THE LOCUS OF HODGE CLASSES IN AN ADMISSIBLE VARIATION OF MIXED HODGE STRUCTURE

PATRICK BROSNAN, GREG PEARLSTEIN, AND CHRISTIAN SCHNELL

Abstract. We generalize the theorem of E. Cattani, P. Deligne, and A. Kaplan to admissible variations of mixed Hodge structure.

1. Introduction

The purpose of this note is to prove the following generalization of the famous theorem of Cattani, Deligne, and Kaplan [2].

Theorem 1. *Let S be a Zariski-open subset of a complex manifold $\bar{S}$, and let $\mathscr{V}$ be a variation of mixed Hodge structure on S. Suppose that $\mathscr{V}$ is defined over $\mathbb{Z}$, graded polarized, and admissible with respect to $\bar{S}$. Let* $\operatorname{Hdg}(\mathscr{V})$ *denote the locus of Hodge classes in $\mathscr{V}$. Then each component of* $\operatorname{Hdg}(\mathscr{V})$ *extends to an analytic space, finite and proper over $\bar{S}$.*

As in the original paper, Chow's theorem implies that the locus of Hodge classes consists of algebraic varieties if S is algebraic.

Corollary 2. *In the situation of Theorem 1, suppose that S is quasi-projective. Then each component of* $\operatorname{Hdg}(\mathscr{V})$ *is a quasi-projective algebraic variety.*

We remind the reader of a few basic definitions. Given a mixed Hodge structure V defined over $\mathbb{Z}$, a *Hodge class* in V is an element of $V_{\mathbb{Z}} \cap W_0 V_{\mathbb{C}} \cap F^0 V_{\mathbb{C}}$, or equivalently, a morphism of mixed Hodge structures $\mathbb{Z}(0) \to V$. Given a variation of mixed Hodge structure $\mathscr{V}$ on a complex manifold S, let $\mathscr{V}_{\mathbb{Z}}$ denote the underlying integral local system. Its étalé space $T(\mathscr{V}_{\mathbb{Z}})$ is a covering space of S with countably many connected components; it naturally embeds into the holomorphic vector bundle $E(\mathscr{V}_{\mathscr{O}})$. The *locus of Hodge classes* in $\mathscr{V}$ can then be described as the intersection

$$\operatorname{Hdg}(\mathscr{V}) = T(\mathscr{V}_{\mathbb{Z}}) \cap E\big(F^0\mathscr{V}_{\mathscr{O}}\big) \cap E\big(W_0\mathscr{V}_{\mathscr{O}}\big).$$

We deduce Theorem 1 from the original result by Cattani, Deligne, and Kaplan with the help of the following difficult theorem; it is the main result of [1], and can also be proved by the methods of [7]. (A similar result has also been announced by Kato, Nakayama, and Usui in [5].) Either proof relies on the SL(2)-orbit theorem of Kato, Nakayama, and Usui [4].

Theorem 3. *Let ν be an admissible higher normal function on S, that is, an admissible extension of $\mathbb{Z}(0)$ by a polarized variation of Hodge structure of negative weight. Let $Z(\nu) = \{s \in S : \nu(s) = 0\}$ denote the zero locus of ν. (C.f. the*

2000 *Mathematics Subject Classification.* 14D05.
Key words and phrases. Hodge class, Admissible variation of mixed Hodge structure.

discussion at the beginning of Section 3.) Then the closure of $Z(\nu)$ in $\bar{S}$ is an analytic subset.

Note that this result includes the case of classical normal functions (where the Hodge structure has weight -1). Theorem 3 in itself is most interesting when S is a quasi-projective complex manifold; we may then take $\bar{S}$ to be any smooth projective compactification, since the notion of admissibility is independent of the particular choice.

Corollary 4. *Suppose that ν is an admissible higher normal function on S, that is, an extension of $\mathbb{Z}(0)$ by a polarized variation of Hodge structure of negative weight. Then the zero locus $Z(\nu)$ is an algebraic subset of S.*

One source for higher normal functions is through families of higher Chow cycles. Let $\pi\colon X \to S$ be a family of complex projective manifolds with S smooth. Then the regulator map from motivic cohomology $H^p_{\mathcal{M}}\big(X, \mathbb{Z}(q)\big) \simeq \mathrm{CH}^q(X, 2q-p)$ to Deligne cohomology $H^p_{\mathcal{D}}\big(X, \mathbb{Z}(q)\big)$ induces a homomorphism

$$\mathrm{CH}^q(X, 2q-p) \otimes \mathbb{Q} \to \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^{p-k}_{\mathrm{MHM}(S)}\big(\mathbb{Q}(0), R^k \pi_* \mathbb{Q}(q)\big),$$

using the decomposition theorem; $\mathrm{MHM}(S)$ is the category of mixed Hodge modules on S. In particular, a higher Chow cycle on X determines an element in $\mathrm{Ext}^1_{\mathrm{MHM}(S)}\big(\mathbb{Q}, R^{p-1}\pi_*\mathbb{Q}(q)\big)$; some multiple is an admissible higher normal function for the variation of Hodge structure $R^{p-1}\pi_*\mathbb{Z}(q)$ of weight $p-2q-1<0$.

