# COHOMOLOGY OPERATIONS AND INVERTING THE MOTIVIC BOTT ELEMENT

PATRICK BROSNAN AND ROY JOSHUA

Abstract. In this note we explore the relationships between the motivic cohomology operations and the (classical) cohomology operations defined on mod-$l$ étale cohomology. More precisely we show that the cohomology operations on motivic cohomology transform to the (classical) cohomology operations on mod-$l$ étale cohomology upon inverting the motivic Bott element.

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## 1. Introduction

The main result of this note is that the motivic cohomology operations transform to the classical cohomology operations on étale cohomology upon inverting the motivic Bott element. Throughout the paper $k$ will denote a fixed perfect field of characteristic $p \geq 0$ and $X$ will denote a smooth scheme of finite type over $k$. We will also assume that $k$ has a primitive $l$-th root of unity, where $l$ will denote a fixed prime different from $p$. $H^n_{\mathcal{M}}(X, \mathbb{Z}(r))$ will denote the motivic cohomology with *degree* $n$ and *weight* $r$; $H^n_{\mathcal{M}}(X, \mathbb{Z}/l(r))$ will denote the corresponding mod-$l$-variant. Similarly $H^n_{et}(X, \mathbb{Z}/l(r))$ will denote the mod-$l$ étale cohomology of $X$. We will restrict to smooth schemes of finite type over $k$.

Let $\beta \varepsilon H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1))$ denote the *Motivic Bott element.* In this situation, let $P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$ and $\beta P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$ denote the motivic cohomology operations defined in [Voev] and recalled below in the next section. As shown in section 3, these operations induce operations on mod-$l$-étale cohomology which we identify with the mod-$l$-motivic cohomology with the Bott element inverted: these will be denoted by the same symbols. By the results of Theorem 1.2 and section 6 of [J] the complex $\mathcal{A} = R\Gamma(X_{et}, \mu_l)$ is an $E_\infty$-algebra over an $E_\infty$-operad. Therefore one obtains certain (*classical*) cohomology operations $Q^r : H^{2q}_{et}(X, \mu_l(q)) \rightarrow H^{2q+2r(l-1)}_{et}(X, \mu_l(q.l))$ and $\beta Q^r : H^{2q}_{et}(X, \mu_l(q)) \rightarrow H^{2q+2r(l-1)+1}_{et}(X, \mu_l(q.l))$

Then the main result of this paper is the following.

**Theorem 1.1.** *Assume the above situation. Then we obtain the relation between the classical operations and the operations on étale cohomology induced by the motivic operations:*

$$Q^r = B^{(q-r).(l-1)}.P^r, \quad \beta Q^r = B^{(q-r).(l-1)}.\beta P^r$$

Here is an outline of the paper. In the next section, we recall the motivic cohomology operations from [Voev]. In the third section we recall the motivic Bott element. In the fourth section we first recall a well-known result that the operadic construction of classical cohomology operations leads to the cohomology operations $Q^r$ and $\beta Q^r$ that may be defined using equivariant cohomology. Then we complete the proof of the main theorem.

## 2. The motivic cohomology operations (after Voevodsky)

The basic reference for this section is [Voev]. We begin with the computation of the motivic cohomology of $B\pi$ where $\pi = \mathbb{Z}/l$ and $\pi = \Sigma_l$ (the symmetric group on $l$ letters) where $l$ is a fixed prime different from the characteristic ($= p$) of the ground field $k$.

The second author was supported by the IHES, the MPI (Bonn) and a grant from the NSA.

