ESSENTIAL DIMENSION OF ABELIAN VARIETIES OVER NUMBER FIELDS

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ABSTRACT. We affirmatively answer a conjecture in the preprint "Essential dimension and algebraic stacks," proving that the essential dimension of an abelian variety over a number field is infinite.

Let k be a field and let Fields_k denote the category whose objects are field extensions L/k and whose morphisms are inclusions $M \hookrightarrow L$ of fields. Let $F : \mathrm{Fields}_k \to \mathrm{Sets}$ be a covariant functor. A *field of definition* for an element $a \in F(L)$ is a subfield M of L over k such that $a \in \mathrm{im}(F(M) \to F(L))$. The *essential dimension* of $a \in F(L)$ is $\mathrm{ed}\, a := \inf\{\mathrm{trdeg}_k M \,|\, M \text{ is a field of definition for } a\}$. The essential dimension of the functor F is $\mathrm{ed}\, F := \sup\{\mathrm{ed}\, a \,|\, L \in \mathrm{Fields}_k, a \in F(L)\}$.

If G is an algebraic group over k, we write $\mathrm{ed}\, G$ for the essential dimension of the functor $L \rightsquigarrow \mathrm{H}^1_{\mathrm{fppf}}(L, G)$. That is $\mathrm{ed}\, G$ is the essential dimension of the functor sending a field L to the set of isomorphism classes of G-torsors over L. The notion of essential dimension of a finite group was introduced by J. Buhler and Z. Reichstein. The definition of the essential dimension of a functor is a generalization given later by A. Merkurjev. In [3] (which the reader could consult for further background), a notion of essential dimension for algebraic stacks was introduced. In the terminology of that paper, $\mathrm{ed}\, G$ is the essential dimension of the stack $\mathscr{B}G$.

The purpose of this paper is to generalize the following result.

Theorem 1 (Corollary 10.4 [3])**.** *Let E be an elliptic curve over a number field k. Assume that there is at least one prime $\mathfrak{p}$ of k where E has semistable bad reduction. Then* $\mathrm{ed}\, E = +\infty$.

Note that another equivalent way of stating the theorem is to say that $\mathrm{ed}\, E = +\infty$ for any elliptic curve E over a number field such that $j(E)$ is not an algebraic integer. The result was proved by showing that Tate curves have infinite essential dimension. This method does not apply to elliptic curves with integral j invariants. Nonetheless, Conjecture 10.5 of [3] guesses that $\mathrm{ed}\, E = +\infty$ for all elliptic curves over number fields. This conjecture is answered by the following.

Theorem 2. *Let A be a non-trivial abelian variety over a number field k. Then* $\mathrm{ed}\, A = +\infty$.

Note that if A is an abelian variety over $\mathbb{C}$, then $\mathrm{ed}\, A = 2 \dim A$. This is the main result of [2].

The theorem is an easy consequence of the following result whose formulation does not involve essential dimension. To state it, for a positive integer m, let μ_m denote the group scheme of m-th roots of unity; and, for a rational prime l, let μ_{l^∞} denote the union $\cup_{n \in \mathbb{Z}_+} \mu_{l^n}$.

Theorem 3. *Let A be a non-trivial abelian variety over a number field k. Then there is an odd prime ℓ and an algebraic field extension L/k such that*

(1) $\mathbb{Q}_\ell/\mathbb{Z}_\ell \subset A(L)$.
(2) $1 < |\mu_{\ell^\infty}(L)| < \infty$.

In the first section, we derive Theorem 2 from Theorem 3. To do this, we use a result of M. Florence concerning the essential dimension of $\mathbb{Z}/\ell^n$. In section 2, we prove Theorem 3. Here the main results used are those of Bogomolov and Serre on the action of the absolute Galois group $\mathrm{Gal}(k)$ on the Tate module $T_\ell A$.