The same methods can be used to describe the locus of points $s \in S$ where V_s splits over $\mathbb{Z}$ (we say that a mixed Hodge structure V *splits* over $\mathbb{Z}$ if $V \simeq \bigoplus_m Gr^W_m V$ in MHS).

Theorem 5. *Let $\mathcal{V}$ be an admissible variation of mixed Hodge structure on S. Then the set of points $s \in S$ where the mixed Hodge structure V_s splits over $\mathbb{Z}$ is an algebraic subset of S.*

Since V_s splits over $\mathbb{Z}$ iff there is a Hodge class in $\mathrm{End}(V_s)$ that induces a splitting of the underlying integral lattice, this result may also be viewed as a special case of Theorem 1.

2. Admissibility

Let $\mathcal{V}$ be a variation of $\mathbb{Z}$-mixed Hodge structure on a Zariski-open subset S of a complex manifold $\bar{S}$. We call $\mathcal{V}$ *admissible with respect to* $\bar{S}$ if $\mathcal{V} \otimes \mathbb{Q}$ is admissible in the sense of Kashiwara [3] (where admissibility is defined by a curve test). It is clear from this definition that admissibility is preserved under holomorphic maps $f\colon \bar{S}' \to \bar{S}$ with the property that $f^{-1}(S)$ is dense in $\bar{S}'$. Moreover, duals and tensor products of admissible variations of mixed Hodge structure are again admissible; this is proved in the appendix to [8].

By work of Saito [6], admissibility can also be phrased in terms of mixed Hodge modules: $\mathcal{V} \otimes \mathbb{Q}$ defines a mixed Hodge module on S, and is admissible if and only if that mixed Hodge module can be extended to $\bar{S}$.

3. The locus of Hodge classes

We now turn to the proof of Theorem 1. Throughout, we let $\mathcal{V}$ be a variation of mixed Hodge structure over S, admissible with respect to $\bar{S}$. We assume that $\mathcal{V}$ is graded polarized, and that the local systems $W_m \mathscr{V}$ making up the weight filtration are defined over $\mathbb{Z}$, with $Gr_m^W \mathscr{V}$ torsion free.

To begin with, we can replace $\mathscr{V}$ by $W_0 \mathscr{V}$, and assume without loss of generality that $\mathscr{V}$ is of weight ≤ 0. We then have

$$\operatorname{Hdg}(\mathscr{V}) = T(\mathscr{V}_{\mathbb{Z}}) \cap E(F^0 \mathscr{V}_{\mathscr{O}}).$$

The next step is to prove a more general version of Theorem 3. Recall that a *generalized normal function* ν is an extension, in the category of variations of mixed Hodge structure, of $\mathbb{Z}(0)$ by a variation of mixed Hodge structure $\mathscr{H}$, all of whose weights are ≤ -1. It is said to be *admissible* if the corresponding variation is admissible. At each point $s \in S$, the extension determines a point $\nu(s) \in \operatorname{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(0), H_s\big)$; the *zero locus* $Z(\nu)$ of the generalized normal function is by definition the set of points where $\nu(s) = 0$. We let

$$\operatorname{NF}(S, \mathscr{H}) = \operatorname{Ext}^1_{\mathrm{VMHS}(S)}\big(\mathbb{Z}(0), \mathscr{H}\big)$$

denote the group of generalized normal functions.

Proposition 6. *Let ν be an admissible generalized normal function on S. Then the closure of $Z(\nu)$ in $\bar{S}$ is an analytic subset.*

Proof. Let $\mathscr{V}$ be the corresponding admissible variation of mixed Hodge structure, and $\mathscr{H} = W_{-1} \mathscr{V}$. If $\mathscr{H}$ is pure, then the result follows from Theorem 3. Otherwise, we let $m \leq -1$ be the smallest integer for which $Gr_m^W \mathscr{V} \neq 0$. Define $\mathscr{V}' = \mathscr{V} / W_m \mathscr{V}$, and let ν_0 be the corresponding generalized normal function induced on $\mathscr{V}'$ by ν. Note that we have $Z(\nu) \subseteq Z(\nu_0)$.

Let S_0 denote the regular locus of an irreducible component of $Z(\nu_0)$. By induction, we know that the closure of S_0 inside of $\bar{S}$ is analytic; let $\pi \colon \bar{S}_0 \to \bar{S}$ be a resolution of singularities of the closure that is an isomorphism over S_0. Since π is proper, we are allowed to replace $\bar{S}$ by $\bar{S}_0$ and ν by its pullback to S_0; we may therefore assume from the beginning that $\nu_0 = 0$. Now the exact sequence

$$0 \longrightarrow \operatorname{NF}\big(S, W_m \mathscr{H}\big) \longrightarrow \operatorname{NF}(S, \mathscr{H}) \longrightarrow \operatorname{NF}\big(S, \mathscr{H} / W_m \mathscr{H}\big)$$

shows that ν induces a generalized normal function $\nu' \in \operatorname{NF}\big(S, W_m \mathscr{H}\big)$. Since $W_m \mathscr{H}$ is pure of weight m, we conclude from Theorem 3 that $Z(\nu')$ has an analytic closure inside $\bar{S}$; but clearly $Z(\nu) = Z(\nu')$, and so the assertion follows. $\square$

We are now ready to prove Theorem 1 in general.