We begin by recalling briefly the construction of the *geometric classifying space of a linear algebraic group*: originally this is due to Totaro and Edidin-Graham - see [Tot] discussed in [J, section 4]. Let G be a linear algebraic group over $S = Spec \quad k$ i.e. a closed subgroup-scheme in $GL_n$ over $S$ for some n. For a (closed) embedding $i : G \rightarrow GL_n$ *the geometric classifying space* $B_{gm}(G; i)$ of $G$ with respect to $i$ is defined as follows. For $m \geq 1$ let $U_m$ be the open sub-scheme of $\mathbb{A}^{nm}_S$ where the diagonal action of $G$ determined by $i$ is free. Let $V_m = U_m/G$ be the quotient $S$-algebraic space of the action of $G$ on $U_m$ induced by the (diagonal) action of $G$ on $\mathbb{A}^{nm}_S$; the projection $U_m \rightarrow V_m$ defines $V_m$ as the quotient algebraic space of $U_m$ by the free action of $G$ and $V_m$ is thus smooth. We have closed embeddings $U_m \rightarrow U_{m+1}$ and $V_m \rightarrow V_{m+1}$ corresponding to the embeddings $Id \times \{0\} : \mathbb{A}^{nm} \rightarrow \mathbb{A}^{nm} \times \mathbb{A}^n$ and we set $EG^{gm} = \lim_{m \rightarrow \infty} U_m$ and $BG^{gm} = \lim_{m \rightarrow \infty} V_m$ where the colimit is taken in the category of sheaves on $(sschms/S)_{Nis}$ or on $(sschms/S)_{et}$. Observe that if $G = \Sigma_n$ (or a subgroup of it) acting on $\mathbb{A}^n$ by permuting the $n$-coordinates, we may take $U_m = \{(x(1)_1, \cdots, x(1)_n, \cdots, x(m)_1, \cdots x(m)_n) | x(i)_j \neq x(i)_k, \quad for \quad all \quad i, \quad j \neq k\}$. (Moreover, in this case, the quotients $V_m$ are in fact schemes.)

2.0.1. The following are proven in [Voev, section 6]:

- the map $i_m : U_m/G \rightarrow U_{m+1}/G$ defines an isomorphism on motivic cohomology of weight less than $m$.
- One has $H^*_{\mathcal{M}}(BG^{gm}, \mathbb{Z}(r)) = \lim_{\infty \leftarrow m} H^*_{\mathcal{M}}(U_m/G, \mathbb{Z}(r))$ where $r \geq 0$ is any weight.

• Let $\mu_l$ denote the group scheme of l-th roots of unity $\mu_l := ker(\mathbf{G}_m \stackrel{z^l}{\rightarrow} \mathbf{G}_m)$. (Observe that since the field $k$ is assumed to have a primitive root of unity, one may identify $\mu_l$ with the constant sheaf $\pi = \mathbb{Z}/l$.) Then one has the identification:

$$B\mu_l = \mathcal{O}(-l)_{\mathbf{P}^\infty} - z(\mathbf{P}^\infty) \tag{2.0.2}$$

We have $U_m = \mathbf{A}^m - \{0\}$.

• Therefore, one has a cofibration sequence of the form

$$X_+ \wedge (B\mu_l)_+ \rightarrow X_+ \wedge (\mathcal{O}(-l)_{\mathbf{P}^\infty})_+ \rightarrow X_+ \wedge Th(\mathcal{O}(-l)) \tag{2.0.3}$$

• $e(\mathcal{O}(-l)) = l\sigma$ where $\sigma \in H^2_{\mathcal{M}}(\mathbf{P}^\infty; \mathbb{Z}(1))$ is the class of the first Chern class of $\mathcal{O}(-1)$ in motivic cohomology. Here $X$ is any smooth scheme. Therefore, the long exact sequence defined by ( 2.0.3) is of the form

$$\ldots \rightarrow H^{*-2}_{\mathcal{M}}(X, \mathbb{Z}(\star-1)[[\sigma]] \stackrel{l\sigma}{\rightarrow} H^*_{\mathcal{M}}(X, \mathbb{Z}(\star-1))[[\sigma]] \rightarrow H^*_{\mathcal{M}}(B\mu_l, \mathbb{Z}(\star)) \rightarrow H^{*-1}(X, \mathbb{Z}(\star-1)[[\sigma]] \rightarrow \ldots \tag{2.0.4}$$

(In the above long-exact-sequence and elsewhere, $*$ ($\star$) will denote the degree (the weight, respectively) in motivic cohomology.)

