a (very minor) abuse of terminology. We adopt the convention of writing y for s in the regular semisimple case.

Remark 15. The Hessenberg varieties are defined on affine open subsets of the complete flag variety by fairly obvious equations. So they are closed subschemes of the complete flag variety in a natural way. In general, they are not irreducible. For example, the regular semisimple Hessenberg variety corresponding to the function $\boldsymbol{\ell} = (1, 2, \ldots, n-1)$ is a collection of $n!$ distinct points. They are also not always reduced. For example, when $n = 2$ and $\mathbf{m} = \boldsymbol{\ell}$ as above, the regular nilpotent Hessenberg variety is defined by the equation $x^2 = 0$ in $\mathbb{A}^1$. (See [3, Theorem 7.6] for a much more general statement.) Hartshorne defines an abstract variety to be an integral separated scheme of finite type over an algebraically closed field [23, p.105]. So, perhaps, it is unfortunate that Hessenberg varieties are called varieties as they are not, in general, integral. However, it happens that we are only interested in the Betti cohomology of these varieties in this paper. So the non-reduced structure will not play a role. Moreover, we will reserve the term “Hessenberg *scheme*” for the families discussed in §8. So we will stick with tradition and continue to call the schemes $\mathscr{H}(\mathbf{m}, s)$ Hessenberg *varieties*.

However, we ask the reader to regard "Hessenberg variety" as one word. The term *variety* by itself will still refer to an integral, separated scheme of finite type over an algebraically closed field.

3 The chromatic quasisymmetric function

Given a graph G whose vertex set is a subset of $\mathbb{P}$, Shareshian and Wachs [43] define the *chromatic quasisymmetric function* $X_G(\mathbf{x}, t)$ of G.

Definition 16. Let G be a graph whose vertex set V is a finite subset of $\mathbb{P}$. Let $C(G)$ denote the set of all proper colorings of G, i.e., the set of all maps $\kappa : V \to \mathbb{P}$ such that adjacent vertices are always mapped to distinct positive integers. Then

$$X_G(\mathbf{x}, t) := \sum_{\kappa \in C(G)} t^{\operatorname{asc} \kappa}\, \mathbf{x}_\kappa, \tag{17}$$

where

$$\operatorname{asc} \kappa := \left| \{\{u, v\} : \{u, v\} \text{ is an edge of } G \text{ and } u < v \text{ and } \kappa(u) < \kappa(v)\} \right|.$$

As Shareshian and Wachs point out, it is obvious that $X_G(\mathbf{x}, t) \in \mathscr{Q}[t]$. On the other hand, while the proof of the following result, [43, Theorem 4.5], is not long, the result itself is not at all obvious:

Theorem 18 (Shareshian–Wachs)**.** *Suppose $G = G(\mathbf{m})$ is the incomparability graph of a natural unit interval order. Then $X_G(\mathbf{x}, t) \in \Lambda[t]$.*

For brevity, we sometimes write $X_G(t)$ for $X_G(\mathbf{x}, t)$. It will be convenient for us to restate the definition of $X_G(t)$ in terms of monomial quasisymmetric functions.

Proposition 19. *Let G be a graph whose vertex set V is a finite subset of $\mathbb{P}$. Then*

$$X_G(\mathbf{x}, t) = \sum_{\sigma = (\sigma_1, \ldots, \sigma_\ell)} t^{\operatorname{asc} \sigma}\, M_{|\sigma_1|, \ldots, |\sigma_\ell|}, \tag{20}$$

where the sum is over all ordered partitions σ of V such that every σ_i is a stable set of G (i.e., there is no edge between any two vertices of σ_i), and $\operatorname{asc} \sigma$ *is the number of edges $\{u, v\}$ of G such that $u < v$ and v appears in a later part of σ than u does.*

Proof. Given a coloring $\kappa \in C(G)$, let σ_i be the set of vertices that are assigned the ith smallest color. Then it is immediate that

1. $\sigma = (\sigma_1, \ldots, \sigma_\ell)$ is an ordered partition of the vertex set of G;
2. σ_i is a stable set for all i; and

3. $\operatorname{asc}\kappa = \operatorname{asc}\sigma$.

It is easy to see that if we sum $\mathbf{x}_\kappa$ over all $\kappa \in C(G)$ that yield the same ordered partition σ, then we obtain the monomial quasisymmetric function M_α where the ith part α_i of the composition α is the cardinality $|\sigma_i|$ of σ_i. The proposition follows. □

We remark that if we set $t = 1$ then the chromatic quasisymmetric function specializes to the *chromatic symmetric function* X_G of Stanley [47].

3.1 Reciprocity

If f is a symmetric function, then a "reciprocity theorem," loosely speaking, is a result that gives a combinatorial interpretation of ωf, where ω is a well-known involution on symmetric functions [48, Section 7.6]. Since Conjecture 3 concerns $\omega X_G(t)$ rather than $X_G(t)$ itself, one might expect a reciprocity theorem to be relevant. This is indeed the case. Specifically, the coefficients of the monomial symmetric function expansion of $\omega X_G(t)$ play an important role in our arguments, so we now introduce some notation for them.

Definition 21. Given a Hessenberg function $\mathbf{m}$, we let $c_{d,\lambda}(\mathbf{m})$ be the coefficients defined by the following expansion of $\omega X_{G(\mathbf{m})}(\mathbf{x}, t)$ in terms of monomial symmetric functions:

$$\omega X_{G(\mathbf{m})}(\mathbf{x}, t) = \sum_d t^d \sum_\lambda c_{d,\lambda}(\mathbf{m}) m_\lambda. \tag{22}$$

It is possible to derive a combinatorial interpretation for $c_{d,\lambda}(\mathbf{m})$ by using the reciprocity theorem of Shareshian and Wachs [43, Theorem 3.1]. However, as we now explain, we shall take a different route.

Our starting point is the observation that Chow [8, Theorem 1] has proved a reciprocity theorem for a symmetric function invariant of a digraph called the *path-cycle symmetric function* Ξ_D. There is a certain precise sense in which Ξ_D is equivalent to Stanley's X_G in the case of posets, but the nice thing about reciprocity for Ξ_D is that it naturally yields a combinatorial interpretation for the coefficients of the monomial symmetric function expansion of $\omega\Xi_D$, which is not immediately evident from Stanley's reciprocity theorem [47, Theorem 4.2] for X_G. This fact suggests the following plan: Generalize Ξ_D to $\Xi_D(t)$ (just as Shareshian and Wachs have generalized X_G to $X_G(t)$), prove reciprocity for $\Xi_D(t)$, and read off the desired combinatorial interpretation of $c_{d,\lambda}(\mathbf{m})$. This plan works, and we now show how to carry it out.

We define the *path quasisymmetric function* $\Xi_D(\mathbf{x}, t)$ of a digraph D; as its name suggests, it enumerates paths only and not cycles (since for our present purposes we do not care about enumerating cycles), and it has a definition analogous to that of the chromatic quasisymmetric function.

Definition 23. Let D be a digraph whose vertex set V is a subset of $\mathbb{P}$. An *ordered path cover* of D is an ordered pair (q, β) such that q is a sequencing of V, $\beta = (\beta_1, \ldots, \beta_\ell)$ is a composition of $n := |V|$, and

$$q(\beta_{i-1}+1) \to q(\beta_{i-1}+2) \to \cdots \to q(\beta_i)$$

is a directed path in D for all $i \in [\ell]$ (adopting the convention that $\beta_0 = 0$). Define

$$\Xi_D(\mathbf{x}, t) := \sum_{(q,\beta)} t^{\operatorname{asc} q} M_\beta \tag{24}$$

where the sum is over all ordered path covers (q, β) of D and $\operatorname{asc} q$ is the number of pairs $\{u, v\}$ of vertices of D such that

1. either $u \to v$ and $v \to u$ are both edges of D or neither one is,
2. $u < v$, and
3. v appears later in the sequencing q than u does.

For brevity, we sometimes write $\Xi_D(t)$ for $\Xi_D(\mathbf{x}, t)$. The chromatic quasisymmetric function and the path quasisymmetric function coincide for posets. More precisely, we have the following proposition.

Proposition 25. *Let P be a poset whose vertex set V is a finite subset of $\mathbb{P}$. Let $D(P)$ be the digraph on V that has an edge $u \to v$ if and only if $v \prec u$ in P. Let $G(P)$ be the incomparability graph of P. Then $\Xi_{D(P)}(\mathbf{x}, t) = X_{G(P)}(\mathbf{x}, t)$.*

Proof (sketch). The proof is mostly a routine verification that the two definitions coincide in this special case. Only a few points require some attention. First, if S is a stable subset in $G(P)$, then S is a totally ordered subset of P, and hence there is exactly one directed path in $D(P)$ through the vertices of S. Hence ordered partitions σ of V such that every σ_i is a stable set of $G(P)$ are in bijective correspondence with ordered path covers (q, α) of $D(P)$. Second, because P is a poset, it is not possible for $u \prec v$ and $v \prec u$ simultaneously, so the condition that "either $u \to v$ and $v \to u$ are both edges of $D(P)$ or neither one is" is equivalent to adjacency in $G(P)$. Third, one might worry that $\operatorname{asc} q$ counts some pairs $\{u, v\}$ where v appears later in the *sequencing* but in the same *path* while $\operatorname{asc} \sigma$ counts only pairs from different parts, but in fact this cannot happen because vertices in the same path are part of the same totally ordered subset of P and thus have a directed edge between them in exactly one direction. □

Although we are ultimately interested in expansions in terms of monomial *symmetric* functions, it turns out that the proofs are more naturally stated in terms of monomial *quasisymmetric* functions. So we need to describe the action of ω on monomial quasisymmetric functions.

Definition 26. The linear map ω on quasisymmetric functions is defined by the following action on monomial quasisymmetric functions.

$$\omega M_\beta := (-1)^{|\beta|} \sum_{\alpha \le \beta} M_\alpha. \tag{27}$$

It is known (e.g., see the proof of [47, Theorem 4.2]) that the usual map ω is characterized by the equation $\omega F_\alpha = F_{\overline{\alpha}}$, so the following proposition confirms that our definition of ω coincides with the standard one.

Proposition 28. *$\omega F_\alpha = F_{\overline{\alpha}}$.*

Proof. Applying ω to Equation (13) and invoking Equation (27) yields

$$\omega F_\alpha = \sum_{\beta \geq \alpha} \omega M_\beta = \sum_{\beta \geq \alpha} (-1)^{|\beta|} \sum_{\gamma \leq \beta} M_\gamma.$$

So the coefficient of M_γ in ωF_α is

$$\sum_{\beta:(\beta \geq \alpha \text{ and } \beta \geq \gamma)} (-1)^{|\beta|} = \sum_{\beta \geq \alpha \cup \gamma} (-1)^{|\beta|} = \begin{cases} 1, & \text{if } \overline{\alpha \cup \gamma} = \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

But $\overline{\alpha \cup \gamma} = \varnothing$ is equivalent to $\gamma \geq \overline{\alpha}$, so $\omega F_\alpha = \sum_{\gamma \geq \overline{\alpha}} M_\gamma = F_{\overline{\alpha}}$. □

We are ready for the reciprocity theorem for $\Xi_D(t)$.

Theorem 29. *Let D be a digraph whose vertex set V is a subset of $\mathbb{P}$. Then $\omega \Xi_D(\mathbf{x}, t) = \Xi_{\overline{D}}(\mathbf{x}, t)$.*

Proof. Let us apply ω to both sides of Equation (24) and invoke the definition of ω.

$$\omega \Xi_D(\mathbf{x}, t) = \sum_{(q,\beta)} t^{\operatorname{asc} q} \omega M_\beta = \sum_{(q,\beta)} t^{\operatorname{asc} q} (-1)^{|\beta|} \sum_{\alpha \leq \beta} M_\alpha.$$

Now we interchange the order of summation; i.e., we want to compute the coefficient of M_α in $\omega \Xi_D(t)$. This involves a sum over all ordered path covers (q, β) such that $\beta \geq \alpha$. The summands involving a fixed sequencing q are

$$\sum_{\beta \geq \alpha} t^{\operatorname{asc} q} (-1)^{|\beta|} = t^{\operatorname{asc} q} \sum_{\beta \geq \alpha} (-1)^{|\beta|}. \tag{30}$$

Now note that if (q, α) is an ordered path cover and β is any composition such that $\beta \geq \alpha$, then (q, β) is also an ordered path cover, because deleting an edge from a directed path simply subdivides it into two smaller directed paths. Therefore the alternating sum in Equation (30) is zero unless the only $\beta \geq \alpha$ for which (q, β) is an ordered path cover is the maximal composition ($\beta_i = 1$ for all i), in which case the alternating sum equals one. But this condition is equivalent to the condition that there is no directed edge in D between any consecutive vertices in the sequencing q that are in the same segment of α, i.e., that (q, α) is an ordered path cover of $\overline{D}$. Finally, note that the definition of $\operatorname{asc} q$ is invariant under taking complements of the digraph. The theorem follows. □

Theorem 29 gives us a nice combinatorial interpretation of $c_{d,\lambda}(\mathbf{m})$.

Corollary 31. *Let $\mathbf{m}$ be a Hessenberg function, and let $D(\mathbf{m})$ denote the digraph on $[n]$ that has an edge $u \to v$ if and only if $v \prec u$ in P. Then for any composition α whose parts are a permutation of the parts of λ, $c_{d,\lambda}(\mathbf{m})$ equals the number of ordered path covers (q, α) of $\overline{D(\mathbf{m})}$ with $\operatorname{asc} q = d$.*

Proof. By Proposition 25, we know that $X_{G(\mathbf{m})}(t) = \Xi_{D(\mathbf{m})}(t)$. By Theorem 18, $X_{G(\mathbf{m})}(t)$ is actually a *symmetric* function (whose coefficients are polynomials in t). Therefore $\omega X_{G(\mathbf{m})}(t) = \omega \Xi_{D(\mathbf{m})}(t)$ is also a symmetric function, and the coefficient of $t^d m_\lambda$ equals the coefficient of $t^d M_\alpha$ for any composition α whose parts are a permutation of the parts of λ. The result then follows from Theorem 29. $\square$

Corollary 32. *For a sequencing q of a digraph whose vertex set is a subset of $\mathbb{P}$, let the definition of $\operatorname{des} q$ be the same as the definition of $\operatorname{asc} q$ except with "$u < v$" replaced by "$v < u$." then Corollary 31 holds with $\operatorname{des} q$ in place of $\operatorname{asc} q$.*

Proof. For a proper coloring κ of a graph whose vertex set is a subset of $\mathbb{P}$, let the definition of $\operatorname{des} \kappa$ be the same as the definition of $\operatorname{asc} \kappa$ except with "$\kappa(u) < \kappa(v)$" replaced by "$\kappa(u) > \kappa(v)$." Shareshian and Wachs prove [43, Corollary 2.7] that the value of $X_G(t)$ is unchanged if "asc" is replaced by "des." It is readily checked that the proofs of Proposition 25 and Theorem 29 go through if "asc" is replaced by "des" everywhere. $\square$

As we remarked before, Corollaries 31 and 32 can be derived from Shareshian–Wachs [43, Theorem 3.1], but we have taken our approach because we believe that Theorem 29 is of independent interest.

