PERVERSE OBSTRUCTIONS TO FLAT REGULAR COMPACTIFICATIONS

PATRICK BROSNAN

ABSTRACT. Suppose $\pi : W \to S$ is a smooth, surjective, proper morphism to a variety S contained as a Zariski open subset in a smooth, complex variety $\bar{S}$. The goal of this note is to consider the question of when π admits a regular, flat compactification. In other words, when does there exists a flat, proper morphism $\bar{\pi} : \overline{W} \to \bar{S}$ extending π with $\overline{W}$ regular? One interesting recent example of this occurs in the preprint [11] of Laza, Saccà and Voisin where π is a family of abelian 5-folds over a Zariski open subset S of $\bar{S} = \mathbb{P}^5$. In that paper, the authors construct $\overline{W}$ using the theory of compactified Prym varieties and show that it is a holomorphic symplectic manifold (deformation equivalent to O'Grady's 10-dimensional example).

In this note I observe that non-vanishing of the local intersection cohomology of $R^1\pi_*\mathbb{Q}$ in degree at least 2 provides an obstruction to finding a $\bar{\pi}$. Moreover, non-vanishing in degree 1 provides an obstruction to finding a $\bar{\pi}$ with irreducible fibers. Then I observe that, in some cases of interest, results of Brylinski, Beilinson and Schnell can be used to compute the intersection cohomology [1, 4, 15]. I also give examples involving cubic 4-folds motivated by [11] and ask a question about palindromicity of hyperplane sections.

1. INTRODUCTION

Let $\bar{S}$ denote a smooth, quasi-projective, complex (irreducible) variety of dimension d, and let S denote a non-empty Zariski open subset of $\bar{S}$. Suppose $\pi : W \to S$ is a smooth, surjective, proper morphism of complex varieties with $n := \dim W - \dim S$. I will call an irreducible, regular scheme $\overline{W}$ equipped with a proper morphism $\bar{\pi} : \overline{W} \to \bar{S}$ a *regular compactification* of π if

(i) $\overline{W}$ contains W as a Zariski dense open subset;
(ii) the restriction of $\bar{\pi}$ to W is π.

Question 1.1. *Under what conditions can we find a regular compactification $\bar{\pi} : \overline{W} \to \bar{S}$ of π which is flat over $\bar{S}$. Also, under what conditions can we find a $\bar{\pi} : \overline{W} \to \bar{S}$ as above with irreducible fibers?*

Remarks 1.2. (a) Mainly for the sake of brevity, I use the word "compactification" in this paper for what many people would call a "partial compactification." In other words, I do not assume that $\bar{S}$ or $\bar{W}$ is compact.

(b) If $\bar{\pi} : \bar{W} \to \bar{S}$ is a morphism between smooth, quasi-projective varieties of relative dimension n as above, then $\bar{\pi}$ is flat near a point $s \in \bar{S}$ if and only if the dimension, $\dim \bar{W}_s$, of the fiber over s is n. This follow, for example, from [8, Ex. III.10.9].

(c) If $d = 1$, then the first part of Question 1.1 always has a positive answer. In other words, we can always find a regular, flat compactification $\bar{\pi} : \bar{W} \to \bar{S}$. To prove this, we can assume that $\bar{S}$ is compact. Using Hironaka [9] and Nagata [12], we can find a smooth, proper variety $\hat{W}$ containing W as a Zariski open subset. From π, we get a rational map $\hat{W} \dashrightarrow \bar{S}$. Then, using Hironaka again, we can resolve the indeterminacy of this rational map to get a morphism $\bar{\pi} : \bar{W} \to \bar{S}$ as desired with $\bar{W}$ smooth. And, since $\bar{W}$ is irreducible and $\bar{S}$ is a curve, $\bar{\pi}$ is flat.