Remark 4. The recent preprint [7] of Karpenko and Merkurjev provides another way to show that the essential dimension of an abelian variety over a number field is infinite. To be precise, by generalizing the results of that paper slightly, one can use them to compute the essential dimension of the group scheme $A[n]$ of n-torsion points of an abelian variety. In fact, using this idea one can show that the essential dimension of an abelian variety over a p-adic field is also infinite. However, the present proof of Theorem 2 is shorter than a proof using [7] would be and we hope that Theorem 3 is independently interesting.

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1. Theorem 3 implies Theorem 2

As mentioned above, we will use the following result [6, Theorem 4.1] of M. Florence.

Theorem 5. *Let ℓ be an odd prime and r a positive integer. Let $L/\mathbb{Q}$ be a field such that $|\mu_{\ell^\infty}(L)| = \ell^r$. Then, for any positive integer k,*

$$\mathrm{ed}_L \mathbb{Z}/\ell^k = \max\{1, \ell^{k-r}\}.$$

Corollary 6. *Let A be an abelian variety over a field L of characteristic 0. Let ℓ be an odd prime and suppose that the statements in the conclusion of Theorem 3 are satisfied; i.e:*

(1) $\mathbb{Q}_\ell/\mathbb{Z}_\ell \subset A(L)$.
(2) $1 < |\mu_{\ell^\infty}(L)| < \infty$.

Then $\mathrm{ed} A = +\infty$.

Proof. Since L satisfies (2), $\mathrm{ed}_L \mathbb{Z}/\ell^n \to \infty$ as $n \to \infty$. By (1), there is an injection $(\mathbb{Z}/\ell^n)_L \to A$. Therefore, by [1, Theorem 6.19], $\mathrm{ed} A \geq \mathrm{ed}_L \mathbb{Z}/\ell^n - \dim A$ for all n. Letting n tend to ∞, we see that $\mathrm{ed} A = +\infty$. □

Proof of Theorem 2 assuming Theorem 3. Let A be a non-trivial abelian variety over a number field k. Using Theorem 3 and Corollary 6, we can find a field extension L/k such that $\mathrm{ed} A_L = +\infty$. This implies that $\mathrm{ed} A = +\infty$ (by [1, Proposition 1.5]).

2. Galois representations and the proof of Theorem 3

Let A be a non-trivial abelian variety over k as in Theorem 3. Before proving Theorem 3, we fix some (standard) notation. We write $\mathrm{Gal}(k) := \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois group of the number field k. For a rational prime ℓ, we write $T_\ell A$ for the Tate-module $\varprojlim A[\ell^n]$ of the abelian variety A. We write $V_\ell A$ for $T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. For an integer n, we write $\mathbb{Z}/n(1)$ for μ_n, and for $j \in \mathbb{Z}$, we write $\mathbb{Z}/n(j)$ for $\mu_n^{\otimes j}$. We write $\mathbb{Z}_\ell(j) := \varprojlim \mathbb{Z}/l^m(j)$.

For any prime $\mathfrak{p}$ of k where A has good reduction, write $T_\mathfrak{p}$ for the corresponding Frobenius torus. (For this notion see [4, Definition 3.1 and p. 326] or [9].) Since A is non-trivial, $T_\mathfrak{p}$ contains a rank 1 torus $D \cong \mathbf{G}_m$ such that, for every rational prime $\ell \notin \mathfrak{p}$, $D(\mathbb{Q}_\ell) \subset \mathbf{GL}(V_\ell A)$ is the set of homotheties (i.e. scalar matrices) [4, Proposition 3.2].

Lemma 7. *Let $\mathfrak{p}$ be a prime of k such that the reduction $A/\mathfrak{p}$ of A at $\mathfrak{p}$ is good but not supersingular. Then the rank of $T_\mathfrak{p}$ is strictly greater than 1.*

Proof. This follows directly from [4, Proposition 3.3]. □

The following lemma was suggested to us by N. Fakhruddin.