The short exact sequence of abelian groups $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}/l \rightarrow 0$ defines a homomorphism $\delta : \tilde{H}^*_{\mathcal{M}}(-, \mathbf{Z}/l(\star)) \rightarrow \tilde{H}^{*+1}_{\mathcal{M}}(-, \mathbf{Z}(\star))$. Let $v$ be Euler class of the line bundle on $X \times B\mu_l$ corresponding to the tautological representation of $\mu_l$. There exists a unique element $u \in H^1(X \times B\mu_l, \mathbf{Z}/l(1))$ such that the restriction of $u$ to $*$ is zero and $\delta(u) = v$. (Here $*$ denotes any $k$-rational point of $B\mu_l$ that lifts to a $k$-rational point of one of the $U_m$ appearing in the definition of the $E\mu_l$.)

• We will denote by $\bar{v}$ the image of the class $v$ in $H^2_{\mathcal{M}}(X \times B\mu_l, \mathbb{Z}/l(1))$. Now the elements $\bar{v}^i$ and $u\bar{v}^i$, $i \geq 0$ form a basis of $H^*_{\mathcal{M}}(X \times (B\mu_l), \mathbb{Z}/l(\star))$ over $H^*_{\mathcal{M}}(X, \mathbb{Z}/l(\star))$.

The *next key observation* is that the same arguments also hold for the mod-$l$ étale cohomology of $B\mu_l$, so that we may conclude:

Let *cycl* denote the cycle map from mod-$l$ motivic to mod-$l$ étale cohomology. Let $c(\bar{v})$ denote the Euler class of of the same bundle on $X \times B\mu_l$ in $H^2_{et}(X \times B\mu_l, \mathbb{Z}/l(1))$. Then $c(\bar{v}) = cycl(\bar{v})$, and there exists a unique class $c(u) \varepsilon H^1_{et}(X \times B\mu_l, \mathbb{Z}/(1))$ so that $\delta(c(u)) = c(\bar{v})$ and $c(u) = cycl(u)$. Then the elements $c(\bar{v})^i$ and $c(u)c(\bar{v})^i$, $i \geq 0$ form basis of $H^*_{et}(X \times B\mu_l, \mathbb{Z}/l(\star))$ over $H^*_{et}(X, \mathbb{Z}/l(\star))$.

Next one may compute the mod-$l$ motivic cohomology and mod-$l$ étale cohomology of the symmetric group $\Sigma_l$ similarly. We recall this from [Voev]:

2.1. $H^*_{\mathcal{M}}(X \times B\Sigma_l; \mathbb{Z}/l(\star))$ is a free module over $H^*_{\mathcal{M}}(X;\mathbb{Z}/l(\star))$with a basis $\{c\bar{d}^i, d^i | i \geq 0\}$ where $\bar{d}$ is a class in $H^{2l-2}_{\mathcal{M}}(X \times B\Sigma_n; \mathbb{Z}/l(l-1))$ which is the mod-$l$ reduction of a class $d\varepsilon H^{2l-2}_{\mathcal{M}}(X \times B\Sigma_n; \mathbb{Z}(l-1))$ and $c$ is a class in $H^{2l-3}_{\mathcal{M}}(X \times B\Sigma_n; \mathbb{Z}/l(l-1))$ so that $\delta(c) = \bar{d}$.

Going over the computation, one observes as in the case of $B\mu_l$, that the same computation carries over to mod-$l$ étale cohomology.

Next we recall the definition of the cohomology operations of Voevodsky. Let $X$ denote a smooth scheme over $k$.