My goal in this note is to write down some necessary conditions for the existence of a $\bar{\pi}$ as in Question 1.1 in terms of local intersection cohomology. For this, let $j : S \to \bar{S}$ denote the inclusion of S in $\bar{S}$. Pick an integer k and set $\mathbf{L} = R^k\pi_*\mathbb{Q}$. Then the intersection complex IC $\mathbf{L}$ is a polarizable

Hodge module on $\bar{S}$ with underlying perverse sheaf given by the intermediate extension of the underlying local system L to $\bar{S}$. The intersection complex is also called the IC complex and is also written as $j_{!*}\mathbf{L}[d]$. The underlying perverse sheaf is a complex of sheaves with cohomology in the interval $[-d, 0)$. The local intersection cohomology of $\mathbf{L}$ at a point $s \in \bar{S}$ is

$$\mathrm{IH}^j_s \mathbf{L} := H^{j-d}(\mathrm{IC}\,\mathbf{L})_s. \tag{1.3}$$

So $\mathrm{IH}^j_s \mathbf{L}$ is the $(j-d)^{\text{th}}$ cohomology of the stalk of $\mathrm{IC}\,\mathbf{L}$ at s. Clearly $\mathrm{IH}^j_s \mathbf{L} = 0$ unless $j \in [0, d)$. Moreover, $\mathrm{IH}^0_s \mathbf{L}$ is the space of local invariants of $\mathbf{L}$ at s. So, $\mathrm{IH}^0_s \mathbf{L}$ is the fiber $\mathbf{L}_s$ for $s \in S$. At points $s \in \bar{S}$, $\mathrm{IH}^0_s \mathbf{L} = \Gamma(B \cap S, \mathbf{L})$ for a sufficiently small ball B in $\bar{S}$ containing s.

The following theorem, which I believe is a well-known consequence of the decomposition theorem of Beilinson, Bernstein and Deligne [2], gives a way to obtain information about possible compactifications $\bar{\pi}$ from the topology of π. For the convenience of the reader I will prove it in Section 2.

Theorem 1.4. *Suppose $\bar{\pi} : \overline{W} \to \bar{S}$ is a regular compactification of π. Then*

(i) *the complex $\oplus_i \mathrm{IC}(R^{n+i}\pi_*\mathbb{Q})[-i]$ includes in $R\bar{\pi}_*\mathbb{Q}[d+n]$ as a direct factor;*
(ii) *for every integer m and each point $s \in \bar{S}$, $\oplus_{j+k=m} \mathrm{IH}^j_s(R^k\pi_*\mathbb{Q})$ includes as a direct factor in the cohomology group $\mathrm{H}^m(\bar{W}_s, \mathbb{Q})$ of the fiber of $\bar{\pi}$ over s.*
(iii) *If the inclusion in (i) is an isomorphism, then so is the inclusion in (ii).*

Corollary 1.5. *If a flat, regular compactification $\bar{\pi}$ of π exists, then, for all $s \in \bar{S}$ and all integers j, k with $j + k > 2n$, $\mathrm{IH}^j_s(R^k\pi_*\mathbb{Q}) = 0$. If the fibers of $\bar{\pi}$ are irreducible, then $\mathrm{IH}^0(R^{2n}\pi_*\mathbb{Q}) = \mathbb{Q}$ and the groups $\mathrm{IH}^j_s(R^{2n-j}\pi_*\mathbb{Q})$ vanish for $j > 0$.*

Proof. If $\bar{\pi}$ is flat, then $\dim \overline{W}_s = n$ for all $s \in \bar{S}$. So $\mathrm{H}^m(\overline{W}_s, \mathbb{Q}) = 0$ for $m > 2n$. Thus Theorem 1.4 (ii) implies the first statement. If the fiber $\overline{W}_s$ is irreducible, then $\mathrm{H}^{2n}(\overline{W}_s) = \mathbb{Q}$. The constant sheaf is a direct factor in $R^{2n}\pi_*\mathbb{Q}$. So $\dim \mathrm{IH}^0_s(R^{2n}\pi_*\mathbb{Q}) \geq 1$ for all $s \in \bar{S}$. The rest of Corollary 1.5 is now immediate from Theorem 1.4. □