Lemma 8. *Let V be an n-dimensional vector space over a field F, and let T be an F-split torus in $\mathbf{GL}_V$ of rank at least 2 containing the homotheties. Then there is a non-zero vector $v \in V$ and a rank 1 subtorus S of T such that*

(1) *S fixes v;*
(2) *the determinant map $\det : S \to \mathbf{G}_m$ is surjective.*

Proof. The proof is elementary linear algebra with the character lattice, $X^*(T)$.

We can find a basis $e_1, \dots, e_n$ of V and characters $\lambda_1, \dots, \lambda_n \in X^*(T)$ such that $te_i = \lambda_i(t)e_i$ for $t \in T, i \in \{1, \dots, n\}$. Since T contains the homotheties, det is a non-trivial character of T. Moreover, since $T \subset \mathbf{GL}_V$, the λ_i generate $X^*(T)$. Since $\dim X^*(T) \otimes \mathbb{Q} \geq 2$, it follows that there exists i such that $\lambda_i^\perp \not\subset \det^\perp$. Thus we can find a cocharacter ν such that $\langle \nu, \lambda_i \rangle = 0$ but $\langle \nu, \det \rangle \neq 0$. Set S equal to the image of ν in T and $v = e_i$. □

Proof of Theorem 3. Let A be a non-trivial abelian variety over a number field k. We can find a prime $\mathfrak{p}$ in k such that A has good reduction at $\mathfrak{p}$ but $A/\mathfrak{p}$ is not supersingular. (This is well-known if $\dim A = 1$: the case where A has CM is standard and otherwise it follows from the exercise on page IV-13 of [8]. When $\dim A > 1$ it can be proved by adapting the exercise as Ogus does in Corollary 2.8 of his notes in [5].) Thus the Frobenius torus $T_\mathfrak{p}$ has rank at least 2. Using Tchebotarev density, it is easy to see that $T_\mathfrak{p} \otimes \mathbb{Q}_\ell$ is a split torus for all rational primes ℓ in a set of positive density. Thus, we can find an odd rational prime ℓ such that $\ell \notin \mathfrak{p}$ and $T_\mathfrak{p} \otimes \mathbb{Q}_\ell$ is split. Now, set $F = k(\zeta_\ell)$ where ζ_ℓ is a primitive ℓ-th root of unity. Note that $T_\mathfrak{p}$ is the Frobenius torus for A_F as Frobenius tori are invariant under finite extension of the ground field.

Now, using Lemma 8, we can can find a rank 1 subtorus $S \subset T_\mathfrak{p} \otimes \mathbb{Q}_\ell$ and a vector $v \in T_\ell A_F$ such that S fixes v and $\det : S \to \mathbf{G}_m$ is surjective. Let $\rho : \mathrm{Gal}(F) \to \mathrm{Aut}(V_\ell A_F)$ denote the Galois representation on the Tate module and let $H = \{g \in \mathrm{Gal}(F) \,|\, \rho(g)v = v\}$. By a theorem of Bogomolov [4, Theorem B] (and the fact that S fixes v), it follows that

$$\mathrm{Lie}(S) \subset \mathrm{Lie}(\rho(H))$$

where $\mathrm{Lie}(S)$ denotes the Lie algebra of S as an algebraic group and $\mathrm{Lie}(\rho(H))$ denotes the Lie algebra as an ℓ-adic group. Therefore the intersection of $S(\mathbb{Q}_\ell)$ with $\rho(H)$ contains an open neighborhood of the identity in $S(\mathbb{Q}_\ell)$. In particular, $\det(H)$ contains a neighborhood of the identity in $\mathbb{Q}_\ell^*$. Set $L := \overline{F}^H$. Then, from the fact that v is fixed by H, it follows that $\mathbb{Q}_\ell/\mathbb{Z}_\ell \subset A(L)$. On the other hand, since $\wedge^{2\dim A} T_\ell A \cong \mathbb{Z}_\ell(\dim A)$, the fact that $\det(H)$ contains a neighborhood of the identity in $\mathbb{Q}_\ell^*$ implies that $\mu_{\ell^\infty}(L)$ is finite. This completes the proof of Theorem 3.

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