4 Betti numbers of regular Hessenberg varieties

The main result of this section is that if $\mathscr{H}(\mathbf{m}, s)$ is a regular Hessenberg variety and s has Jordan type λ, then its Betti number β_{2d} equals $c_{d,\lambda}(\mathbf{m})$.

Tymoczko [51, Theorem 7.1] has already done a lot of the work needed to prove this result, by showing that Hessenberg varieties admit a paving (or cellular decomposition) by affine spaces, and obtaining a combinatorial interpretation of the dimensions of the cells. For regular Hessenberg varieties, Tymoczko's theorem simplifies as follows. If λ is an integer partition of n then by a *tableau of shape* λ we mean any filling of the boxes of the Young diagram of λ with one copy each of the numbers $1, 2, \ldots, n$.

Theorem 33 (Tymoczko)**.** *Let $\mathscr{H}(\mathbf{m}, s)$ be a regular Hessenberg variety and let the partition λ encode the sizes of the Jordan blocks of s. Then $\mathscr{H}(\mathbf{m}, s)$ is paved by affines. The nonempty cells are in bijection with tableaux T of shape λ with the property that k appears in the box immediately to the left of j only if $k \le m_j$. The dimension of a nonempty cell is the sum of:*

1. *the number of pairs i, k in T such that*

(a) *i and k are in the same row,*

(b) *i appears somewhere to the left of k,*

(c) *$k < i$, and*

(d) *if j is in the box immediately to the right of k then $i \le m_j$;*

2. *the number of pairs i, k in T such that*

(a) *i appears in a lower row than k, and*

(b) *$k < i \le m_k$.*

It remains for us to establish a correspondence between the combinatorics of Theorem 33 and the combinatorics of $\omega X_{G(\mathbf{m})}(t)$, or equivalently (by the results of the previous section) the combinatorics of ordered path covers.

Definition 34. If X is a topological space and i is an integer, we write β_i or $\beta_i(X)$ for the ith Betti number $\dim \mathrm{H}^i(X, \mathbb{C})$ of X.

Theorem 35. *Let $\mathscr{H}(\mathbf{m}, s)$ be a regular Hessenberg variety and let the Jordan type of s be λ. Then the Betti number β_{2d} of $\mathscr{H}(\mathbf{m}, s)$ equals $c_{d,\lambda}(\mathbf{m})$, and $\beta_i = 0$ for i odd.*

Proof. As Tymoczko [51, Proposition 2.2] mentions, it is well known that if we have a paving by affines, then β_{2d} is just the number of nonempty cells with dimension d. Furthermore, $\beta_i = 0$ for i odd [51, Corollary 6.2]. On the other hand, by Corollaries 31 and 32, we know that $c_{d,\lambda}(\mathbf{m})$ is the number of ordered path covers (q, α) of $\overline{D(\mathbf{m})}$ with $\operatorname{des} q = d$, where we may take the parts of the composition α to be any permutation of the parts of λ. So it suffices to show, firstly, that there is a bijection between nonempty cells and ordered path covers (q, α), and secondly, that under this bijection, the dimension of the nonempty cell is equal to $\operatorname{des} q$.

First we should specify α. If λ has ℓ parts $\lambda_1, \ldots, \lambda_\ell$, we set $\alpha_i := \lambda_{\ell+1-i}$. That is, the parts of α are the parts of λ in *reverse* order.

Instead of nonempty cells, we use the tableaux T of Theorem 33 to describe our bijection. Given an ordered path cover (q, α) of $\overline{D(\mathbf{m})}$, take the elements of the ith path

$$q(\alpha_{i-1} + 1) \to q(\alpha_{i-1} + 2) \to \cdots \to q(\alpha_i)$$

and place them from left to right in the ith row (from the bottom) of T. We need to verify that Tymoczko's condition $k \le m_j$ is equivalent to the condition that $k \to j$ is a directed edge in $\overline{D(\mathbf{m})}$. By definition, there is a directed edge $k \to j$ in $\overline{D(\mathbf{m})}$ if and only if there is *not* a directed edge $k \to j$ in $D(\mathbf{m})$, i.e., if and only if either k and j are incomparable in $P(\mathbf{m})$, or $k \prec j$ in $P(\mathbf{m})$. The only way this property can fail is if $j \prec k$ in $P(\mathbf{m})$, i.e., if $k > m_j$. So indeed the conditions are equivalent.

Let us call a pair i, k satisfying the conditions in Theorem 33 a "T-inversion." Using the above bijection, we can think of T-inversions as certain pairs i, k in an ordered path cover (q, λ). The statistic des can also be thought of as counting certain pairs i, k of (q, λ), namely those satisfying

1. either $i \to k$ and $k \to i$ are both edges of $\overline{D(\mathbf{m})}$ or neither is,
2. $i > k$, and
3. k appears later in the sequencing q than i does.

Call such a pair an "SW-inversion." We claim that for any ordered path cover, the number of T-inversions equals the number of SW-inversions. This will prove the theorem.

First let us note that the condition $k < i$ implies that $k \le m_i$ (since $\mathbf{m}$ is a Hessenberg function) and therefore, by the argument we gave above, $k \to i$ is an edge of $\overline{D(\mathbf{m})}$. That is, if $k < i$ then it is not possible for neither $i \to k$ nor $k \to i$ to be an edge of $\overline{D(\mathbf{m})}$, so in fact both must be, and in particular we must have $i \to k$, or in other words $i \le m_k$. Therefore an SW-inversion can be redefined as a pair i, k such that

1. i appears earlier in the sequencing q than k does, and
2. $k < i \le m_k$.

It is now immediate that if i and k are in different paths then i, k is a T-inversion if and only if i, k is an SW-inversion, because by construction, i appearing in an earlier path than k is equivalent to appearing in a lower row than k in the tableau.

If i and k are in the same path then the situation is more complicated because T-inversions and SW-inversions do not necessarily coincide. However, we now give a bijection from the set of SW-inversions to the set of T-inversions, thereby showing that they are equinumerous.

Given an SW-inversion i, k, let $k_1, k_2, \ldots, k_r$ denote the remaining elements, in order, that succeed k in the path. For convenience, set $k_0 := k$ and $k_{r+1} := \infty$. Now let j be the smallest number such that $i \le m_{k_{j+1}}$. Then we claim that i, k_j is a T-inversion, and that this is a bijection.

First let us verify that i, k_j is a T-inversion. Condition 1(d) is satisfied almost by definition because what the construction is doing is scanning to the right *until* condition 1(d) is satisfied, and it will always succeed, since we just take $j = r$ in the worst case. So we just need to verify that $i > k_j$. If $j = 0$ then we are done, because $(i, k_0) = (i, k)$ is an SW-inversion by assumption, and in particular $i > k$. Otherwise, by minimality of j, we know that $i > m_{k_j} \ge k_j$.

Thus the construction scans rightwards from k *until the first T-inversion* i, k_j *is reached.*

To see that this map is injective, observe that by minimality of j, we have $i > m_{k_{j'}}$ for every $0 \le j' \le j$, so $(i, k_{j'})$ is *not* an SW-inversion. Thus, as we scan rightwards from k in search of the first T-inversion i, k_j, we do not encounter any other SW-inversions en route. If more than one SW-inversion were mapped to the same T-inversion, then the leftmost one would have to cross over the other ones en route.

To see that the map is surjective, we can define an inverse map, that scans *leftwards* from a T-inversion until it finds a pair that satisfies $i \le m_k$. Such a scan always succeeds because in the worst case it ends up at the successor i' of i, and $i \leq m_{i'}$ because they are consecutive elements of a path. Then by minimality, if we arrive at a pair i, k with k' being the successor of k, we must have $k > m_{k'} \geq k$, so what we have arrived at is indeed an SW-inversion. $\square$

Let us remark that our proof shows that at least in the case of regular Hessenberg varieties, the two cases of Theorem 33 can be unified, namely that the dimension is just the number of pairs i, k such i appears to the left of k or in a lower row than k, and $k < i \leq m_k$.

Corollary 36. *Let $\mathscr{H}(\mathbf{m}, s)$ be a regular Hessenberg variety with s of type λ as in Theorem 35. Set*

$$q = q_{\mathscr{H}(\mathbf{m},s)} := \sum_{i \in \mathbb{Z}} \beta_i t^{i-|\mathbf{m}|}.$$

Then $q(t) = q(t^{-1})$.

Proof. First note that $|\mathbf{m}|$ (as defined in Equation (5)) is the number $|E|$ of edges in the incomparability graph $G = G(\mathbf{m})$ of $P(\mathbf{m})$. This follows directly from the description of $G(\mathbf{m})$ given in §2.1. By [43, Corollary 4.6], $X_G(\mathbf{x}, t)$ is palindromic. More precisely, we have $X_G(\mathbf{x}, t) = t^{|\mathbf{m}|} X_G(x, t^{-1})$. Therefore, for each partition λ, we have

$$\sum_d c_{d,\lambda}(\mathbf{m}) t^d = t^{|\mathbf{m}|} \sum_d c_{d,\lambda} t^{-d}.$$

So

$$\begin{aligned} q(t) &= \sum_i \beta_i t^{i-|\mathbf{m}|} = \sum_d c_{d,\lambda}(\mathbf{m}) t^{2d-|\mathbf{m}|} \\ &= t^{-|\mathbf{m}|} \sum_d c_{d,\lambda}(\mathbf{m}) t^{2d} = t^{-|\mathbf{m}|} t^{2|\mathbf{m}|} \sum_d c_{d,\lambda}(\mathbf{m}) t^{-2d} \\ &= t^{|\mathbf{m}|} \sum_d c_{d,\lambda}(\mathbf{m}) t^{-2d} = \sum_d c_{d,\lambda}(\mathbf{m}) t^{|\mathbf{m}|-2d} \\ &= q(t^{-1}). \end{aligned}$$

$\square$

Corollary 37. *Suppose s is a regular matrix. Then, for all $i \in \mathbb{Z}$, we have $\beta_i = \beta_{2|\mathbf{m}|-i}$. Consequently, $\dim \mathscr{H}(\mathbf{m}, s) = |\mathbf{m}|$.*

Proof. The first assertion follows (after a few algebraic manipulations) from Corollary 36. It is well known that, for a complex, projective variety X, we have $\dim X = \max\{i : \mathrm{H}^{2i}(X, \mathbb{C}) \neq 0\}$. So, the second assertion is a direct consequence of the first. $\square$

5 Local monodromy and local fundamental groups

5.1 Local systems

In this subsection, we review some terminology concerning local systems. This material is standard (going back in some ways to Riemann [38]), but we realized that including it might help to make our paper more broadly accessible. Moreover, since we are making considerable use of local systems, it seems appropriate to be as precise as possible about what they are. To have a specific (modern) reference, we follow the dictionary on page 3 of Deligne's book on differential equations [11].

By a locally constant sheaf on a topological space X, we simply mean a sheaf of sets $\mathcal{F}$ which is locally isomorphic to a constant sheaf of sets. In other words, each $x \in X$ has an open neighborhood U such that the restriction of $\mathcal{F}$ to U is constant. We can consider the class of locally constant sheaves as a full subcategory of the class of all sheaves. On the other hand, it is well known and easy to see that the category of locally constant sheaves on X is equivalent to the category of covering spaces of X (c.f. [53, Definition and Proposition 3.41]).

By a local system on a topological space X, we mean a sheaf of finite dimensional $\mathbb{C}$-vector spaces $\mathcal{F}$ on X which is locally isomorphic to a constant sheaf of $\mathbb{C}$-vector spaces. Note that this definition differs slightly from Deligne's in [11] in that Deligne requires the dimension of the stalks to be constant. However, this is guaranteed by our definition if X is connected, which is the most important case, and the added flexibility is useful.

For any ring R, we could equally well define R-local systems by replacing $\mathbb{C}$ with R (and finite dimensional vector spaces by finitely generated R-modules). But, to avoid cluttering up the notation, we refer the reader to [14] for this notion. (For most of the paper we only use $\mathbb{C}$-local systems, however, we do use $\mathbb{Z}$-local systems in §8 and A-local systems for A a polynomial ring in §9.)

We view the class of local systems as a full-subcategory of the category of sheaves of $\mathbb{C}$-vector spaces on X. Clearly, there is a forgetful functor from the category of local systems on X to the category of locally constant sheaves on X (by forgetting the $\mathbb{C}$-vector space structures).

Suppose now that X is non-empty. Pick a point $x_0 \in X$, which we call a "basepoint." Then the fundamental group $\pi_1(X, x_0)$ acts on the fiber $\mathcal{F}_{x_0}$ of any locally constant sheaf $\mathcal{F}$ giving us a homomorphism $\rho : \pi_1(X, x_0) \to \operatorname{Aut} \mathcal{F}_{x_0}$. If $\mathcal{F}$ is a local system then ρ respects the $\mathbb{C}$-vector space structure giving us a group homomorphism

$$\rho : \pi_1(X, x_0) \to \mathbf{GL}(\mathcal{F}_{x_0}) \tag{38}$$

which is usually called the *monodromy representation.* The fundamental fact about local systems and locally constant sheaves is then the following standard result (which can be found on pages 3 and 4 of [11]).

Theorem 39. *Suppose that X is a locally path connected, locally simply connected, connected topological space equipped with a point x_0. Then the functor $\mathcal{F} \rightsquigarrow \mathcal{F}_{x_0}$ induces an equivalence from the category of locally constant sheaves* *(resp. local systems) on X to the category of $\pi_1(X, x_0)$-sets (resp. finite dimensional complex representations of $\pi_1(X, x_0)$).*

Sketch. Since Deligne does not actually prove Theorem 39 in [11], we give a sketch.

The main point is that, under the assumption that X is locally path connected, locally simply connected and connected, there exists a universal cover $\tilde{X}$ of X. For a proof, see the discussion starting on page 64 of Hatcher's book [24], where $\tilde{X}$ is constructed as a space of homotopy classes of paths starting from the point x_0. Moreover $\pi_1(X, x_0)$ acts freely on $\tilde{X}$ with quotient X. Given a $\pi_1(X, x_0)$-set E (resp. a finite dimensional $\pi_1(X, x_0)$-representation E), we consider the quotient $\mathcal{F}_E := (\tilde{X} \times E)/\pi_1(X, x_0)$ where the fundamental group acts on the product by $\gamma(\tilde{x}, e) = (\gamma\tilde{x}, \gamma^{-1}e)$. That is, we form the Borel construction, where here E is given the discrete topology.