In writing this note, I was mainly motivated by a recent preprint of Laza, Saccà and Voisin which concerns the situation where $\pi : A \to S$ is an abelian scheme of relative dimension n [11]. In this case, set $\mathbf{H} := R^1\pi_*\mathbb{Q}(1)$. It is a polarized variation of Hodge structure of weight -1 on S which is isomorphic to $R^{2n-1}\pi_*\mathbb{Q}(n)$ by Hard Lefschetz. We get the following.

Corollary 1.6. *Set $k = \max\{j : \mathrm{IH}^j_s \mathbf{H} \neq 0\}$. Suppose a flat regular compactification $\bar{\pi} : \bar{A} \to \bar{S}$ of π exists. Then $k \leq 1$. If the fiber $\bar{A}_s$ is irreducible, then $k = 0$.*

Proof. This follows directly from Corollary 1.5 applied to $R^{2n-1}\pi_*\mathbb{Q}$. □

Suppose X is a smooth, closed, $2m$-dimensional subvariety of $P := \mathbb{P}^N$ for some positive integers m and N. By cutting X with hyperplanes, we get a family $\mathcal{X} \to P^\vee$ over the dual projective space, which is smooth over a Zariski dense open subset $U \subset P^\vee$. (See §3.) Set $n := 2m - 1$ so that the general member of the family $\mathcal{X} \to P^\vee$ is, by Bertini, a smooth n-dimensional variety. We get a variation of Hodge structure $\mathbf{H}_\mathbb{Z}$ over U such that the fiber over $H \in P^\vee$ is $\mathrm{H}^n(X \cap H, \mathbb{Z}(m))$. Let $J(\mathbf{H}_\mathbb{Z}) \to U$ denote the family of Griffiths intermediate Jacobians of $\mathbf{H}_\mathbb{Z}$. In very special cases, it turns out to be an abelian scheme. Write $\mathbf{H}$ for the $\mathbb{Q}$-variation of Hodge structure obtained by tensoring $\mathbf{H}_\mathbb{Z}$ with $\mathbb{Q}$. In Section 3, I will prove the following theorem (which, along with Corollary 1.8, assumes the notation of the preceding paragraph).

Theorem 1.7. *Suppose that* $\mathbf{H}$ *is non-constant. Let* $H \in P^\vee$ *be a hyperplane and write* $Y := H \cap X$ *for the hyperplane section. Write* $b_k Y := \dim \mathrm{H}^k(Y, \mathbb{Q})$ *for the k-th Betti number. Then, for* $k > 0$,

$$b_{n+k}Y - b_{n-k}Y = \dim \mathrm{IH}^k_H\, \mathbf{H}.$$

Call Y *palindromic* (resp. *weakly palindromic*) if $b_{n+k}Y = b_{n-k}Y$ for all k (resp. for all $k > 1$).

Corollary 1.8. *Suppose that* $\pi : J(\mathbf{H}_{\mathbb{Z}}) \to U$ *is a non-constant abelian scheme admitting a flat, regular compactification* $\bar{\pi} : \bar{J} \to P^\vee$. *Fix* $H \in P^\vee$ *and set* $Y = X \cap H$. *Then* Y *is weakly palindromic. If the fiber of* $\bar{\pi}$ *over* $H \in P^\vee$ *is irreducible, then* Y *is a palindromic.*

Proof. Since $R^1\pi_*\mathbb{Q}(1) \cong \mathbf{H}$, Corollary 1.8 follows from Theorem 1.7 and Corollary 1.6. □

In [11], the authors produce a flat, regular compactification $\bar{A}$ of a family $\pi : A \to U$ of abelian 5-folds over an open subset of $\mathbb{P}^5$. In Section 4, I will give examples where Corollary 1.8 can be used to rule out the existence of a flat, regular compactification, or a flat, regular compactification with irreducible fibers. I will also give a consequence (Corollary 4.3) of the main result of [11] and state a conjecture (Conjecture 4.4) about palindromicity partially motivated by the results of [11].