Now the symmetric group $\Sigma_l$ acts on $X^{\times^l}$ by permutations. In this context, one has the total power operation:

(2.1.1)
$$\tilde{\mathcal{P}}_l : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{il}_{\mathcal{M}}(E\Sigma_l \underset{\Sigma_l}{\times} X^{\times^l}, \mathbb{Z}/l(jl))$$

Next one uses the pull-back by the diagonal $\Delta^* : H^{il}_{\mathcal{M}}(E\Sigma_l \underset{\Sigma_l}{\times} X^{\times^l}, \mathbb{Z}/l(jl)) \rightarrow H^{il}_{\mathcal{M}}(B\Sigma_l \times X, \mathbb{Z}/l(jl))$. We will denote the composition $\Delta^* \circ \tilde{\mathcal{P}}_l$ by $\mathcal{P}_l$. By the above results, $\underset{i,j}{\oplus} H^{in}_{\mathcal{M}}(B\Sigma_l \times X, \mathbb{Z}/l(jl))$ is a free module over $H^*_{\mathcal{M}}(X, \mathbb{Z}/(\star))$ with basis given by the elements $\bar{d}^r$ and $c\bar{d}^r$, $r \geq 0$. The operation $P^r$ $(\beta P^r)$ is defined by the formula:

(2.1.2)
$$\mathcal{P}_l(w) = \Sigma_{r \geq 0} P^r(w)\bar{d}^{d-r} + \beta P^r(w)c\bar{d}^{d-r-1}, \quad w\varepsilon H^{2d}(\quad, \mathbb{Z}/l(d))$$

(A crucial observation is that, since the motivic cohomology operations are stable with respect to shifting degrees by 1, and also both degrees and weights by 1, this defines the operations $P^r$ and $\beta P^r$ on all $H^i_{\mathcal{M}}(\quad, \mathbb{Z}/l(j))$.)

Observe that so defined $P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$ and $\beta P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$.

In view of the observations above, exactly the same definitions will define the cohomology operations in mod-$l$ étale cohomology as well. We will denote the operations $P^r : H^i_{et}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)}_{et}(X, \mathbb{Z}/l(j + r(l-1)))$ $(\beta P^r : H^i_{et}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)+1}_{et}(X, \mathbb{Z}/l(j + r(l-1))))$ by $P^r_{et}$ ($\beta P^r_{et}$, respectively). Therefore, we obtain the following result:

**Theorem 2.1.** *Denoting the cycle map from motivic cohomology to étale cohomology by cycl, we obtain:* $cycl \circ P^r = P^r_{et} \circ cycl$ *and* $cycl \circ \beta P^r = \beta P^r_{et} \circ cycl$.

## 3. Inverting the Motivic Bott element

Recall that if $k$ is a field as above, we have:

(3.0.3)
$$\begin{aligned} H^p_{\mathcal{M}}(Spec \quad k, \mathbb{Z}(1)) &= 0, p \neq 1 \\ &= k^*, p = 1 \end{aligned}$$

Now the universal coefficient sequence associated to the short exact sequence $0 \rightarrow \mathbb{Z}(1) \overset{\times l}{\rightarrow} \mathbb{Z}(1) \rightarrow \mathbb{Z}/l(1) \rightarrow 0$ of motivic complexes, provides the isomorphism

(3.0.4)
$$H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1)) \cong \mu_l(k)$$

The *Motivic Bott element* is the class in $H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1))$ corresponding under the above isomorphism to the primitive $l$-th root of unity $\zeta$. We will denote this element by $B$. Since $cycl(B) = \zeta$ in $H^*_{et}(\quad, \mu_l(*))$, multiplication by the class $cycl(B)$ induces an isomorphism: $H^*_{et}(\quad, \mu_l(r)) \rightarrow H^*_{et}(\quad, \mu_l(r+1))$. It follows that the cycle map $cycl$ induces a map of cohomology functors:

(3.0.5)
$$cycl(B^{-1}) : H^*_{\mathcal{M}}(\quad, \mathbb{Z}/l(\star))[B^{-1}] \rightarrow H^*_{et}(\quad, \mu_l(\star))$$

It is shown in [Lev] that this map is an isomorphism on smooth schemes.