The space $\mathcal{F}_E$ is naturally a covering space of X via the map $\mathcal{F}_E \to X$ induced by projection on the first factor in the product $\tilde{X} \times E$. If E is a $\mathbb{C}$-vector space, the sheaf corresponding to $\mathcal{F}_E$ has the natural structure of a sheaf of $\mathbb{C}$-vector spaces. We leave the rest of the verification to the reader. $\square$

Corollary 40. *Suppose $\mathcal{F}$ is a local system on a topological space X as in Theorem 39. Then there is a natural isomorphism $\mathrm{H}^0(X, \mathcal{F}) = \mathcal{F}_{x_0}^{\pi_1(X,x_0)}$.*

Proof. Write $\mathbb{C}_X$ for the constant local system on X, which corresponds to the trivial representation of $\pi_1(X, x_0)$. Then we have $\mathcal{F}^{\pi_1(X,x_0)} = \mathrm{Hom}(\mathbb{C}_X, \mathcal{F})$ by Theorem 39. But it is easily seen that the natural map $\mathrm{Hom}(\mathbb{C}_X, \mathcal{F}) \to \mathcal{F}(X)$ is an isomorphism. $\square$

Remark 41. By Proposition A.4 on page 531 of Hatcher's book [24], CW complexes are locally contractible. It follows that, if X is a CW complex and Y is a closed subcomplex, then $X \setminus Y$ is locally contractible. In particular, if $X \setminus Y$ is connected then it satisfies the hypotheses of Theorem 39.

5.2 Local homotopy type

In this section we review the definition and some of the main properties of local homotopy type. This material is probably well known to some readers, but we feel that it will be convenient to review it. Our treatment follows the book by Looijenga [29], a paper by Kumar [27] and another paper by Prill [37].

Suppose X is a topological space and $x \in X$. A *fundamental system of neighborhoods $\mathscr{U}$* of x is a system of open neighborhoods such that any open neighborhood V of x contains a $U \in \mathscr{U}$.

The following Lemma is [27, Lemma 1.1].

Lemma 42. *Suppose X is a CW complex, $x \in X$ and Y is a closed subcomplex of X containing x. Then there exists a fundamental system $\{U\}_{U \in \mathscr{U}}$ of open neighborhoods of x in X such that the following condition is satisfied:*

$$\text{For any } U, V \in \mathscr{U} \text{ with } V \subset U, \text{ the inclusion } V \setminus Y \hookrightarrow U \setminus Y \text{ is a homotopy equivalence.} \tag{43}$$

A system of neighborhoods $\mathscr{U}$ as in Lemma 42 is called a *good fundamental system of neighborhoods relative to* Y.

Lemma 44. *Suppose $\mathcal{C}$ is a category and*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is a sequence of morphisms. Assume that $g \circ f$ and $h \circ g$ are isomorphisms. Then f, g and h are all isomorphisms.

Proof. Easy exercise. □

We have adapted the proof of the following Proposition from Looijenga's [29, p. 114], and Prill's Proposition 2 [37].

Proposition 45. *Suppose $\{\mathscr{U}_\alpha\}_{\alpha \in I}$ is a non-empty collection of good fundamental systems of neighborhoods as in Lemma 42. Then so is $\mathscr{U} := \cup_{\alpha \in I} \mathscr{U}_\alpha$. Consequently, the union of all good fundamental systems of neighborhoods is itself a good fundamental system of neighborhoods.*

Proof. Take $U \in \mathscr{U}_\alpha$ and $V \in \mathscr{U}_\beta$ with $V \subset U$. We can find $U' \in \mathscr{U}_\alpha$ such that $U' \subset V$, and $V' \in \mathscr{U}_\beta$ such that $V' \subset U'$. Then apply Lemma 44 to the sequence of inclusions

$$V' \setminus Y \to U' \setminus Y \to V \setminus Y \to U \setminus Y.$$

This shows that the inclusion $V \setminus Y \to U \setminus Y$ is a homotopy equivalence. □

Definition 46. Suppose X is a CW complex and Y is a closed subcomplex containing a point x. We say an open neighborhood U of x is *good relative to* Y if U is an element of a good fundamental system of neighborhoods. The *local homotopy type of $X \setminus Y$ at x* is the homotopy type of $U \setminus Y$ where U is any good neighborhood.

If A and B are objects in any category $\mathcal{C}$, we say that A is a *retract* of B if there are morphism $i : A \to B$ and $r : B \to A$ such that $r \circ i = \mathrm{id}_A$. In other words, we follow Mac Lane's terminology in [30, p. 19].

Suppose U is a good neighborhood of x and W is an arbitrary (not necessarily good) open neighborhood of x contained in U. Then we can find a good neighborhood V such that $V \subset W$. Since V is good, Proposition 45 shows that the composition

$$V \setminus Y \to W \setminus Y \to U \setminus Y$$

is a homotopy equivalence. In other words, the local homotopy type of $X \setminus Y$ at x is a retract of the homotopy type of $W \setminus Y$.

Now suppose X is an analytic space, Y is a Zariski closed subspace and $x \in Y$. We can find an analytic open neighborhood W of x in X such that W has the topological structure of a CW complex with $W \cap Y$ a subcomplex. (See, for example, [28].) Consequently, there exist good neighborhoods of x in X relative to Y.

The following fact is certainly well known (see, e.g., [37, Corollary 1]), but we give a proof because it is short.

Fact 47. *Suppose X is a complex manifold and Y is a closed, nowhere dense, analytic subspace of X containing a point x. Then $U \setminus Y$ is non-empty and connected for any good neighborhood of x.*

Proof. Let U be a good neighborhood of x and let V be any connected neighborhood of x contained in U. Then $V \setminus Y$ is connected (for example, by the Criterion for Connectedness on page 133 of [18]). It is also non-empty. But the homotopy type of $U \setminus Y$ is a retract of the homotopy type of $V \setminus Y$. So $U \setminus Y$ is connected and non-empty as well. $\square$

If X is smooth at x, we can find a contractible good neighborhood U of x. (See [37].) In fact, we can take a sufficiently small ball as in the following theorem, which follows from Theorem 5.1 of Dimca's [12].

Theorem 48. *Suppose X is a complex manifold of dimension n at x and Y is a closed analytic subspace of X containing x. For each positive real number r, write B_r for the ball of radius r centered at 0 in $\mathbb{C}^n$. Then there exists a good neighborhood U of x relative to Y and biholomorphism $\varphi : U \to B_1$ such that the following holds: For each $r \in (0,1)$, $\varphi^{-1}B_r$ is a good neighborhood of x.*

5.3 Local fundamental group

Fact 47 leads to the following definition.

Definition 49. Suppose X is a complex manifold, and Y is a closed, nowhere dense, analytic subspace of X containing a point x. Then the *local fundamental group* of $X \setminus Y$ at x is the isomorphism class of the group $\pi_1(U \setminus Y, p)$ where U is any good neighborhood of x (with respect to Y) and $p \in U \setminus Y$.

Since the smoothness of X in the Definition 49 (together with Fact 47) implies that $U \setminus Y$ is connected, the isomorphism class of $\pi_1(U \setminus Y, p)$ is indeed well-defined. But, since we have not given a way to fix a base point, it is only defined up to a non-canonical isomorphism.

On the other hand, suppose X as in Definition 49 is connected. Pick any point $q \in X \setminus Y$. Given a good neighborhood U of x relative to Y, we can find a point $p \in U \setminus Y$ and a path γ from p to q. From this, we get a group homomorphism

$$\varphi_\gamma : \pi_1(U \setminus Y, p) \to \pi_1(X \setminus Y, q).$$

Changing γ has the effect of conjugating φ_γ by an element of $\pi_1(X \setminus Y, q)$. So the conjugacy class of the subgroup $\varphi_\gamma(\pi_1(U \setminus Y, p))$ is independent of γ.

Proposition 50. *Suppose $f : X' \to X$ is a morphism of complex analytic spaces admitting a section $\epsilon : X \to X'$. Let Y be a closed, nowhere dense, analytic subspace of X containing $x \in X$, and set $Y' := f^{-1}Y$. Then the local homotopy type of $X \setminus Y$ at x is a retract of the local homotopy type of $X' \setminus Y'$ at $\epsilon(x)$. In particular, if X and X' are complex manifolds, then the local fundamental group of $X \setminus Y$ at x is a retract of the local fundamental group of $X' \setminus Y'$ at $\epsilon(x)$.*

Proof. Pick a good neighborhood U of x. Then find a good neighborhood V of $\epsilon(x)$ contained in $f^{-1}U$. Finally, find a good neighborhood U' of x contained in $\epsilon^{-1}V$. We then have a composition

$$U' \setminus Y \xrightarrow{\epsilon} V \setminus Y' \xrightarrow{f} U \setminus Y$$

which is a homotopy equivalence. The result follows. $\square$

5.4 Local systems and local invariant cycles

Suppose X is a CW complex containing a closed subcomplex Y which contains a point x, and $\mathcal{L}$ is a local system of complex vector spaces on $X \setminus Y$. For any two good neighborhoods U_1 and U_2 of x and any integer k, the sheaf cohomology groups $\mathrm{H}^k(U_i \setminus Y, \mathcal{L}), i = 1, 2$ are canonically isomorphic. To see this, take a good neighborhood $V \subset U_1 \cap U_2$, and note that the restriction maps $\mathrm{H}^k(U_i \setminus Y, \mathcal{L}) \to \mathrm{H}^k(V \setminus Y, \mathcal{L})$ are isomorphisms. So we write $\mathrm{H}^k(x, \mathcal{L})$ for the group $\mathrm{H}^k(U \setminus Y, \mathcal{L})$ where U is any good neighborhood of x. It is isomorphic to the group $\operatorname{colim} \mathrm{H}^k(U \setminus Y, \mathcal{L})$ where the colimit is taken over all open neighborhoods of x. The group $\mathrm{H}^0(x, \mathcal{L})$ is called the space of *local invariants*.

If X is a complex manifold and Y is a nowhere dense analytic subspace, then $U \setminus Y$ is connected for any good neighborhood of x relative to Y. Pick a basepoint $p \in U \setminus Y$. Then the data of the local system $\mathcal{L}$ defines an action of $\pi_1(U \setminus Y, p)$ on the fiber $\mathcal{L}_p$ at p. Moreover, via Corollary 40, the space of local invariants is given by the invariants of the action:

$$\mathrm{H}^0(x, \mathcal{L}) = \mathcal{L}_p^{\pi_1(U \setminus Y, p)}. \tag{51}$$

Corollary 52. *Suppose X is smooth and B is any connected neighborhood of x contained in a good neighborhood U. Then $\mathrm{H}^0(x, \mathcal{L}) = \mathrm{H}^0(B \setminus Y, \mathcal{L})$.*

Proof. Pick a point $b \in B \setminus Y$. We have $\mathrm{H}^0(B \setminus Y, \mathcal{L}) = \mathcal{L}_b^{\pi_1(B \setminus Y, b)}$. But $\pi_1(U \setminus Y, b)$ is a retract of $\pi_1(B \setminus Y, b)$. So $\mathcal{L}_b^{\pi_1(B \setminus Y, b)} = \mathcal{L}_b^{\pi_1(U \setminus Y, b)} = \mathrm{H}^0(x, \mathcal{L})$. $\square$

We can also describe the space $\mathrm{H}^k(x, \mathcal{L})$ sheaf theoretically. Write $j : X \setminus Y \to X$ for the inclusion. Then the group $\mathrm{H}^k(x, \mathcal{L})$ is naturally isomorphic to $(R^k j_* \mathcal{L})_x$; i.e., to the stalk at x of the kth higher direct image $R^k j_* \mathcal{L}$.

The following is certainly well known, but we sketch a short proof.

Lemma 53. *Suppose X is a connected complex manifold and Y is a nowhere dense closed analytic subspace. Then, for $p \in X \setminus Y$, the homomorphism $\pi_1(X \setminus Y, p) \to \pi_1(X, p)$ is surjective.*

Sketch. Let $\pi : \tilde{X} \to X$ denote the universal cover of X. Then $\pi^{-1}(X \setminus Y) = \tilde{X} \setminus \pi^{-1}(Y)$ is connected because $\tilde{X}$ is a complex manifold and $\pi^{-1}(Y)$ is a closed, nowhere dense, complex analytic subspace [18, p. 133]. It follows that $\pi_1(X \setminus Y, p)$ acts transitively on $\pi^{-1}(p)$. If we pick a point $\tilde{p}$ in $\pi^{-1}(p)$, we get an identification of $\pi^{-1}(p)$ with $\pi_1(X, p)$. Moreover, the action of $\pi_1(X, p)$

on $\pi^{-1}(p)$ corresponds to the left regular action of $\pi_1(X, p)$ on itself. From this, we see that the action of $\pi_1(X \setminus Y, p)$ on $\pi_1(X, p)$ induced by the group homomorphism $\pi_1(X \setminus Y, p) \to \pi_1(X, p)$ is transitive. Therefore the map of fundamental groups is surjective. $\square$

Now, suppose X is a connected, complex manifold, Y is a closed, nowhere dense, analytic subspace, $\mathcal{L}$ is a local system on $X \setminus Y$, $x \in Y$ and $q \in X \setminus Y$. The *monodromy group of* $\mathcal{L}$ is the image M of the group homomorphism $\pi_1(X \setminus Y, q) \to \mathbf{GL}(\mathcal{L}_q)$. By Lemma 53, M is unchanged if we replace Y by a larger closed, nowhere dense analytic subset Y'. That is, if Y' contains Y (but not q), then the image of the homomorphism $\pi_1(X \setminus Y', q)$ is also M.

Suppose U is a good fundamental neighborhood of x relative to Y and $p \in U \setminus Y$. Then the *local monodromy group of* $\mathcal{L}$ *at* y is the image $H = H(y)$ of the composition

$$\pi_1(U \setminus Y, p) \stackrel{\varphi_\gamma}{\to} \pi_1(X \setminus Y, q) \to M$$

where γ is a path from p to q. It depends on the choice of U, γ and p, but only up to conjugacy by an element of M. Like M itself, H is independent of Y in the sense that enlarging Y does not change H.

6 Palindromic Betti numbers and the local invariant cycle theorem

6.1 Main Theorems

A crucial tool in our argument is the local invariant cycle theorem of Beilinson, Bernstein, and Deligne (BBD), which we state here in the generality relevant to this paper.