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The interaction between palindromicity and intersection cohomology comes up in a similar way to the way it is used here in my joint paper [3] written with T. Chow. I thank Chow for many conversations about the notion of palindromicity.

2. Proof of Theorem 1.4

Proof. Let d denote the dimension of $\bar{S}$ and let n denote the dimension of the generic fiber of π. By the decomposition theorems of Beilinson—Bernstein—Deligne [2] and Saito [14], we have $R\bar{\pi}_*\mathbb{Q}[d+n] = \oplus_{i\in\mathbb{Z}}\mathcal{F}_i[-i]$ where the $\mathcal{F}_i$ are direct sums of intersection complexes underlying polarizable Hodge modules coming from local systems on various strata. The restriction of $\mathcal{F}_i$ to S is equal to $R^{n+i}\pi_*\mathbb{Q}[d]$. So, by the semi-simplicity of the category of polarizable Hodge modules, each $\mathcal{F}_i$ contains $\mathrm{IC}(R^{n+i}\pi_*\mathbb{Q})$ as a direct factor, and this implies Theorem 1.4 (i).

Let $\iota : \{s\} \to \bar{S}$ denote the inclusion of the point s. By proper base change, we have $\mathrm{H}^m(\overline{W}_s, \mathbb{Q}) = H^m(\iota^* R\bar{\pi}_*\mathbb{Q}) = H^{m-d-n}(\iota^* R\bar{\pi}_*\mathbb{Q}[d+n]) = \oplus_i H^{m-d-n}(\iota^*\mathcal{F}_i[-i])$. This vector space contains as a direct factor the space

$$\begin{aligned}\oplus_i H^{m-d-n}(\iota^*\,\mathrm{IC}(R^{n+i}\pi_*\mathbb{Q})[-i]) &= \oplus_i H^{m-d-n-i}(\iota^*\,\mathrm{IC}(R^{n+i}\pi_*\mathbb{Q}))\\ &= \oplus_i\, \mathrm{IH}^{m-n-i}_s(R^{n+i}\pi_*\mathbb{Q}))\\ &= \oplus_{j+k=m}\, \mathrm{IH}^j_s(R^k\pi_*\mathbb{Q}).\end{aligned}$$

Moreover, if the inclusion in (i) is an isomorphism, the two spaces are equal. This proves (ii) and (iii). □

3. Proof of Theorem 1.7

Now we fix the notation from the introduction that X is a smooth $2m$ dimensional closed subvariety of $P = \mathbb{P}^N$ and $P^\vee$ is the dual projective space. Let $\mathcal{X} := \{(x, H) \in X \times P^\vee : x \in H\}$ denote the incidence variety. Write q and p for the projections on the first and second factors respectively. Then q is a $\mathbb{P}^{N-1}$-bundle. So $\mathcal{X}$ is smooth and irreducible of dimension $d_{\mathcal{X}} = n + N$ with $n = 2m - 1$. On the other hand, the fiber of $p : \mathcal{X} \to P^\vee$ over a hyperplane H is the hyperplane section $Y_H := H \cap X$. Write U for the locus of hyperplanes H such that Y_H is smooth, and set $\mathcal{X}_U = p^{-1}(U)$. Then the restriction of p to $\mathcal{X}_U$ gives a smooth, proper morphism $p_U : \mathcal{X}_U \to U$.