3.1. Observe that by the multiplicative properties of the operations and the observation that $P^r(B) = 0$ if $r \geq 1$ ([Voev, Lemma 9.8]):

$$P^r(B^j\alpha) = B^j P^r(\alpha), \tag{3.1.1}$$

$$\beta P^r(B^j\alpha) = B^j \beta P^r(\alpha) \tag{3.1.2}$$

The above relations show that the motivic cohomology operations above induce operations on $H^*(\quad, \mathbb{Z}/l(\star))[B^{-1}]$ in the obvious manner: we define $P^r(\alpha.B^{-1}) = P^r(\alpha).B^{-1}$ and $\beta P^r(\alpha.B^{-1}) = \beta P^r(\alpha).B^{-1}$. Since we have already observed that the cohomology operations commute with the cycle map, it follows that the induced operations on $H^*(\quad, \mathbb{Z}/l(\star))[B^{-1}]$ may be identified with the cohomology operations on mod-$l$ étale cohomology.

3.2. These operations on mod$-l$ étale cohomology will be denoted $P^r$ and $\beta P^r$.

## 4. Comparison of cohomology operations in étale cohomology

We will begin by defining classical cohomology operations in étale cohomology. For this we start with a smooth scheme $X$ and let $\mathcal{A} = R\Gamma(X_{et}, \mu_l)$. We let $\{N\mathbb{Z}E\Sigma_n|n\}$ denote the simplicial Barratt-Eccles operad defined in [J, Definition 4.1]. By the results in Theorem 1.1 and section 6 of [J], this acts on the complex $\mathcal{A} = R\Gamma(X_{et}, \mu_l)$. We will let $\mathcal{H}om(K, \mathbb{Z}/l)$ be denoted by $K^\vee$, if $K$ is a complex of $\mathbb{Z}/l$-sheaves on $(smt.schms)_{et}$ or a complex of $\mathbb{Z}/l$-vector spaces. Let $\pi$ denote the cyclic group $\mathbb{Z}/l$ imbedded as a subgroup of the symmetric group $\Sigma_l$.

If $H$ is any subgroup of the symmetric group $\Sigma_l$, we will define the *equivariant cohomology* of $\mathcal{A}^{\otimes^l}$ with respect to $H$ as follows: $H^*(\mathcal{A}^{\otimes^l}, H; \mathbb{Z}/l)$ will be the cohomology of the complex $(N\mathbb{Z}E\Sigma_l)^\vee \underset{ZH}{\otimes} \mathcal{A}^{\otimes^l}$.

4.0.1. For our comparison purposes, it is important to realize that the equivariant cohomology defined above is nothing other than equivariant étale cohomology. We proceed to explain this identification. First of all let $H$ also denote the obvious constant group-scheme defined over the field $k$ and associated to the sub-group $H$ of $\Sigma_l$. Now $H$ acts as a group scheme on the scheme $X^l$; therefore we may form the simplicial scheme $EH\underset{H}{\times}X^l$ in the obvious manner. We define the $H$-equivariant mod-$l$ étale cohomology of $X^l$ to be the mod-$l$ étale cohomology of the simplicial scheme $EH\underset{H}{\times}X^l$. This identifies with the equivariant cohomology $H^*(\mathcal{A}^{\otimes l}, H, \mathbb{Z}/l)$. (In fact one may identify the complex $R\Gamma(EH\underset{H}{\times}X^l, \mathbb{Z}/l)$, upto quasi-isomorphism, with the complex $(N\mathbb{Z}E\Sigma_l)^\vee \underset{ZH}{\otimes} \mathcal{A}^{\otimes^l}$.)