Theorem 54 ([4, Corollaire 6.2.9])**.** *Suppose $f : X \to Y$ is a proper morphism of smooth, separated, irreducible complex schemes. Let $y \in Y(\mathbb{C})$, and set $X_y := f^{-1}(y)$. Suppose that U is a Zariski dense, Zariski open subset of Y such that the restriction of f to $f^{-1}U$ is smooth. Then, for every sufficiently small ball $B = B(y)$ centered at y as in Theorem 48, the natural map*

$$\mathrm{H}^i(X_y, \mathbb{C}) \longrightarrow \mathrm{H}^0(B(y) \cap U, R^i f_* \mathbb{C}) \tag{55}$$

is a surjection. Moreover, $B(y) \cap U$ is nonempty, and we have

$$\mathrm{H}^0(B(y) \cap U, R^i f_* \mathbb{C}) = \mathrm{H}^i(X_z)^{\pi_1(B(y) \cap U, z)} \tag{56}$$

for any $z \in B(y) \cap U$.

The vector space $\mathrm{H}^0(B(y) \cap U, R^i f_* \mathbb{C})$ is called the space of *local invariant cycles*, and we call the map in (55) (which we will explain in some detail below) the *local invariant cycle map*. The assumption that f is smooth and proper over U implies that the sheaves $R^i f_* \mathbb{C}$ restrict to local systems on U. It follows from

Corollary 52 that, for a fixed U, the spaces $\mathrm{H}^0(B(y)\cap U, R^i f_*\mathbb{C})$ are canonically isomorphic for $B(y)$ sufficiently small. Moreover, as BBD point out, up to a canonical isomorphism, (55) is independent of U.

When Z is a scheme, we write $d_Z := \dim Z$ to save space. This notation is useful in the main result of this section, which is the following.

Theorem 57. *Suppose that $f : X \to Y$ is a projective morphism between smooth, separated, irreducible complex schemes; and let y be a closed point of Y. Set $d = d_X - d_Y$. Then the local invariant cycle map* (55) *is an isomorphism for all $i \in \mathbb{Z}$ if and only if*

$$\dim \mathrm{H}^i(X_y, \mathbb{C}) = \dim \mathrm{H}^{2d-i}(X_y, \mathbb{C}) \tag{58}$$

for all i.

The rest of this section is devoted to a proof of Theorem 57. Our proof uses the ideas behind the proof of Theorem 54 extensively. Somewhat unfortunately for us, in [4], the proof of Theorem 54 in the case of complex varieties is more or less left to the reader to construct using the proof given earlier in the book of the ℓ-adic analogue of the theorem for schemes of finite type over a finite field. While it is probably fairly clear how to do this for anyone who has made it to the last few pages of [4] where Theorem 54 appears, it makes it difficult for us to cite passages in the text where specific results we need are proved. In the original arXiv version of this paper [6], we handled this essentially by assuming that the reader was familiar with the proof of Theorem 54. However, we realized that this approach has serious disadvantages. So, to help make our proof of Theorem 57 as clear and precise as possible, we have decided to include a proof of Theorem 54 in the complex case.

One advantage of this is that we are able to use the Kashiwara conjecture for semisimple perverse sheaves [25], which has been proved by T. Mochizuki and independently by combining work of A.J. de Jong, Drinfeld, Gaitsgory, and Böckle–Khare [31, 32, 9, 13, 15, 5]. This allows us to point out strong forms of Theorems 54 and Theorem 57. See Theorem 84 and Theorem 102 below.

6.2 The local invariant cycle map

6.2.1 Geometric definition

The map (55) in the statement of Theorem 54 can be defined in two equivalent ways, geometrically and sheaf theoretically. We start with the geometric component of the definition. To explain it, write X_S for $f^{-1}S$ when $S \subset X$, and write f_S for the map $X_S \to S$ coming from the restriction of f. Then, for $B = B(y)$ a sufficiently small ball, the restriction morphism

$$\mathrm{H}^i(X_B, \mathbb{C}) \to \mathrm{H}^i(X_y, \mathbb{C}) \tag{59}$$

is an isomorphism. This follows from proper base change. On the other hand, we have a map

$$\mathrm{H}^i(X_{B\cap U}, \mathbb{C}) \to \mathrm{H}^0(B \cap U, R^i f_*\mathbb{C}) \tag{60}$$

coming from the edge homomorphism in the Leray–Serre spectral sequence applied to the fibration $f_{B\cap U}: X_{B\cap U} \to B\cap U$. Composing the map in (60) with the inverse of the map in (59) gives the local invariant cycle map (55).

6.2.2 General definition

Theorem 54 is proved (and even stated) in [4] in a much more general sheaf-theoretic context. This allows for greater generality in the statements, but it also allows for more flexibility in the proof. As we will use this generality to prove Theorem 57, we now explain how to generalize the local invariant cycle map for complexes of sheaves on Y. This essentially involves unwinding the definition of the edge homomorphism in the hypercohomology spectral sequence.

Suppose Y is any scheme of finite type over $\mathbb{C}$. We write $D^b_c Y$ for the bounded derived category of sheaves of complex vector spaces on Y with constructible cohomology. Given a complex $K \in D^b_c Y$ and an integer i, we write $H^i K$ for the ith cohomology sheaf of K. For $j \in \mathbb{Z}$, we write $K[j]$ for the shift of K by j units to the left. So $H^i K[j] = H^{i+j} K$. If $y \in Y(\mathbb{C})$ and $\mathcal{F}$ is a sheaf on Y, then, as usual, $\mathcal{F}_y$ denotes the stalk of $\mathcal{F}$ over y. Similarly, if K is a complex, K_y denotes the object in the derived category of $\mathbb{C}$ vectors spaces obtained by taking stalks. Since taking stalks is exact, we have a canonical isomorphism $H^i(K_y) = (H^i K)_y$. Usually, we simply write $H^i K_y$ for this vector space.

Now, let $j: U \hookrightarrow Y$ denote the inclusion of a Zariski open subset and let $y \in Y(\mathbb{C})$ be a closed point. Adjunction then gives us maps

$$\lambda : H^i K \to j_* j^* H^i K \tag{61}$$

$$\lambda(y) : H^i K_y \to (j_* j^* H^i K)_y \tag{62}$$

which we call the *(generalized) local invariant cycle maps*. Here we get (62) from (61) by taking stalks, and j_* denotes the pushforward of sheaves (not the derived pushforward as it often does in [4]). The functor j^* is just the restriction to U. So for $M \in D^b_c Y$, $j^* M$ is synonymous with $M_{|U}$.

When $K = Rf_*\mathbb{C}$ as in Theorem 54, then $H^i K_y = \mathrm{H}^i(X_y, \mathbb{C})$ by proper base change, and $(j_* j^* H^i K)_y = (j_* j^* R^i f_* \mathbb{C})_y$ which is equal to $\mathrm{H}^0(B(y)\cap U, R^i f_* \mathbb{C})$ for $B(y)$ sufficiently small by the constructibility of the sheaves involved. (Compare with the statement of the local invariant cycle theorem in [4, Corollaire 6.2.9]). Moreover, it is easy to see that $\lambda(y)$ agrees with the geometric description of (55) in §6.2.1.

Note that λ and $\lambda(y)$ are natural in K. To make this explicit, write $\mathrm{Shv}_c Y$ for the category of constructible sheaves of $\mathbb{C}$ vector spaces on Y. Then $K \rightsquigarrow H^i K$ and $K \rightsquigarrow j_* j^* H^i K$ are both additive functors from $D^b_c Y$ to $\mathrm{Shv}_c Y$, and λ is a natural transformation from the first to the second (as it comes from the adjunction, which is itself natural). So write λ^i_K for the map in (61) to keep track of the index and the complex. Then, $\lambda^i_{K[j]} = \lambda^{i+j}_K$, and, for $K_1, K_2 \in D^b_c Y$, $\lambda^i_{K_1\oplus K_2} = \lambda^i_{K_1} \oplus \lambda^i_{K_2}$. Similar remarks hold obviously for $\lambda(y)$.

Following [4], we are going to isolate a class of objects K in $D^b_c\,Y$ on which λ, and thus $\lambda(y)$, turn out to be surjections. However, it might help to start out with a trivial example along with a trivial non-example.

Example 63. Let $Y = \mathbb{A}^1_{\mathbb{C}}$, the affine line and let $j : U \hookrightarrow Y$ denote the inclusion of the complement of the origin. Consider the sheaves $\mathcal{F} = j_!\mathbb{C}_U$ and $\mathcal{G} = \mathbb{C}_Y$ as objects in $D^b_c\,Y$. For $i \neq 0$, both the source and target of λ are 0 for both $\mathcal{F}$ and $\mathcal{G}$. So there is nothing interesting happening. For $i = 0$, λ is an isomorphism for $\mathcal{G}$. But for $\mathcal{F}$ it is the inclusion of $j_!\mathbb{C}_U$ in $\mathbb{C}_Y$, which is not a surjection (in $\mathrm{Shv}_c\,Y$): we have $\mathcal{F}_0 = 0$ while $(j_*j^*\mathcal{F})_0 = \mathbb{C}$.

6.3 Recollections concerning Perverse Sheaves

The definition of the class of objects in $D^b_c\,Y$ we are looking for is tied up with the theory of perverse sheaves. So we will explain the part of that theory that we need here.

For the rest of this subsection, we fix a scheme Y which is reduced and of finite type over $\mathbb{C}$. We point out that everything we are going to do goes on in $D^b_c\,Y$ and is essentially topological with respect to $Y(\mathbb{C})$. So we do not really need the assumption that Y is reduced: if W is a scheme of finite type over $\mathbb{C}$, then $D^b_c\,W$ is the same as $D^b_c\,W_{\mathrm{red}}$. But it makes it slightly more convenient to say certain things. On the other hand, while we do not require Y to be separated, this is mostly to conform to the setup of [4]. In the end, the theorems we are really interested in are local near a closed point in Y. So we could just as well assume that Y is separated (or even affine).

We write $\mathrm{Perv}\,Y$ for the category of perverse sheaves on Y (for the middle perversity). This is a full subcategory of the derived category $D^b_c\,Y$. For $K \in D^b_c\,Y$ and $i \in \mathbb{Z}$, we write ${}^pH^i\,K$ for the ith perverse cohomology sheaf. So, while H^iK is a usual sheaf on Y, ${}^pH^i\,K$ is an object in $\mathrm{Perv}\,Y$. We have ${}^pH^i\,K[j] = {}^pH^{i+j}\,K$.

Suppose $j : V \hookrightarrow Y$ is a (locally closed) immersion of schemes, and K is a perverse sheaf on V. Then we write $j_{!*}K$ for the intermediate extension of K to Y, a perverse sheaf on Y supported on the Zariski closure of V. The intermediate extension is an extension of K in the sense that there is a natural isomorphism $j^*j_{!*}K = K$. In other words, the restriction of the intermediate extension to V is just K. In fact, we have the following characterization of the intermediate extension.

Theorem 64 (BBD)**.** *The intermediate extension $j_{!*}K$ is the unique extension of K in* $\mathrm{Perv}\,Y$ *supported on $\overline{V}$ with no nontrivial sub or quotient object supported on $\overline{V} \setminus V$.*

Proof. This follows from [4, Corollaire 1.4.25]. □

Proposition 65. *Suppose V is a (locally closed) subscheme of Y and K is a perverse sheaf on V. Then*

$$H^i(j_{!*}K) = 0 \text{ for } i \notin [-d_V, 0]. \tag{66}$$

Moreover, if we write $j : V \hookrightarrow Y$ for the inclusion, then

$$H^{-d_V}(j_{!*}K) = j_* H^{-d_V} K. \tag{67}$$

Proof. Since K is perverse, so is $j_{!*}K$, and it is supported on the Zariski closure $\overline{V}$ of V. Therefore $H^i j_{!*}K = 0$ for $i > 0$ by [4, Definition 2.1.2]. But, by the discussion in the two paragraphs just after Definition 2.1.2, $H^i j_{!*}K = 0$ for $i < -d_V$. So (66) is proved.

Since $j_{!*}K$ is an extension of K supported on $\overline{V}$, $j^* H^{-d_V}(j_{!*}K) = H^{-d_V}K$. So adjunction gives a natural morphism $H^{-d_V}(j_{!*}K) \to j_* H^{-d_V}K$. Now (67) follows easily from Deligne's formula [4, Proposition 2.1.11], which computes $j_{!*}K$ in terms of a series of derived pushforwards and truncations. $\square$

If V is smooth and irreducible and $\mathcal{L}$ is a local system on V, then $\mathcal{L}[d_V]$ is a perverse sheaf on V. So the intermediate extension $\mathrm{IC}\,\mathcal{L} := j_{!*}\mathcal{L}[d_V]$ is a perverse sheaf on Y, which is colloquially called the *IC* sheaf or the *intersection cohomology* sheaf. For $d_V > 0$, Deligne's formula actually implies that $H^i\,\mathrm{IC}(\mathcal{L}) = 0$ for $i \notin [-d_V, 0)$. So we get a slightly stronger vanishing statement than (66). Note that, if V' is a Zariski dense, Zariski open subset of V, then $\mathrm{IC}(\mathcal{L}_{|V'}) = \mathrm{IC}\,\mathcal{L}$, i.e., the two perverse sheaves are canonically isomorphic [4, Lemme 4.3.2].

By [4, Theorem 4.3.1], a perverse sheaf K on Y is simple (as an object in the abelian category $\mathrm{Perv}\,Y$) if and only if it is isomorphic to $\mathrm{IC}\,\mathcal{L}$ where $\mathcal{L}$ is an irreducible local system on a smooth, irreducible subscheme V as above. So, a perverse sheaf K is semisimple if and only if it is a direct sum of such sheaves. (Such a direct sum is necessarily finite because the category of perverse sheaves is artinian.) Suppose Z is a closed subvariety of Y (that is, Z is an integral, closed subscheme). Following M. Saito's notation from [40], we say that a perverse sheaf K has *strict support* Z if it is supported on Z and has no proper sub or quotient object supported on a proper subscheme of Z. We write $\mathrm{Perv}_Z\,Y$ for the full subcategory of $\mathrm{Perv}\,Y$ consisting of perverse sheaves with strict support Z. By Theorem 64, if $\mathcal{L}$ is a local system on a non-empty, smooth, Zariski open subscheme V of Z, then $\mathrm{IC}\,\mathcal{L}$ has strict support Z. It follows that any semisimple perverse sheaf K on Y can be written as a direct sum

$$K = \oplus_Z K_Z \tag{68}$$

where Z ranges over all closed subvarieties of Y and $K_Z \in \mathrm{Perv}_Z\,Y$ (with $K_Z = 0$ for all but finitely many Z). This decomposition is easily seen to be unique (as there are no nonzero morphisms between objects in $\mathrm{Perv}_Z\,Y$ and $\mathrm{Perv}_{Z'}\,Y$ for $Z \neq Z'$).

Obviously, if K is simple, then we must have $K_Z = 0$ for all but one irreducible closed subscheme Z of Y. We call this subscheme the *strict support of* K.