Set $\mathbf{H} := R^n p_{U*}\mathbb{Q}(m)$. This is a weight -1 variation of pure Hodge structure on U. By weak Lefschetz, it follows that the sheaves $R^{n-k} p_{U*}\mathbb{Q}$ are constant for $k > 0$. In fact they are the constant sheaves given by $\mathrm{H}^{n-k}(X, \mathbb{Q})$. Then, Hard Lefschetz shows that, for $k > 0$, $R^{n+k} p_{U*}\mathbb{Q}(k) \cong R^{n-k} p_{U*}\mathbb{Q}$. By Deligne's degeneracy theorem [5], $Rp_{U*}\mathbb{Q}(n) = \oplus_k R^k p_{U*}\mathbb{Q}[-k]$. So $Rp_{U*}\mathbb{Q}(n)$ is a direct sum of shifted constant sheaves and $\mathbf{H}[-n]$.

The following theorem, which is Theorem C of C. Schnell's paper [15], shows that an analogous decomposition holds on the level of $Rp_*\mathbb{Q}$ provided that $\mathbf{H}$ is non-constant. As explained by Beilinson in [1], the result is also a direct consequence of a much older paper of Brylinski on the Radon transform and perverse sheaves [4].

Theorem 3.1 (Beilinson, Brylinski, Schnell)**.** *Suppose* $\mathbf{H}$ *is non-constant. Then*

$$\begin{aligned} Rp_*\mathbb{Q}[d_{\mathcal{X}}] &= \bigoplus_k \mathrm{IC}(R^{n+k} p_{U*}\mathbb{Q})[-k] \\ &= \mathrm{IC}(\mathbf{H}) \oplus \bigoplus_{k\neq 0} \mathrm{IC}(R^{n+k} p_{U*}\mathbb{Q})[-k] \\ &= \mathrm{IC}(\mathbf{H}) \oplus (\bigoplus_{k<0} \mathrm{IC}(\mathrm{H}^{n+k}(X))[-k]) \oplus (\bigoplus_{k<0} \mathrm{IC}(\mathrm{H}^{n+k}(X))[k]). \end{aligned}$$

Here $\mathrm{IC}(\mathrm{H}^{n+k}(X))$ *simply denotes the constant perverse sheaf on* $P^\vee$ *with group* $\mathrm{H}^{n+k}(X)$*. In particular,* $Rp_*\mathbb{Q}[d_{\mathcal{X}}]$ *is the direct sum of* $\mathrm{IC}(\mathbf{H})$ *and (shifted) constant sheaves on* $P^\vee$*.*

Proof of Theorem 1.7. Pick $H \in P^\vee$ and set $Y = X \cap H$. We have $\mathrm{IH}^j_H(R^{n+i}\pi_*\mathbb{Q}) = 0$ for $ij \neq 0$. So, using Theorem 1.4 (iii), we have

$$\mathrm{H}^{n+j}(Y, \mathbb{Q}) = \mathrm{IH}^j_H(\mathbf{H}) \oplus \mathrm{IH}^0(R^{n+j}\pi_*\mathbb{Q}) \tag{3.2}$$

for $j \neq 0$. By Hard-Lefschetz, $R^{n+j}\pi_*\mathbb{Q}(j) \cong R^{n-j}\pi_*\mathbb{Q}$. So, since $\mathrm{IH}^j_H(\mathbf{H}) = 0$ for $j < 0$, we get that $b_{n+j}Y - b_{n-j}Y = \dim \mathrm{IH}^j_H(\mathbf{H})$ for $j > 0$ as desired. □

4. Examples

Terminology Reminder. If S is an irreducible complex scheme of finite type and P is a property of closed points of S, then P holds for the *general* (resp. *very general*) point of S if P holds outside of a finite (resp. countable) union of proper closed subschemes of S.

Cubic 4-folds. The paper [11] starts with a smooth cubic 4-fold X embedded in $P = \mathbb{P}^5$ and considers the family $p : \mathcal{X} \to P^\vee$. The family $p_U : \mathcal{X}_U \to U$ of smooth cubic 3 folds gives rise to a variation of Hodge structure $\mathbf{H}_\mathbb{Z} = R^3 p_{U*}\mathbb{Z}$ as in the introduction and a family $\pi : J(\mathbf{H}_\mathbb{Z}) \to U$ which, in this case, turns out to be a family of 5-dimensional abelian varieties.