Proof of Theorem 1. Let $\mathscr{V}$ be the admissible variation of mixed Hodge structure; as explained above, we may suppose that it has weights ≤ 0. For any point $s \in S$, let V_s be the corresponding mixed Hodge structure; then we have an exact sequence

$$0 \longrightarrow \operatorname{Hdg}(V_s) \longrightarrow \operatorname{Hdg}\big(Gr_0^W V_s\big) \longrightarrow \operatorname{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(0), W_{-1} V_s\big). \tag{1}$$

It follows that the locus of Hodge classes for $\mathscr{V}$ is embedded into that for $Gr_0^W \mathscr{V}$. Let Z be an irreducible component of $\operatorname{Hdg}(\mathscr{V})$, and let Y be the irreducible component of $\operatorname{Hdg}\big(Gr_0^W \mathscr{V}\big)$ containing Z. By the theorem of Cattani, Deligne, and Kaplan [2], Y can be extended to an analytic space $\bar{Y}$ that is proper and finite over $\bar{S}$. Let $\mu \colon \bar{Y}' \to \bar{Y}$ be a resolution of singularities of the analytic space $\bar{Y}$ and denote by $\mathscr{V}'$ the pullback of $\mathscr{V}$ to Y.

By construction, we have a section $\mathbb{Z}(0) \to Gr_0^W \mathscr{V}'$. It induces a generalized normal function $\nu' \in \operatorname{NF}(Y, \mathscr{H}')$, where $\mathscr{H}' = W_{-1} \mathscr{V}'$. Moreover, it is clear from

(1) that $Z = Z(\nu')$. Since ν' is easily seen to be admissible with respect to $\bar{Y}'$, we conclude from Proposition 6 that the closure of $Z(\nu')$ in $\bar{Y}'$ is analytic. Because μ is proper, it follows that Z has an analytic closure inside of $\bar{Y}$; this completes the proof. □

4. The split locus

The proof of Theorem 5 is similar to that of Theorem 1.

Proof. It suffices to prove the statement with coefficients in $\mathbb{Q}$. So let $\mathscr{V}$ be an admissible variation of mixed Hodge structure on S, where S is Zariski-open in a complex manifold $\bar{S}$. Let m be the largest integer for which $Gr_m^W \mathscr{V} \neq 0$. By induction, we know that the locus of points $s \in S$ where $W_{m-1}V_s$ splits over $\mathbb{Q}$ has an analytic closure inside of $\bar{S}$. Arguing as before, we may therefore assume from the beginning that $W_{m-1}\mathscr{V}$ is split. Now $\mathscr{V}$ determines an element of

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{VMHS}(S)}\bigl(Gr_m^W \mathscr{V}, W_{m-1}\mathscr{V}\bigr) &\simeq \bigoplus_{k<m} \mathrm{Ext}^1_{\mathrm{VMHS}(S)}\bigl(Gr_m^W \mathscr{V}, Gr_k^W \mathscr{V}\bigr) \\ &\simeq \bigoplus_{k<m} \mathrm{Ext}^1_{\mathrm{VMHS}(S)}\bigl(\mathbb{Q}(0), (Gr_m^W \mathscr{V})^\vee \otimes Gr_k^W \mathscr{V}\bigr).\end{aligned}$$

Since admissibility is preserved under tensor products, the problem is reduced to the case of admissible higher normal functions; applying Theorem 3 completes the proof. □

References

[1] P. Brosnan and G. Pearlstein, *On the algebraicity of the zero locus of an admissible normal function* (2009). preprint, first version at arXiv:0910.0628v1.

[2] E. Cattani, P. Deligne, and A. Kaplan, *On the locus of Hodge classes*, Journal of the American Mathematical Society **8** (1995), no. 2, 483–506.

[3] M. Kashiwara, *A study of variation of mixed Hodge structure*, Publications of the Research Institute for Mathematical Sciences **22** (1986), no. 5, 991–1024.

[4] K. Kato, C. Nakayama, and S. Usui, SL(2)-*orbit theorem for degeneration of mixed Hodge structure*, Journal of Algebraic Geometry **17** (2008), no. 3, 401–479.

[5] ———, *Moduli of log mixed Hodge structures* (2009), available at `arXiv:0910.4454`.

[6] M. Saito, *Admissible normal functions*, Journal of Algebraic Geometry **5** (1996), no. 2, 235–276.

[7] C. Schnell, *Complex analytic Néron models for arbitrary families of intermediate Jacobians* (2009), submitted, available at `arXiv:0910.0662`.

[8] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I*, Inventiones Mathematicae **80** (1985), no. 3, 489–542.

Department of Mathematics, The University of British Columbia, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2

E-mail address: `brosnan@math.ubc.ca`

Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: `gpearl@math.msu.edu`

Department of Mathematics, Statistics & Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607

E-mail address: `cschnell@math.uic.edu`