Lemma 69. *Suppose K is a perverse sheaf on a scheme Y of finite type over $\mathbb{C}$, and let $j : V \hookrightarrow Y$ denote the inclusion of a Zariski dense, Zariski open* *subset. Then j^*K is perverse on V. If K is simple with strict support equal to an irreducible component of Y, then*

$$K = j_{!*}j^*K. \tag{70}$$

Proof. If j is an open immersion (or even an étale morphism), then j^* always takes perverse sheaves to perverse sheaves [4, Corollaire 2.2.6 (ii)]. This proves the first assertion.

Now suppose K is simple with strict support equal to an irreducible component Z of Y. Then K is an extension of j^*K with no non-trivial sub or quotient object supported on $Z \setminus V$. Since, by Theorem 64, $j_{!*}j^*K$ is the unique such extension, it follows that $K = j_{!*}j^*K$. $\square$

Lemma 71. *Suppose Y is a scheme of finite type over $\mathbb{C}$ and let $\{\mathcal{F}\}_{i=1}^n$ be a finite collection of sheaves on Y. If $\mathcal{F} := \oplus\mathcal{F}_i$ is a local system, then each $\mathcal{F}_i$ is as well.*

Proof. Any idempotent $p \in \operatorname{End}\mathcal{F}$ is locally constant on $\mathcal{F}$. In particular, p has locally constant rank. So $\ker p$ and $\ker 1-p$ are local systems. The result then follows by induction. $\square$

Definition 72. Suppose U is a Zariski open subset of Y and $K \in D_c^b Y$. We say that U is a *mollifying* subset for K if U is Zariski dense in Y and, for all $i \in \mathbb{Z}$, $H^iK_{|U}$ is a local system on U.

Lemma 73. *Suppose $K \in D_c^b Y$. Then K has a mollifying subset. In fact, we can even find one which is smooth.*

Proof. This follows from generic smoothness and the definition of a constructible sheaf. $\square$

Remark 74. It turns out that we will not really need the existence of smooth mollifying subsets, and in [4, Corollaire 6.2.9], BBD do not use it.

Proposition 75. *Suppose K is a simple perverse sheaf on Y with strict support Z, and $j : U \hookrightarrow Y$ is the inclusion of a mollifying subset for K.*

1. *If Z is an irreducible component of Y, then $K_{|U} = \mathcal{M}[d_Z]$ where $\mathcal{M} := H^{-d_Z}K_{|U}$, and $K = j_{!*}\mathcal{M}[d_Z]$. Moreover, $j_*j^*H^iK = 0$ for $i \neq -d_Z$ and the local invariant cycle map $\lambda_K^{-d_Z}$ is an isomorphism.*

2. *Otherwise $j^*K = 0$. Therefore $j_*j^*H^iK = 0$ for all i.*

Proof. Suppose V is an irreducible component of U. If Z does not contain V, then, for each i, $H^iK_{|V}$ is a local system on V supported on the closed, proper subscheme $Z \cap V$ of V. Since V is connected, it follows that $H^iK_{|V} = 0$ for all i. This proves (2).

So assume Z is an irreducible component of Y and let $V = Z \cap U$. Since U is dense in Y, V is dense in Z. So $d_V = d_Z$. We then have $H^iK = 0$

for $i \notin [-d_Z, 0]$ by Proposition 65. On the other hand, if $i > -d_Z$, then $\dim \operatorname{supp} H^i K_{|V} \le -i < d_Z$. So again $H^i K_{|V} = 0$ as it is a local system. Therefore $H^i K_{|U} = 0$ for all $i \neq -d_Z$, and it follows that $K_{|U} = \mathcal{M}[d_Z]$ where $\mathcal{M} = H^{-d_Z} K_{|U}$. This shows that $j_* j^* H^i K = 0$ for $i \neq -d_Z$.

By Lemma 69, we then get that $K = j_{!*}\mathcal{M}[d_Z]$. Then, by Proposition (67), $H^{-d_Z} K = H^{-d_Z}(j_{!*}\mathcal{M}[d_Z]) = j_* H^{-d_Z}(\mathcal{M}[d_Z]) = j_* j^* H^{-d_Z} K$. So the local invariant cycle map $\lambda_K^{-d_Z}$ is an isomorphism. □

Corollary 76. *Suppose K is a simple perverse sheaf on Y, and $j : U \hookrightarrow Y$ is the inclusion of a mollifying open subset for K.*

1. *The local invariant cycle map $\lambda_K^i : H^i K \to j_* j^* H^i K$ is a surjection for all i.*

2. *Suppose every irreducible component of Y has dimension d_Y and suppose i is an integer not equal to $-d_Y$. Let $y \in Y(\mathbb{C})$ be a closed point of Y, and write $\lambda_K^i(y)$ for the map in (62). Then $\lambda_K^i(y)$ is an isomorphism if and only if $H^i K_y = 0$.*

3. *If Y is equidimensional as in (2), then $\lambda_K^{-d_Y}$ is an isomorphism.*

Proof. Proposition 75 shows that either λ_K^i is an isomorphism or $j_* j^* H^i K = 0$. So (1) holds, because, in either case, λ_K^i is a surjection. Similarly, (3) holds because, in case (1) of Proposition 75, $\lambda_K^{-d_Z} = \lambda_K^{-d_Y}$ was proven to be an isomorphism and, otherwise, $H^{-d_Y} K = 0$ by Proposition 65, which, by (1), trivially implies that $\lambda_K^{-d_Y}$ is an isomorphism.

For (2), suppose $i \neq -d_Y$. Then $j_* j^* H^i K = 0$, again by Proposition 75. Therefore, $\lambda_K^i(y)$ is an isomorphism if and only if $H^i K_y = 0$.

□

Corollary 77. *Suppose K is a simple perverse sheaf on Y and U and V are two mollifying open subsets for K with inclusions j_U (resp. j_V) into Y. Then the local invariant cycle maps for U and V are canonically isomorphic. More precisely, $U \cap V$ is also a mollifying subset, and, if we let $j_{U\cap V} : U \cap V \hookrightarrow Y$ denote the inclusion, then, for each $i \in \mathbb{Z}$, we have have a commutative diagram*

$$\begin{tikzcd}
 & j_{U*}j_U^* H^i K \arrow[dr, "\mathrm{Res}"] & \\
H^i K \arrow[ur, "\lambda_K^i"] \arrow[rr, "\lambda_K^i"] \arrow[dr, "\lambda_K^i"'] & & j_{(U\cap V)*}j_{U\cap V}^* H^i K \\
 & j_{V*}j_V^* H^i K \arrow[ur, "\mathrm{Res}"'] &
\end{tikzcd}$$

where Res *denotes restriction. Moreover, each map labeled* Res *is an isomorphism.*

Proof. It is obvious that $U \cap V$ is a mollifying subset, and it is also very easy to see that the diagram above commutes. So, let Z be the strict support of K. If Z is not an irreducible component of Y or if $i \neq -d_Z$, then, by Proposition 75, the right three vertices are all 0. So there is nothing to prove.

Otherwise, all of the arrows labeled λ_K^i are isomorphisms. So the commutativity of the diagram shows that the maps labeled Res are all isomorphisms. $\square$

Definition 78. Suppose Y is a scheme of finite type over $\mathbb{C}$. An object $K \in D_c^b Y$ is said to be *semisimple* if $K \cong \oplus_{i \in \mathbb{Z}} ({}^p H^i K)[-i]$ with each summand a semisimple perverse sheaf.

Remark 79. If $K \in D_c^b Y$, then it is not hard to see (directly from the definitions in [4]) that ${}^p H^i K = 0$ for all but finitely many $i \in \mathbb{Z}$.

Lemma 80. *Suppose K is a semisimple object in $D_c^b Y$ and U is a Zariski dense, Zariski open subset of Y. Then U is mollifying for K if and only if it is mollifying for every sub perverse sheaf of every perverse cohomology sheaf ${}^p H^i K$.*

Proof. Since the direct sum of local systems is a local system, it is obvious that, if U mollifies all the ${}^p H^i K$, it mollifies K as well. This proves one direction of the assertion. The converse direction follows from Lemma 71. $\square$

Theorem 81. *Suppose K is a semisimple object in $D_c^b Y$ and let U be mollifying for K. Then, for each i, the local invariant cycle map λ_K^i is a surjection.*

Proof. Using Lemma 80 along with the naturality of λ explained in §6.2.2, we can assume that K is a simple perverse sheaf. Then the result follows from Corollary 76 (1). $\square$

Theorem 81 and the decomposition theorem, [4, Théorème 6.2.5], are the main ingredients in the proof of Theorem 54. Analogously, the main ingredients of the proof of Theorem 57 are the decomposition theorem, the perverse hard Lefschetz theorem and the next result.

Theorem 82. *Suppose that Y is equidimensional and K is a semisimple object in $D_c^b Y$. Let $y \in Y(\mathbb{C})$ be a closed point, and let $j : U \hookrightarrow Y$ be the inclusion of a mollifying subset for K. Then the following are equivalent.*

1. *$\lambda(y) : H^i K_y \to (j_* j^* H^i K)_y$ is an isomorphism for all i.*

2. *For all j and all $i \neq -d_Y$, $H^i({}^p H^j K)_y = 0$.*

Proof. By shifting and passing to direct summands via 80, we can assume that K is a simple perverse sheaf. The theorem is then a direct consequence of Corollary 76 (2) and (3). $\square$

6.4 Proof of the Local Invariant Cycle Theorem

Suppose now that X is a reduced scheme of finite type over $\mathbb{C}$ and $K \in D^b_c X$. If $i : W \to X$ is the inclusion of a subscheme, we write $\mathrm{H}^j(W, K) := \mathrm{H}^j(W, i^*K)$.

Suppose Y is another reduced scheme of finite type over $\mathbb{C}$ and $y \in Y(\mathbb{C})$ is a closed point. Then by a *ball* centered at y, we mean any open neighborhood of y in $Y(\mathbb{C})$ which is obtained by intersecting an affine open neighborhood V of y embedded in $\mathbb{A}^n_{\mathbb{C}}$ with a ball in $\mathbb{C}^n$.

Remark 83. We have to distinguish this notion of a ball from the notion of a ball as in Theorem 48 which we use when Y is smooth. The issue is that, if Y is singular, then we can not necessarily find an open neighborhood of y which is homeomorphic to an actual ball in $\mathbb{C}^{d_Y}$.

In the proof of the next theorem, we are going to use the Kashiwara conjecture. As we mentioned above in §6.1, this is now a theorem owing to the work of many authors. We will not cite these authors again here, but we point out that Drinfeld's paper [13] is short and has a very efficient statement of the part of the conjecture having to do with perverse sheaves (which is the part that we use).

Theorem 84. *Suppose $f : X \to Y$ is a proper morphism of reduced schemes of finite type over $\mathbb{C}$, and let K be a semisimple object in $D^b_c X$. Let $y \in X(\mathbb{C})$ be a closed point and set $X_y = f^{-1}(y)$. Let $U \subset Y$ be a mollifying open subset for the complex $Rf_*\mathbb{C}$. Then, for every sufficiently small ball $B(y)$ centered at y, the local invariant cycle map induces a natural surjection*

$$\mathrm{H}^i(X_y, K) \twoheadrightarrow \mathrm{H}^0(B(y) \cap U, R^i f_* K). \tag{85}$$

Moreover, the target of (85) *is independent of the mollifying open subset U.*

Proof. First note that, since the whole theorem is a Zariski local statement near y, we can easily reduce to the situation where Y (and, therefore, X) are separated. For example, we can replace Y with an affine Zariski open neighborhood of y and X with the inverse image of that neighborhood. So we assume X and Y are separated. (We do this in order to freely use work on Kashiwara's conjecture.)

The source of (85) is naturally isomorphic to $H^i K_y$ by proper base change. The target is naturally isomorphic to $(j_*j^*H^iK)_y$ for $B(y)$ sufficiently small. Since K is semisimple, it follows from the Kashiwara conjecture that Rf_*K is semisimple. Therefore the surjectivity result follows from Theorem 81.

The independence of U follows from Corollary 77. □

Proof of Theorem 54. Since X is smooth, the constant sheaf $\mathbb{C} = \mathbb{C}_X$ is semisimple. In fact, we even assumed that X is irreducible, so $\mathbb{C}[d_X]$ is the IC sheaf of the simple local systems $\mathbb{C}_X$. Therefore $\mathbb{C}[d_X]$ is a simple perverse sheaf. Since f is smooth and proper over U, U is a mollifying subset for Rf_*K. So take $K = \mathbb{C}$ in Theorem 84. This proves that (55) of Theorem 54 holds.

Since a ball $B(y)$ as in Theorem 48 is homeomorphic to an open ball in $\mathbb{C}^{d_Y}$, $B(y) \cap U$ is non-empty and connected. This follows from Fact 47 in the case

$d_Y > 0$ and it is obvious otherwise. Then (56) follows from proper base change and Corollary 40. $\square$

Remark 86. In [4], BBD prove Theorem 84 for K semisimple of geometric origin. (These are essentially the complexes that can be obtained from the constant sheaf by the standard operations of sheaf theory, e.g., Grothendieck's six operations.) The reason for the restriction was that they were only able to prove the decomposition theorem [4, Théorème 6.2.5] for such complexes.

BBD also state Theorem 84 for arbitrary schemes of finite type (without the restriction that X or Y be reduced). But, as we mentioned at the beginning of §6.3, there is actually no loss in generality in assuming that the schemes involved are reduced.

6.5 The Palindromicity Theorem

It will be convenient to introduce some notation concerning palindromic polynomials.

Definition 87. Suppose $p \in \mathbb{R}[t, t^{-1}]$ is a Laurent polynomial. We say that p is *palindromic* if $p(t) = p(t^{-1})$.

Lemma 88. *Suppose $p \in \mathbb{R}[t, t^{-1}]$ is palindromic. Then $p'(1) = 0$.*

Proof. By palindromicity and the chain rule,

$$p'(t) = \frac{d}{dt} p(t^{-1}) = -p'(t^{-1})t^{-2}.$$

So $p'(1) = -p'(1)$. Therefore, $p'(1) = 0$. $\square$

If p is any Laurent polynomial, we think of $p'(1)$ as the *center of mass* of p.