Theorem 4.1 (Laza—Saccà—Voisin). *Suppose the cubic 4-fold X is very general. Then there is flat regular compactification $\bar{\pi} : \bar{J} \to P^\vee$ with irreducible fibers.*

Explanation. The fact that there exists a regular flat compactification $\bar{\pi} : \bar{J} \to P^\vee$ is part of the main theorem of [11]. The irreducibility of the fibers is not explicitly stated in [11], but it is an important part of the construction. Proving it amounts to tracing through several definitions and intermediate results in [11], which I now do.

By [11, Definition 4.11], the compactified relative Prym variety, $\overline{\mathrm{Prym}}\,\widetilde{\mathcal{C}}_B/\mathcal{C}_B$, is irreducible (as it is defined as an irreducible component of a larger scheme). By [11, Proposition 4.16], this definition is stable under base change so that the fiber over a point $b \in B$ is also irreducible. Then [11, Proposition 5.1] states that $\overline{\mathrm{Prym}}\,\widetilde{\mathcal{C}}_B/\mathcal{C}_B \to B$ is flat when B is a certain Fano variety $\mathcal{F}^0$ of lines.

Section 5 of [11] descends the family of Prym varieties over $\mathcal{F}^0$ (which maps surjectively to $\mathbb{P}^5$) to $P^\vee = \mathbb{P}^5$. As explained in the paragraph between Lemmas 5.3 and 5.4 of [11], the result is a family $\bar{J} \to P^\vee$ whose pullback to $\mathcal{F}^0$ is the above family of compactified Prym varieties. Since the compactified Prym varieties are irreducible, the fibers of $\bar{J} \to P^\vee$ are as well. □

Remark 4.2. In [11], the authors, in fact, prove that the morphism $\pi : J \to U$ has a flat, regular compactification $\bar{\pi} : \bar{J} \to P^\vee$ with irreducible fibers for X *Hodge* general. In other words, the conclusion holds for all X outside of the inverse image of a countable union of divisors in the image of the period map. The authors inform me that *a postiori* it follows that the conclusion holds for general X. That is, it holds for all X in some Zariski open subset of the space of cubic 4-folds. I will not give an argument proving this fact (which is not needed in what follows). However, the proof of Corollary 4.3 below goes from very general to general in a similar way.

Corollary 4.3. *Suppose X is a general cubic 4-fold and $H \in P^\vee$ is any hyperplane. Then $Y := X \cap H$ is palindromic.*

Proof. By Theorem 4.1 and Corollary 1.8, we see that Y is palindromic for X very general and Y arbitrary. The conclusion with X general now follows from the constructibility of comhomology sheaves.

To be explicit about this last point, we can consider the space B of pairs (X, L) with X a cubic 4-fold and L a hyperplane in $\mathbb{P}^5$. Over B, we have a universal family $f : \mathcal{Y} \to B$ whose fiber over (X, L) is the intersection $Y = X \cap L$. If V denotes the projective space of all cubic 4-folds, we have a morphism $g : B \to V$ which forgets L. Write C for the locus of points in B consisting of pairs (X, L) where $H^*(X \cap L)$ fails to be palindromic. By the constructibility of the sheaves $R^k f_*\mathbb{C}$, C is a (locally closed) subscheme of B. By the results cited above, $H^*(X \cap L)$ is palindromic for all pairs $(X, Y) \in B$ with X outside of a countable union $Z = \cup_{i=1}^\infty Z_i$ of divisors in V. It follows that $g(C)$ is contained in Z. Therefore, since $g(C)$ is constructible, $g(C)$ is contained in a divisor D in V. So $H^*(X \cap L)$ is palindromic for all pairs (X, L) with X outside of D. □

This motivates the following conjecture.

Conjecture 4.4. *Suppose X is a general complete intersection in $P = \mathbb{P}^N$ of multi-degree $(d_1, d_2, \dots, d_k)$ with $d_1 \leq \cdots \leq d_k$. Assume that $d_1 \gg 0$. Then, for any hyperplane $H \in P^\vee$, the hyperplane section $Y := X \cap H$ is palindromic.*

Remark 4.5. Perhaps "$\gg$" could be replaced with a reasonable lower bound.

Here is a simple argument proving the conjecture when X is a very general surface complete intersection of multi-degree not equal to $(2),(3)$ or $(2,2)$. In that case, the Noether-Lefschetz theorem says that the Néron-Severi group of X is $\mathbb{Z}$ with generator $[X\cap H]$ (for any hyperplane H). (See Voisin's book [16, Theorem 3.32] and [10, Theorem 1] for modern proofs.) It follows easily that $Y := X\cap H$ is irreducible (since $[Y]$ cannot be the direct sum of two non-trivial effective divisors). But, since Y is a curve, this implies that Y is palindromic. Note that we can then deduce that the conjecture hold for general (as opposed to very general) surface complete interesections X using the arguments similar to the proof of Corollary 4.3 above.

Proposition 4.6. *Suppose X is a smooth hypersurface in $P = \mathbb{P}^N$ and $H \in P^\vee$ is any hyperplane section. Then $Y := X\cap H$ is weakly palindromic.*

Sketch. This is well-known (see [6, Theorem 2.1]). So I only give a sketch. The main point is that Y has isolated singularities. From this one can either use the Clemens-Schmid exact sequence or an argument comparing the intersection cohomology with the ordinary cohomology. □

Suppose X is an arbitrary smooth cubic 4-fold. Since every hyperplane section is weakly palindromic, Corollary 1.8 does not rule out the existence of a flat regular compactification $\bar{\pi} : \bar{J} \to P^\vee$. However, it can rule out the existence of a flat regular compactification with irreducible fibers: If X is a cubic fourfold containing a non-palindromic cubic 3-fold $Y = X\cap H$, then the fiber $\bar{\pi}^{-1}\{H\}$ is not irreducible. To find such a cubic 4-fold we use the following Lemma.

Lemma 4.7. *Suppose $Y = V(f)$ is a degree d hypersurface in $\mathbb{P}^{N-1}$. Fix a (linear) embedding $\mathbb{P}^{N-1} \subset \mathbb{P}^N$. Then there is a smooth degree d hypersurface X in $\mathbb{P}^N$ such that $Y = X\cap \mathbb{P}^{N-1}$ if and only if Y has isolated singularities.*

Sketch. The "only if" part is easy (and was already used above in the proof of Proposition 4.6). For the "if" part, suppose $f(x_1,\dots,x_N)$ is a degree d homogeneous polynomial. Consider the linear subspace V in $\mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ spanned by f and x_0h as h runs over all degree $d-1$ homogeneous polynomials in the $N+1$ variables. The base locus of the linear system $|V|$ is Y. So the general member of $|V|$ is smooth off of Y by Bertini. But the singularities of $g = f + x_0h$ on Y are contained in the intersection of $V(h)$ with the singularities of Y. Therefore, the general member of $|V|$ is smooth. □

Now, I use a result of Segre and Fano as interpreted by Dolgachev.

Theorem 4.8. *There exists a cubic threefold Y smooth outside of 10 ordinary double points and with $b_4Y = 6$.*

Proof. See [7, Proposition 1.1] and the discussion shortly before and shortly after. □

Corollary 4.9. *There exists a smooth, cubic 4-fold X containing a cubic 3-fold Y with $b_4Y = 6$. For such an X, there is no flat regular compactification $\bar{\pi} : \bar{J} \to P^\vee$ with irreducible fibers.*

Proof. The cubic 3-fold Y with $b_4Y = 6$ is not palindromic since $b_2Y = 1$ by weak Lefschetz. Using Lemma 4.7, we can find a smooth cubic 4-fold X containing Y. The result then follows from Corollary 1.8. □