Lemma 89. *Suppose $q = \sum_{\ell \geq 0} p_\ell t^\ell$ where the p_ℓ are palindromic Laurent polynomials with real coefficients. Assume that, for $\ell > 0$, the coefficients of the p_ℓ are non-negative. Then*

$$q'(1) \geq 0. \tag{90}$$

Moreover, the following are equivalent:

1. *We have equality in* (90).
2. *$p_\ell = 0$ for all $\ell > 0$.*
3. *q is palindromic.*

Proof. Since the p_ℓ are all palindromic, we have $p'_\ell(1) = 0$ for all $\ell \geq 0$. Therefore

$$q'(1) = \sum_{\ell > 0} \ell p_\ell(1). \tag{91}$$

Since the coefficients of the p_ℓ are non-negative for $\ell > 0$, (91) implies (90). From (91), it is also immediate that (1) implies (2). Then (2)⇒ (3) is obvious, and (3)⇒ (1) follows from Lemma 88. $\square$

Theorem 92. *Suppose Y is a scheme of finite type over $\mathbb{C}$ and $K \in D^b_c Y$ is a semisimple complex. Assume that*

$$ {}^pH^{-i} K \cong {}^pH^i K \tag{93}$$

for all $i \in \mathbb{Z}$. Fix $y \in Y(\mathbb{C})$ and $r \in \mathbb{Z}$, and suppose that

$$H^j({}^pH^i K)_y = 0 \text{ for } i, j \in \mathbb{Z} \text{ such that } j < r. \tag{94}$$

Set

$$q(t) := \sum_{k\in\mathbb{Z}} t^k \dim H^k(K[r])_y. \tag{95}$$

Then $q'(1) \geq 0$. Moreover, the following are equivalent:

1. *$q'(1) = 0$.*
2. *We have*
$$H^j({}^pH^i K)_y = 0 \text{ for } j \neq r. \tag{96}$$
3. *q is palindromic.*

Proof. Since K is semisimple, we have

$$K = \oplus({}^pH^i K)[-i]. \tag{97}$$

Here we sum over $i \in \mathbb{Z}$, but, since this is clear from the context, we leave this out of summation notation to save clutter (as we will do in the rest of the proof as well).

Substituting in (97) for K in (95), we get

$$\begin{aligned}
q(t) &= \sum t^k \dim H^k(K[r])_y \\
&= \sum_{k\in\mathbb{Z}} t^k \sum_{i\in\mathbb{Z}} \dim H^k\left(({}^pH^i K)[r-i]\right)_y \\
&= \sum t^k \dim H^{k+r-i}({}^pH^i K)_y && (98)\\
&= \sum_{\ell,i\in\mathbb{Z}} t^{i+\ell} \dim H^{\ell+r}({}^pH^i K)_y, && (99)\\
&= \sum_{\ell} t^\ell \sum_i t^i \dim H^{\ell+r}({}^pH^i K)_y. && (100)
\end{aligned}$$

where here we go from (98) to (99) by substituting $k = i + \ell$.

Now, for $\ell \in \mathbb{Z}$, set

$$p_\ell := \sum_{i\in\mathbb{Z}} t^i \dim H^{\ell+r}({}^pH^i K)_y. \tag{101}$$

By (94), $p_\ell = 0$ for $\ell < 0$. So, by 99, $q = \sum_{\ell\geq 0} t^\ell p_\ell$. But, by (93), each p_ℓ is palindromic. The theorem now follows from Lemma 89. $\square$

Theorem 102. *Suppose $f : X \to Y$ is a projective morphism of reduced schemes of finite type over $\mathbb{C}$ with Y equidimensional. Let $y \in Y(\mathbb{C})$ be a closed point and let K be a semisimple complex in* $\operatorname{Perv} X$*. Set*

$$q(t) := \sum_{k\in\mathbb{Z}} t^k \dim \mathrm{H}^{k-d_Y}(X_y, K). \tag{103}$$

Then $q'(1) \geq 0$. Moreover, the following are equivalent:

1. *$q'(1) = 0$.*
2. *The local invariant cycle maps in* (85) *are isomorphisms for all i.*
3. *q is palindromic.*

Proof. As in the proof of Theorem 84, we can (and do) assume that Y is separated. Then, by Kashiwara's conjecture, Rf_*K is a semisimple complex and, for each i, we have an isomorphism ${}^pH^{-i}\, Rf_*K \cong {}^pH^i\, Rf_*K$. (This is the "Hard Lefschetz" part of the Kashiwara conjecture, Item 2 in Drinfeld's statement [13].) On the other hand, we have a canonical isomorphism $\mathrm{H}^{k-d_Y}(X_y, K) = \mathrm{H}^k(X_y, Rf_*K[-d_Y]) = H^k(Rf_*K[-d_Y])_y$. Finally, for any complex C on Y, we have $H^j({}^pH^i\, C) = 0$ for $j < -d_Y$ by Proposition 65. So, in particular, $H^j({}^pH^i\, Rf_*K)_y = 0$ for $j < -d_Y$.

Now, by Theorem 82 we have $H^j({}^pH^i\, Rf_*K)_y = 0$ for all $j \neq -d_Y$ for all j if and only if the local invariant cycle maps are all isomorphisms. So the theorem now follows from Theorem 92. ☐

Proof of Theorem 57. Since X is smooth and irreducible, $\mathbb{C}[d_X]$ is semisimple perverse. So let $K = \mathbb{C}[d_X]$ in Theorem 102. Set

$$q(t) = \sum_{i\in\mathbb{Z}} t^i \dim \mathrm{H}^i(X_y, \mathbb{C}[d]) = \sum_{i\in\mathbb{Z}} t^i \dim \mathrm{H}^i(X_y, \mathbb{C}[d_x - d_Y]).$$

Then q is palindromic if and only if (58) holds. So the result follows from Theorem 102. ☐

7 Galois covers

The purpose of this section is to prove a lemma about the local monodromy groups of Galois covers. We use the concepts of §5, but we have changed some of the notation (partially to avoid running out of capital letters towards the end of the alphabet).

7.1 Covers and monodromy

Suppose U is a smooth, connected, complex, quasi-projective variety and G is a finite group acting freely on U. Let $V = U/G$, and write $\pi : U \to V$ for the quotient morphism. (It is well known that U/G is a scheme. See, for example, [46, Tag 0725].)

For each (closed) point $u \in U$, we get a surjective group homomorphism

$$\psi_u : \pi_1(V, \pi(u)) \twoheadrightarrow G. \tag{104}$$

If $\gamma : [0,1] \to V$ represents an element of $\pi_1(V, \pi(u))$ and $\tilde{\gamma}$ is a lift of γ to U with $\tilde{\gamma}(0) = u$, then $\psi_u(\gamma)u = \tilde{\gamma}(1)$. From this description, we see that, if $\pi(u') = \pi(u)$, then ψ_u and $\psi_{u'}$ differ by conjugation by an element of $\pi_1(V, \pi(u))$. (See [24] for a complete discussion of these matters.)

Now, suppose that V is contained as a Zariski open subset of a smooth, quasi-projective variety Y. Set $Z = Y \setminus V$, and suppose z is a closed point of Z. Let W be a good neighborhood of z in Y relative to Z, and let w be a point in $W \setminus Z$. The choice of a path from w to $\pi(u)$ gives us a sequence of group homomorphisms

$$\pi_1(W \setminus Z, w) \to \pi_1(V, \pi(u)) \twoheadrightarrow G. \tag{105}$$

Moreover, up to conjugation by an element of G, this map is independent of u, the path from w to $\pi(u)$, W and w. We call the image $H(z)$ of the composition in (105) the *local monodromy subgroup at z*. (The conjugacy class of $H(z)$ is independent of any choices.) Note that, if we replace V with any non-empty Zariski open subset V' of V containing $\pi(u)$ and we replace U with $\pi^{-1}(V')$, then $H(z)$ does not change. This follows from Lemma 53.

Proposition 106. *Let G be a finite group acting on a smooth, quasi-projective variety X, and suppose G acts freely on a Zariski dense open subset U of X. Suppose $\pi : X \to Y$ is the quotient of X by G and let $V = \pi(U)$. Pick a closed point $x \in X \setminus U$, and suppose that Y is smooth. Then $H(\pi(x))$ is the stabilizer G_x of the point x.*

Proof. Take a good neighborhood B of $y := \pi(x)$ with respect to $Z := Y \setminus V$. Pick $b \in B \cap V$ and set $A = \pi^{-1}B$. Let A_x denote the component of A containing x. There exists $a \in A_x$ such that $\pi(a) = b$. Let H denote the image of the composition $\pi_1(B \cap V, b) \to \pi_1(V, b) \twoheadrightarrow G$, where the last homomorphism is ψ_a. Then $H = H(a)$. The group G acts transitively on the connected components of $A \cap U$, and the stabilizer of the component of $A \cap U$ containing a is H. Since $A_x \cap U$ is connected, $A_x \cap U$ is this component. So the stabilizer in G of this component is the same as the stabilizer of the A_x. But, by possibly shrinking B, we can arrange that this is just G_x. □

Remark 107. We use the assumption that Y is smooth because we have only defined the local fundamental group in that case. However, since Y is a quotient of a smooth variety, it is automatically normal. And good neighborhoods of normal quasi-projective varieties are connected. (See Chapter 3 of Mumford's [33]). It follows that the assumption that Y is smooth can be dropped.

8 Geometry of Hessenberg Schemes

In this section, we study the geometry of the family of Hessenberg varieties over the space of regular matrices. We also study the family of maximal tori defined by centralizers of regular, semisimple matrices. Ngô's paper on the Hitchin fibration [34] significantly influenced our thinking about these matters, and we have consequently borrowed Ngô's notation.

8.1 Regular matrices

Fix a positive integer n and write $\mathfrak{g}$ for the Lie algebra $\mathfrak{gl}_n$. Recall that a matrix $s \in \mathfrak{g}$ is regular if the Jordan blocks of s have distinct eigenvalues. A matrix s is regular if and only if its centralizer has dimension n. As in §1, we say that s is regular of type λ for a partition λ of n if the Jordan blocks of s are of sizes $\lambda_1, \dots \lambda_r$. We write $\mathfrak{g}^{\mathrm{r}}$ for the subset of regular matrices and $\mathfrak{g}^{\mathrm{r}}_\lambda$ for the subset of regular matrices of type λ. We write $\mathfrak{g}^{\mathrm{rs}}$ for the subset of regular semisimple matrices. This is a dense open subset of $\mathfrak{g}$.

8.2 Hessenberg schemes

Fix a Hessenberg function $\mathbf{m} = (m_1, \dots, m_{n-1})$ with $\mathbf{m}(i) := m_i$, and set $m_n = n$. Write $\mathscr{X}$ for the variety of complete flags in $\mathbb{C}^n$, and set

$$\mathscr{H}(\mathbf{m}) := \{(F, s) \in \mathscr{X} \times \mathfrak{g} : sF_i \subseteq F_{m_i} \text{ for } 1 \le i \le n\}.$$

Note that the projection pr_1 on the first factor makes $\mathscr{H}(\mathbf{m})$ into a vector bundle of rank $\sum_{i=1}^n m_i$ over $\mathscr{X}$. So $\mathscr{H}(\mathbf{m})$ is a smooth, connected scheme with

$$\dim \mathscr{H}(\mathbf{m}) = \dim \mathscr{X} + \sum_{i=1}^n m_i = \frac{n(n-1)}{2} + \sum_{i=1}^n m_i. \tag{108}$$

Let $\pi : \mathscr{H}(\mathbf{m}) \to \mathfrak{g}$ denote the projection on the second factor. Then the fiber of π over a matrix $s \in \mathfrak{g}$ is simply the Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$. Note that π is smooth over $\mathfrak{g}^{\mathrm{rs}}$.

Theorem 109. *The map $\pi : \mathscr{H}(\mathbf{m}) \to \mathfrak{g}$ is flat over the locus $\mathfrak{g}^{\mathrm{r}}$ of regular matrices.*

Proof. Both $\mathscr{H}(\mathbf{m})$ and $\mathfrak{g}$ are smooth over $\mathbb{C}$, and, by Corollary 37, all fibers of π over $\mathfrak{g}^{\mathrm{r}}$ have the same dimension, $|\mathbf{m}|$. It follows from the theorem which is sometimes called "miracle flatness" that the restriction of π to the inverse image of $\mathfrak{g}^{\mathrm{r}}$ is flat. (See [23, Ex. III.10.9] for miracle flatness.) □

8.3 Diagonal matrices and characteristics

Write $G := \mathbf{GL}_n$ and write $\mathbf{D}$ for the diagonal subgroup of G. Write $\mathbf{d}$ for the Lie algebra of $\mathbf{D}$, and $\mathbf{d}_r$ for the regular elements of $\mathbf{d}$. The symmetric group S_n acts on $\mathbf{d} = \mathbb{A}^n$ in the obvious way: $\sigma(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$.

The quotient is the characteristic variety $\mathbf{car} = \mathbf{car}_G = \mathbf{d}/S_n$. We can view $\mathbf{car}$ as the variety of monic polynomials $t^n + a_{n-1}t^{n-1} + \cdots a_0$ of degree n. The *Chevalley morphism* $\chi : \mathfrak{g} \to \mathbf{car}$ is the morphism sending a matrix $s \in \mathfrak{g}$ to its characteristic polynomial $\chi(s)$.

8.4 The smallest Hessenberg scheme

Let $\boldsymbol{\ell}$ denote the Hessenberg function $\boldsymbol{\ell}(i) = i$. Then the restriction $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \to \mathfrak{g}^{\mathrm{rs}}$ of $\pi : \mathscr{H}(\boldsymbol{\ell}) \to \mathfrak{g}$ to the inverse image of $\mathfrak{g}^{\mathrm{rs}}$ is an étale cover of degree $n!$. Since $\mathscr{H}(\boldsymbol{\ell})$ is connected and smooth, so is $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$. So, since $\mathfrak{g}^{\mathrm{rs}}$ is also connected, $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \to \mathfrak{g}^{\mathrm{rs}}$ is an étale cover corresponding to an index $n!$ subgroup of the fundamental group of $\mathfrak{g}^{\mathrm{rs}}$.

We call $\mathscr{H}(\boldsymbol{\ell})$ the smallest Hessenberg scheme because, for any Hessenberg function $\mathbf{m}$, there is a canonical inclusion $\mathscr{H}(\boldsymbol{\ell}) \to \mathscr{H}(\mathbf{m})$ (which is a closed immersion).

We have a morphism $\tilde{\chi} : \mathscr{H}(\boldsymbol{\ell}) \to \mathbf{d}$ sending a pair (s, F) to the diagonal matrix with $\mathrm{Gr}_i^F s$ in the (i,i)-entry. This gives rise to a commutative diagram

$$\begin{tikzcd} \mathscr{H}(\boldsymbol{\ell}) \arrow[r, "\tilde{\chi}"] \arrow[d, "\pi"] & \mathbf{d} \arrow[d, "\chi_{|\mathbf{d}}"] \\ \mathfrak{g} \arrow[r, "\chi"] & \mathbf{car}, \end{tikzcd} \tag{110}$$

which coincides with Grothendieck's simultaneous resolution of χ. (For a discussion of Grothendieck's resolution, see Springer [45, §4.1] or Slodowy [44, §4.7].) The restriction of (110) to the inverse image of $\mathbf{car}^{\mathrm{rs}} := \mathbf{d}_{\mathrm{r}}/S_n$ is a pullback diagram. In other words, $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) = \mathfrak{g}^{\mathrm{rs}} \times_{\mathbf{car}^{\mathrm{rs}}} \mathbf{d}_{\mathrm{r}}$. This shows that $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$ is a (connected) Galois cover of $\mathfrak{g}^{\mathrm{rs}}$ with Galois group S_n.

We can describe this Galois covering a little bit more explicitly if we introduce the closed subscheme Z of $(\mathbb{P}^{n-1})^n \times \mathfrak{g}^{\mathrm{rs}}$ consisting of ordered tuples $([v_1], \dots, [v_n]; y)$ where the v_i form a basis of eigenvectors of y. Given a point $z = ([v_1], \dots, [v_n]; y)$ in Z, we can define a complete flag $F(z)$ by setting $F_i = \langle v_1, \dots, v_i \rangle$. This defines a morphisms $Z \to \mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$ given by $z \mapsto (F(z), y)$. Using the fact that Z and $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$ are both étale covers of $\mathfrak{g}^{\mathrm{rs}}$ of the same degree, it is easy to see that $Z \to \mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$ is an isomorphism. Then S_n acts on Z by permuting the v_i: $\sigma([v_1], \dots, [v_n]; y) = ([v_{\sigma^{-1}(1)}], \dots, [v_{\sigma^{-1}(n)}], y)$. It is easy to see that the map $\tilde{\chi} : \mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \to \mathbf{d}_{\mathrm{r}}$ is S_n-equivariant.

8.5 The fundamental groups

Suppose $z = (F, y) \in \mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$. We get a surjection $\psi_z : \pi_1(\mathfrak{g}^{\mathrm{rs}}, y) \twoheadrightarrow S_n$ corresponding to the Galois covering $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \to \mathfrak{g}^{\mathrm{rs}}$ with Galois group S_n. Similarly, for a regular diagonal matrix $u \in \mathbf{d}_r$, we have a surjection $\psi_u : \pi_1(\mathbf{car}^{\mathrm{rs}}, \chi(u)) \to$

S_n. Since

$$\begin{tikzcd} \mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \arrow[r, "\tilde{\chi}"] \arrow[d] & \mathbf{d}_{\mathrm{r}} \arrow[d, "\chi|_{\mathbf{d}_{\mathrm{r}}}"] \\ \mathfrak{g}^{\mathrm{rs}} \arrow[r, "\chi"] & \mathbf{car}^{\mathrm{rs}}. \end{tikzcd}$$

is a pullback diagram of Galois étale covers with Galois group S_n, we have $\psi_z = \psi_{\tilde{\chi}(z)} \circ \chi_*$.

Definition 111. We say a polynomial $p \in \mathbf{car}$ is of type λ if $p = \prod_{i=1}^{\ell}(x-x_i)^{\lambda_i}$ where $x_1, \ldots, x_\ell$ are distinct.

Lemma 112. *Suppose $p = \prod_{i=1}^{\ell}(x-x_i)^{\lambda_i}$ is a polynomial of type λ. Then the local monodromy subgroup $H(p)$ of S_n at p for the S_n-cover $\mathbf{d_r} \to \mathbf{car}^{\mathrm{rs}}$ is S_λ.*

Proof. Let τ denote the diagonal matrix

$$\mathrm{diag}(x_1, \ldots, x_1, x_2 \ldots, x_2, \ldots, x_r, \ldots, x_r).$$

Then the stabilizer in S_n of τ is precisely S_λ. The results then follows from Proposition 106. □

8.6 The Kostant section

The Kostant section is a morphism $\epsilon : \mathbf{car} \to \mathfrak{g}^{\mathrm{r}}$ which is a section of χ; i.e., $\chi \circ \epsilon = \mathrm{id}$. We give the definition of ϵ following Ngô's paper [34, Theorem 2.1]. We remark, however, that, while the general definition makes sense for any reductive Lie algebra, we only discuss it for $\mathfrak{gl}_n$.

Let $\mathbf{x}_-$ (resp. $\mathbf{x}_+$) denote the $n \times n$ matrix with 1's just below (resp. just above) the diagonal and 0's everywhere else. Then let $\mathfrak{g}^{\mathbf{x}_+}$ denote the centralizer of $\mathbf{x}_+$ in $\mathfrak{g}$. In [26], Kostant showed that the subspace $\mathbf{x}_- + \mathfrak{g}^{\mathbf{x}_+}$ is contained in $\mathfrak{g}^{\mathrm{r}}$. Moreover, he showed that the restriction of χ to $\mathbf{x}_- + \mathfrak{g}^{\mathbf{x}_+}$ induces an isomorphism onto $\mathbf{car}$. The Kostant section is the inverse morphism $\epsilon : \mathbf{car} \to \mathbf{x}_- + \mathfrak{g}^{\mathbf{x}_+}$. In the case of $\mathfrak{gl}_n$,

$$\mathbf{x}_- + \mathfrak{g}^{\mathbf{x}_+} = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ 1 & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\ 0 & 1 & x_0 & \ldots & x_{n-4} & x_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & & \ldots & 1 & x_0 \end{pmatrix} \right\}.$$

From this, it is elementary to compute the Kostant section. For example, for $n = 2$, it sends the characteristic polynomial $p = x^2 + a_1 x + a_0$ to the matrix of the form above with $x_0 = -a_1/2$, $x_1 = a_1^2/4 - a_0$.

Proposition 113. *Suppose $s \in \mathfrak{g}^r$ is a regular matrix of type λ. Then the local monodromy $H(s)$ at s for the S_n-cover $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \to \mathfrak{g}^{\mathrm{rs}}$ is conjugate to the Young subgroup S_λ.*

Proof. We can assume that $s = \epsilon(p)$ for some $p \in \mathbf{car}$. Then, by Proposition 50, the local fundamental group at p is a retract of the local fundamental group at s. Since the S_n-cover $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) \to \mathfrak{g}^{\mathrm{rs}}$ is a pullback of the S_n-cover $\mathbf{d}^r \to \mathbf{car}^{\mathrm{rs}}$, it follows that the local monodromy subgroup at s is equal to the local monodromy subgroup at p. By Lemma 112, this subgroup is S_λ. □

Remark 114. We could have used any (continuous) section of the map $\chi : \mathfrak{g} \to \mathbf{car}$ to prove Proposition 113.

8.7 The commuting group scheme

Write $I := \{(g, x) \in G \times \mathfrak{g} : \mathrm{Ad}\, g(x) = x\}$. The projection $p : I \to \mathfrak{g}$ is a group scheme in a more or less obvious way. Write $p^{\mathrm{rs}} : \mathscr{T} \to \mathfrak{g}^{\mathrm{rs}}$ for the restriction of p to the inverse image of $\mathfrak{g}^{\mathrm{rs}}$. Then $\mathscr{T}$ is a torus bundle: the fiber over a point $y \in \mathfrak{g}^{\mathrm{rs}}$ is the maximal torus in G centralizing y.

Identify the scheme Z from §8.4 with $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$ and form a pullback diagram

$$\begin{tikzcd} \mathscr{T}_Z \arrow[r] \arrow[d] & \mathscr{T} \arrow[d, "p^{\mathrm{rs}}"] \\ Z \arrow[r, "\pi"] & \mathfrak{g}^{\mathrm{rs}}. \end{tikzcd}$$

Then $\mathscr{T}_Z$ is equipped with an isomorphism $\mathscr{T}_Z \to \mathbb{G}^n_{mZ}$ to the split torus over Z. To see this, suppose $z = ([v_1], \dots, [v_n]; y)$ and $g \in G$ is an element commuting with y. Then g preserves the eigenspaces of y. So for each $i = 1, \dots n$, there is a unique character $t_i \in X^*(\mathscr{T}_Z)$ such that $gv_i = t_i(g)v_i$. The n-tuple of characters $\vec{t} := (t_1, \dots, t_n) : \mathscr{T}_Z \to \mathbb{G}^n_{m,Z}$ is easily seen to give an isomorphism.

Over $\mathfrak{g}^{rs}$, the torus $\mathscr{T}$ is determined up to isomorphism by its group of characters $X^*(\mathscr{T})$ viewed as a $\mathbb{Z}$-local system over $\mathfrak{g}^{\mathrm{rs}}$. Moreover, this local system is canonically isomorphic to $R^1 p^{\mathrm{rs}}_* \mathbb{Z}$. For any point $y \in \mathfrak{g}^{\mathrm{rs}}$, the fundamental group $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y)$ acts on the fiber of $X^*(\mathscr{T})$ lying over y by permuting the characters $t_1, \dots, t_n$.

9 Monodromy and Tymoczko's dot action

9.1 Fiberwise cohomology of $B\mathscr{T}$

For each $y \in \mathfrak{g}^{\mathrm{rs}}(\mathbb{C})$, we have a torus $\mathscr{T}_y$ and its associated classifying space $B\mathscr{T}_y$. The cohomology of $B\mathscr{T}_y$ is naturally a polynomial ring $\mathbb{C}[t_1, \dots, t_n] = \mathbb{C}[X^*(\mathscr{T}_y)]$ generated in degree 2 by the characters of $\mathscr{T}_y$. As we vary y, these glue together to form a local system $\mathscr{A}$ of polynomial algebras over $\mathfrak{g}^{\mathrm{rs}}$. In fact, since $\mathscr{T}$ is étale locally trivial, we can construct a fiber bundle $a : B\mathscr{T} \to \mathfrak{g}^{\mathrm{rs}}$ over $\mathfrak{g}^{\mathrm{rs}}$ such that the fiber over each $y \in \mathfrak{g}^{\mathrm{rs}}$ is $B\mathscr{T}_y$. Then we have $\mathscr{A} = \oplus_{k \geq 0} \mathscr{A}^k$ where $\mathscr{A}^k = R^{2k} a_* \mathbb{C}$.

Let y_0 denote the regular semisimple matrices with diagonal entries 1, 2, $3, \dots, n$ (written in order). Let F^0 denote the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots$$

where e_i is the standard basis of $\mathbb{C}^n$. Set $z_0 = (F^0, y_0) \in \mathscr{H}(\boldsymbol{\ell}, y_0)$. This gives rise to a surjection

$$\psi : \pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0) \to S_n, \tag{115}$$

where, for simplicity, we write $\psi := \psi_{z_0}$.

Let T denote the fiber of $\mathscr{T}$ over y_0. So T is simply the diagonal subgroup of G. Then, by the discussion in §8.7, $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ acts on $X^*(T)$ by permuting the characters. Explicitly, if we let $\sigma(t_i) = t_{\sigma^{-1}(i)}$ for $\sigma \in S_n$, then $\gamma(t_i) = (\psi(\gamma))(t_i)$. Consequently, if we let $A = \mathscr{A}_{y_0} = \mathbb{C}[t_1, \dots, t_n]$, then $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ acts on the polynomials in A by $\gamma(p) = (\psi(\gamma))(p)$, where S_n acts on A in the standard way:

$$(\sigma p)(t_1, \dots, t_n) = p(t_{\sigma(1)}, \dots, t_{\sigma(n)}). \tag{116}$$

9.2 Fiberwise equivariant cohomology of Hessenberg varieties

Now, for each Hessenberg function $\mathbf{m}$, the torus $\mathscr{T}$ acts on the Hessenberg scheme $\mathscr{H}^{\mathrm{rs}}(\mathbf{m}) \to \mathfrak{g}^{\mathrm{rs}}$. So for each $y \in \mathfrak{g}^{\mathrm{rs}}$, we can take the equivariant cohomology groups $\mathrm{H}^*_{\mathscr{T}_y}(\mathscr{H}(\mathbf{m}), y)$ (with complex coefficients). By localization, we know that $\mathrm{H}^*_{\mathscr{T}_y}(\mathscr{H}(\mathbf{m}), y))$ is a free module of rank $n!$ over $\mathscr{A}_y = \mathrm{H}^*(B\mathscr{T}_y)$. Moreover, the canonical inclusion $\mathscr{H}(\boldsymbol{\ell}) \to \mathscr{H}(\mathbf{m})$ induces an inclusion $\mathrm{H}^*_{\mathscr{T}_y}(\mathscr{H}(\mathbf{m}, y)) \to \mathrm{H}^*_{\mathscr{T}_y}(\mathscr{H}(\boldsymbol{\ell}, y))$. (See Tymoczko's paper [52] for results on localization applied to Hessenberg varieties.) The modules $\mathrm{H}^*_{\mathscr{T}_y}(\mathscr{H}(\mathbf{m}, y))$ glue together to form a local system $\mathscr{L}(\mathbf{m})$ over $\mathfrak{g}^{\mathrm{rs}}$ of $\mathscr{A}$-modules. This can be seen explicitly using Tymoczko's description of the equivariant cohomology of Hessenberg varieties in terms of moment graphs.

Proposition 117. *Write $\pi_{\mathbf{m}} : \mathscr{H}^{\mathrm{rs}}(\mathbf{m}) \to \mathfrak{g}^{\mathrm{rs}}$ for the projection morphism, and let $\mathscr{A}_+$ denote the sheaf of ideals in $\mathscr{A}$ generated by the positive degree elements. Then we have an isomorphism of sheaves*

$$\mathscr{L}(\mathbf{m})/\mathscr{L}(\mathbf{m})\mathscr{A}_+ \to R^*\pi_{\mathbf{m}*}\mathbb{C}.$$

Proof. This follows from the fact that Hessenberg varieties are GKM spaces. (See [52, §2 and Proposition 5.4]). □

For each $\mathbf{m}$, localization induces an inclusion $\mathscr{L}(\mathbf{m}) \to \mathscr{L}(\boldsymbol{\ell})$ of $\mathscr{A}$-modules. Write $L(\mathbf{m})$ for the fiber, $\mathrm{H}^*_T(\mathscr{H}(\mathbf{m}, y_0))$ of $\mathscr{L}(\mathbf{m})$ over y_0. Then $L(\mathbf{m})$ is free as an A-module, and both A and $L(\mathbf{m})$ are equipped with compatible actions of $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$. If we write A_+ for the ideal of positive degree polynomials, then we have

$$\mathrm{H}^*(\mathscr{H}(\mathbf{m}, y_0)) = L/A_+L(\mathbf{m}), \tag{118}$$

and the monodromy action of $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ on both sides is compatible.