Quadrics in Cubic 4-folds. Suppose X is a smooth $2m$-dimensional subvariety in $P = \mathbb{P}^N$ as in the beginning of §3. The family $\pi : J(\mathbf{H}_\mathbb{Z}) \to U$ of intermediate Jacobians will be an abelian scheme provided the Hodge structure $\mathrm{H}^{2m-1}Y$ of a smooth hyperplane section Y is level ≤ 1. This means that $\mathrm{H}^{2m-1}(Y,\mathbb{C}) = F^m\mathrm{H}^{2m-1}(Y,\mathbb{C})$. As in §3, we set $n = 2m-1$.

I do not have a very clear idea how often the situation above occurs for arbitrary X. However, in [13], Rapoport has a table of all complete intersections Y for which the Hodge level of the middle dimensional cohomology is 1. Write $V_n(d_1, \ldots, d_k)$ for the family of smooth complete intersections of dimension n coming from intersecting k hypersurfaces of degrees $d_1, \ldots, d_k$ in $\mathbb{P}^{n+k}$. Then, according to Rapoport's table, the only non-empty families with middle dimensional cohomology of level one (and n odd) are: $V_n(2,2)$, $V_n(2,2,2)$, $V_3(3)$, $V_3(2,3)$, $V_5(3)$ and $V_3(4)$.

The case where Y is a cubic 3-fold, $V_3(3)$, was the subject of the last subsection. In this subsection, I want to consider $V_3(2,3)$.

So fix a cubic 4-fold X embedded in $\mathbb{P}^5$. Set $\mathcal{L} := \mathcal{O}_{\mathbb{P}^5}(1)_{|X}$. An easy computation shows that $|\mathcal{L}^2| = \dim H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)) - 1 = 20$. So, the complete linear system $\mathcal{L}^2$, gives an embedding of X into $P := \mathbb{P}^{20}$. Cutting X with hyperplanes $H \subset P$, we get a family $p : \mathcal{X} \to P^\vee$ as in the beginning of §3 which is smooth over an open subset $U \subset P^\vee$. Since the smooth hyperplane sections are complete intersections of type $V_3(2,3)$, they have level 1. Therefore, the family $\pi : J(\mathbf{H}_\mathbb{Z}) \to U$ is an abelian scheme. In fact, Rapoport's table also gives $b_3 Y = 40$ for Y of type $V_3(2,3)$. So the family $\pi : J(\mathbf{H}_\mathbb{Z}) \to U$ is, in fact, a family of 20-dimensional abelian varieties over a 20 dimensional base.

Theorem 4.10. *Let X be a cubic 4-fold as above embedded in $P = \mathbb{P}^{20}$. There is no regular flat compactification $\bar{\pi} : \bar{J} \to P^\vee$ of the family $\pi : J(\mathbf{H}_\mathbb{Z}) \to U$ of intermediate Jacobians.*

Proof. The elements $H \in P^\vee$ are in 1-1 correspondence with quadrics in $\mathbb{P}^5$. Pick two hyperplanes L_1 and L_2 in $\mathbb{P}^5$ such that the cubic 3 folds $Y_i := X \cap L_i$ are smooth and distinct. Let H be the point in $P^\vee$ corresponding to the union $L_1 \cup L_2$. Then $Y := X \cap H$ has two irreducible components. Therefore $b_6 Y = 2$. So Y is not weakly palindromic. The result follows from Corollary 1.8. □

Remark 4.11. In [11, §1.3], Laza, Saccà and Voisin point out that the total space of the family $J(\mathbf{H}_\mathbb{Z}) \to U$ admits a holomorphic symplectic form which would extend to any compactification $\bar{\pi} : \bar{J} \to P^\vee$. They also show that this form is non-degenerate above a quadric if and only if the quadric is non-degenerate [11, Lemma 1.20].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD USA
E-mail address: pbrosnan@math.umd.edu