Proposition 119. *The action of $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ on $L(\mathbf{m})$ factors through the homomorphism $\psi : \pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0) \twoheadrightarrow S_n$.*

Proof. The pullback $\mathscr{T}_Z$ of $\mathscr{T}$ to the S_n-cover $Z \to \mathfrak{g}^{\mathrm{rs}}$ is a constant group scheme, and the pullback of $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell}) = Z$ to Z is disjoint union of copies of Z indexed by elements of S_n. It follows that the action of $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ on $\mathrm{H}_T^*(\mathscr{H}(\boldsymbol{\ell}, y_0))$ is trivial on the image of the map $\pi_1(Z, z_0) \to \pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$. In other words, the action of $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ on $\mathrm{H}_T^*(\mathscr{H}(\boldsymbol{\ell}, y_0))$ factors through S_n.

Since $\mathscr{L}(\mathbf{m}) \to \mathscr{L}(\boldsymbol{\ell})$ is an inclusion of local systems, we have a $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$-equivariant inclusion $L(\mathbf{m}) \to L(\boldsymbol{\ell})$. The result follows. □

Corollary 120. *The action of $\pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0)$ on $\mathrm{H}^*(\mathscr{H}^{\mathrm{rs}}(\mathbf{m}, y_0))$ induced from the local system $R^*\pi_{\mathbf{m}*}\mathbb{C}$ factors through $\psi : \pi_1(\mathfrak{g}^{\mathrm{rs}}, y_0) \twoheadrightarrow S_n$.*

Proof. This follows directly from Propositions 119 and 117. □

Definition 121. The action of S_n on $L(\mathbf{m})$ (resp. $\mathrm{H}^*(\mathscr{H}^{\mathrm{rs}}(\mathbf{m}, y_0))$) coming from Proposition 119 (resp. Corollary 120) is called the *monodromy action of S_n*.

9.3 Monodromy action for $\mathscr{H}^{\mathrm{rs}}(\boldsymbol{\ell})$

To make the monodromy action of S_n on $L(\boldsymbol{\ell})$ explicit, recall from §9.1 that z_0 denotes the element of $Z_0 := \mathscr{H}(\boldsymbol{\ell}, y_0)$ corresponding to y_0 with the standard ordering of its eigenspaces. So $z_0 = ([e_1], \ldots, [e_n], y_0)$. Given $\sigma \in S_n$, we have $\sigma z_0 = ([e_{\sigma^{-1}1}], \ldots, [e_{\sigma^{-1}(n)}]; y_0)$. The cohomology group $\mathrm{H}^* Z_0 = \mathrm{H}^0 Z_0$ is simply the group of functions $f : Z_0 \to \mathbb{C}$. If, for $w \in S_n$, we write δ_w for the function taking wz_0 to 1 and all other elements of Z_0 to 0, then we have $(\sigma\delta_w)(z) = \delta_w(\sigma^{-1}z)$. From this it easily follows that $\sigma\delta_w = \delta_{\sigma w}$.

Lemma 122. *As an A-module, $L(\boldsymbol{\ell})$ is isomorphic to the module $A^{|S_n|}$ of functions from the set S_n to A. The monodromy action of S_n on $L(\boldsymbol{\ell})$ is given by*

$$((wp)(v))(t_1, \ldots, t_n) = (p(w^{-1}v))(t_{w(1)}, \ldots, t_{w(n)})$$

where $v, w \in S_n$, $p \in A^{|S_n|}$ and $t_1, \ldots, t_n$ are variables.

Proof. Under the identification $S_n \to Z_0$ given by $w \mapsto wz_0$, the δ_w form a $\mathbb{C}$-basis of $\mathrm{H}^0(Z_0)$. Moreover, the map $\mathrm{H}_T^0(Z_0) \to \mathrm{H}^0(Z_0)$ is an S_n-equivariant isomorphism, and, under this identification, the δ_w freely generate $\mathrm{H}_T^*(Z_0)$ as an A-module. The result now follows by direct verification using the fact that S_n acts on A as in (116). □

Corollary 123. *The monodromy action of S_n agrees with Tymoczko's dot action of S_n on $\mathrm{H}_T^*(\mathscr{H}(\boldsymbol{\ell}, y_0))$.*

Proof. This follows immediately by comparing the description of the monodromy action in Lemma 122 with Tymoczko's description of the dot action [52, §3.1]. □

Theorem 124. *Let* $\mathbf{m}$ *be a Hessenberg function. The monodromy action of* S_n *on* $\mathrm{H}_T^*(\mathscr{H}(\mathbf{m}, y_0))$ *is the same as Tymoczko's dot action.*

Proof. Under Tymoczko's dot action, $\mathrm{H}_T^*(\mathscr{H}(\mathbf{m}, y_0))$ is an S_n-equivariant A-submodule of $\mathrm{H}_T^*(\mathscr{H}(\boldsymbol{\ell}, y_0))$. The same is true of the monodromy action of S_n. Therefore, by Corollary 123, the two actions must coincide. □

Corollary 125. *Tymoczko's dot action of* S_n *on the non-equivariant cohomology group* $\mathrm{H}^*(\mathscr{H}(\mathbf{m}, y_0))$ *coincides with the monodromy action.*

Proof. Tymoczko defines the dot action on $\mathrm{H}^*(\mathscr{H}(\mathbf{m}, y_0))$ as the dot action on the quotient $L(\mathbf{m})/A_+L(\mathbf{m})$. The monodromy action is also given by this quotient. □

Since Tymoczko's dot action and the monodromy action of S_n coincide, we will not distinguish between them from now on: it will be the only action of S_n appearing in the remainder of the paper.

Theorem 126. *Let* $s \in \mathfrak{g}^{\mathrm{r}}$ *be a regular element of type* λ *and let* $\pi = \pi_{\mathbf{m}} : \mathscr{H}(\mathbf{m}) \to \mathfrak{g}$*. Let* $B(s)$ *be a sufficiently small ball in* $\mathfrak{g}$ *centered at* s*. Then, for each* $k \in \mathbb{Z}$*, there is a* $\mathbb{C}$*-vector space isomorphism*

$$\mathrm{H}^0(B(s) \cap \mathfrak{g}^{\mathrm{rs}}, R^k\pi_*\mathbb{C}) \cong \mathrm{H}^k(\mathscr{H}(\mathbf{m}, y_0))^{S_\lambda}.$$

Proof. By (51) applied with $\mathcal{L} = R^k\pi_*\mathbb{C}$, we have

$$\mathrm{H}^0(B(s) \cap \mathfrak{g}^{\mathrm{rs}}, R^k\pi_*\mathbb{C}) = \mathrm{H}^k(\mathscr{H}(\mathbf{m}, b))^{\pi_1(B(s)\cap\mathfrak{g}^{\mathrm{rs}}, b)}$$

where b is any point in $B(s) \cap \mathfrak{g}^{\mathrm{rs}}$. The last vector space is isomorphic to the invariants of $\mathrm{H}^k(\mathscr{H}(\mathbf{m}, y))$ under the local monodromy at s. The result then follows from Proposition 113. □

Theorem 127. *Suppose* $s \in \mathfrak{g}^r$ *is a regular element of type* λ*. Then, for each* $k \in \mathbb{Z}$*,*

$$\dim \mathrm{H}^k(\mathscr{H}(\mathbf{m}, s)) = \dim \mathrm{H}^k(\mathscr{H}(\mathbf{m}, y_0))^{S_\lambda}. \tag{128}$$

Proof. We are going to apply Theorem 57 to the morphism $\pi : \mathscr{H}(\mathbf{m}) \to \mathfrak{g}$. Both the source and the target of π are smooth, quasi-projective varieties. Moreover, π has relative dimension $|\mathbf{m}|$. (One way to check this is to use the fact that the projection $\mathrm{pr}_1 : \mathscr{H}(\mathbf{m}) \to \mathscr{X}$ has relative dimension $\sum_{i=1}^n m_i$, while $\dim \mathscr{X} = \sum_{i=1}^{n-1} i$. Another way to see it, is to use the fact that the regular semisimple Hessenberg varieties have dimension $|\mathbf{m}|$.)

By Corollary 36, we have

$$\dim \mathrm{H}^i(\mathscr{H}(\mathbf{m}, s), \mathbb{C}) = \dim \mathrm{H}^{2|\mathbf{m}|-i}(\mathscr{H}(\mathbf{m}, s), \mathbb{C})$$

for all i.

It follows then from Theorem 57 that the local invariant cycle map

$$\mathrm{H}^i(\mathscr{H}(\mathbf{m}, s)) \to \mathrm{H}^0(B(s) \cap \mathfrak{g}^{\mathrm{rs}}, R^i\pi_*\mathbb{C})$$

is an isomorphism, where $B(s)$ is any sufficiently small ball centered at s in $\mathfrak{g}$. The result now follows from Theorem 126. □

Finally we can put all the pieces together to prove Conjecture 3.

Theorem 129. *If $\chi_{\mathbf{m},d}$ denotes the dot action on the cohomology group H^{2d} of the regular semisimple Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$, then $\operatorname{ch}\chi_{\mathbf{m},d}$ equals the coefficient of t^d in $\omega X_{G(\mathbf{m})}(t)$.*

Proof. By Theorem 35, the left-hand side of Equation (128) (in Theorem 127) equals $c_{d,\lambda}(\mathbf{m})$ when $k = 2d$. On the other hand, by Proposition 10, the right-hand side of Equation (128) equals the coefficient of m_λ in $\operatorname{ch}\chi_{\mathbf{m},d}$. □

References

[1] H. Abe, M. Harada, T. Horiguchi, and M. Masuda. The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A. *arXiv:1512.09072*.

[2] H. Abe, M. Harada, T. Horiguchi, and M. Masuda. The equivariant cohomology rings of regular nilpotent Hessenberg varieties in Lie type A: a research announcement. *arXiv:1411.3065*. Accepted for publication in MORFISMOS (special volume in honor of Samuel Gitler).

[3] Dave Anderson and Julianna Tymoczko. Schubert polynomials and classes of Hessenberg varieties. *J. Algebra*, 323(10):2605–2623, 2010.

[4] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[5] Gebhard Böckle and Chandrashekhar Khare. Mod l representations of arithmetic fundamental groups. II. A conjecture of A. J. de Jong. *Compos. Math.*, 142(2):271–294, 2006.

[6] P. Brosnan and T. Y. Chow. Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties. *arXiv:1511.00773*, November 2015.

[7] T.-H. Chen, K. Vilonen, and T. Xue. Hessenberg varieties, intersections of quadrics, and the Springer correspondence. *arXiv:1511.00617*.

[8] Timothy Y. Chow. The path-cycle symmetric function of a digraph. *Adv. Math.*, 118(1):71–98, 1996.

[9] A. J. de Jong. A conjecture on arithmetic fundamental groups. *Israel J. Math.*, 121:61–84, 2001.

[10] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg varieties. *Trans. Amer. Math. Soc.*, 332(2):529–534, 1992.

[11] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[12] Alexandru Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992.

[13] Vladimir Drinfeld. On a conjecture of Kashiwara. *Math. Res. Lett.*, 8(5-6):713–728, 2001.

[14] Fouad El Zein and Jawad Snoussi. Local systems and constructible sheaves. In *Arrangements, local systems and singularities*, volume 283 of *Progr. Math.*, pages 111–153. Birkhäuser Verlag, Basel, 2010.

[15] D. Gaitsgory. On de Jong's conjecture. *Israel J. Math.*, 157:155–191, 2007.

[16] Vesselin Gasharov. Incomparability graphs of $(3 + 1)$-free posets are s-positive. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 193–197, 1996.

[17] Ira M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[18] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.

[19] M. Guay-Paquet. A modular relation for the chromatic symmetric functions of (3+1)-free posets. *arXiv:1306.2400*.

[20] M. Guay-Paquet. A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra. *arXiv:1601.05498*.

[21] Mark Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, 6(3):569–595, 1993.

[22] M. Harada and M. Precup. The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture. *arXiv:1709.06736*.

[23] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[24] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[25] Masaki Kashiwara. Semisimple holonomic $\mathscr{D}$-modules. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, volume 160 of *Progr. Math.*, pages 267–271. Birkhäuser Boston, Boston, MA, 1998.

[26] Bertram Kostant. Lie group representations on polynomial rings. *Amer. J. Math.*, 85:327–404, 1963.

[27] Shrawan Kumar. Finiteness of local fundamental groups for quotients of affine varieties under reductive groups. *Comment. Math. Helv.*, 68(2):209–215, 1993.

[28] S. Lojasiewicz. Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa (3)*, 18:449–474, 1964.

[29] Eduard J. N. Looijenga. *Isolated singular points on complete intersections*, volume 5 of *Surveys of Modern Mathematics*. International Press, Somerville, MA; Higher Education Press, Beijing, second edition, 2013.

[30] Saunders Mac Lane. *Categories for the working mathematician*. Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5.

[31] Takuro Mochizuki. Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. I. *Mem. Amer. Math. Soc.*, 185(869):xii+324, 2007.

[32] Takuro Mochizuki. Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. II. *Mem. Amer. Math. Soc.*, 185(870):xii+565, 2007.

[33] David Mumford. *Algebraic geometry. I*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Complex projective varieties, Reprint of the 1976 edition.

[34] Bao Châu Ngô. Fibration de Hitchin et endoscopie. *Invent. Math.*, 164(2):399–453, 2006.

[35] M. Precup. The Betti numbers of regular Hessenberg varieties are palindromic. *arXiv:1603.07662*. To appear in *Transformation Groups*.

[36] Martha Precup. Affine pavings of Hessenberg varieties for semisimple groups. *Selecta Math. (N.S.)*, 19(4):903–922, 2013.

[37] David Prill. Local classification of quotients of complex manifolds by discontinuous groups. *Duke Math. J.*, 34:375–386, 1967.

[38] B. Riemann. Theorie der Abel'schen Functionen. *J. Reine Angew. Math.*, 54:115–155, 1857.

[39] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[40] Morihiko Saito. Decomposition theorem for proper Kähler morphisms. *Tohoku Math. J. (2)*, 42(2):127–147, 1990.

[41] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[42] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration spaces*, volume 14 of *CRM Series*, pages 433–460. Ed. Norm., Pisa, 2012.

[43] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, 295:497–551, 2016.

[44] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[45] T. A. Springer. Quelques applications de la cohomologie d'intersection. In *Bourbaki Seminar, Vol. 1981/1982*, volume 92 of *Astérisque*, pages 249–273. Soc. Math. France, Paris, 1982.

[46] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2018.

[47] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.

[48] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[49] Richard P. Stanley and John R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *J. Combin. Theory Ser. A*, 62(2):261–279, 1993.

[50] Nicholas James Teff. *The Hessenberg representation*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The University of Iowa.

[51] Julianna S. Tymoczko. Linear conditions imposed on flag varieties. *Amer. J. Math.*, 128(6):1587–1604, 2006.

[52] Julianna S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 365–384. Amer. Math. Soc., Providence, RI, 2008.

[53] Torsten Wedhorn. *Manifolds, sheaves, and cohomology*. Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, 2016.