4.0.2. We need to also compare the $H$-equivariant mod-$l$ étale cohomology defined above with the equivariant étale cohomology obtained by inverting the Bott element in $H$-equivariant mod-$l$ motivic cohomology. Recall that the definition of $H$-equivariant mod-$l$-motivic cohomology uses the *geometric model* for the classifying space for $H$ as opposed to the simplicial model. However, it is shown in [MV] that the two variants give isomorphic mod-$l$ étale cohomology, i.e. $H^*_{et}(BG^{gm}, \mathbb{Z}/l) \cong H^*_{et}(BG, \mathbb{Z}/l)$ where $BG$ denotes the simplicial classifying space for $G$ considered above.

Let $\Delta^* : H^*(\mathcal{A}^{\otimes^l}, \pi; \mathbb{Z}/l) \rightarrow H^*(\mathcal{A}, \pi; \mathbb{Z}/l) \cong H^*(B\pi; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ denote the obvious map induced by the diagonal $:X \rightarrow X^l$. One may also observe readily that the $l$-th power map defines a map $H^*(\mathcal{A}) \rightarrow H^*(\mathcal{A}^{\otimes^l}, \pi; \mathbb{Z}/l)$, $a \mapsto a^l$. Let $\{w^i, vw^i | i \geq 0\}$ denote a basis of the $\mathbb{Z}/l$-vector space $H^*(B\pi; \mathbb{Z}/l)$. Here $v$ has degree 1 and $w$ has degree two. Since the cohomology operations are assumed to be stable, they are stable with respect to suspension so that it suffices to define these on classes of even degree. One defines cohomology operations $Q^s, \beta Q^s$ on $H^{2q}(\mathcal{A})$ by the formula: if $l = 2$, we let:

$$\Delta^*(x^2) = \Sigma_s Q^s(x) w^{(q-s)} + \beta Q^s(x) v w^{q-s-1} \tag{4.0.3}$$

and if $l > 2$, we let:

$$\Delta^*(x^l) = \Sigma_s (-1)^{d-s} Q^s(x) w^{(q-s)(l-1)} + (-1)^{d-s} \beta Q^s(x) v w^{(q-s)(l-1)-1} \tag{4.0.4}$$

In [J, Section 7.1] we provided the action of the operad $\{NE\Sigma_n|n\}$ on the motivic complex $\mathbb{Z}/{l^\nu}^{mot}_{et}$ which is the mod-$l^\nu$ motivic complex sheafified on the big étale site of smooth schemes. One may identify the complex $\mathbb{Z}/l^\nu(r)^{mot}_{et}$ with $\mu_{l^\nu}(r)[0]$ upto quasi-isomorphism: see [MVW, Theorem 10.3]. These lead to a somewhat different definition of the *classical* cohomology operations on mod$-l^\nu$ -étale cohomology as discussed in [J, Section 8]. We will explain in outline that these operations are in fact identical. (Since most of this is folklore, we will be brief.)

**Proposition 4.1.** *The cohomology operations defined above coincide with the classical cohomology operations defined on mod-l étale cohomology in* [J, Section 8].

*Proof.* For the rest of this section we will denote $\mathbb{Z}/l^{\nu}{}^{mot}_{et}$ by $\mathcal{A}$. The above action of the operad $\{NE\Sigma_n|n\}$ on the above complex provides us maps

(4.0.5) $$\theta_n : N\mathbb{Z}E\Sigma_n \otimes \mathcal{A}^{\otimes^n} \to \mathcal{A}$$

We will let $\mathcal{H}om(K, \mathbb{Z}/l)$ be denoted by $K^{\vee}$, if $K$ is a complex of $\mathbb{Z}/l$-sheaves on $(smt.schms)_{et}$. From the above pairing we obtain

$$\theta_n^* : N\mathbb{Z}E\Sigma_n \otimes \mathcal{A}^{\vee} \to (\mathcal{A}^{\vee})^{\otimes^n}$$

where we define $\theta_n^*(h, a^{\vee})(a_1 \otimes \cdots \otimes a_n) =< a^{\vee}, \theta_n(h \otimes a_1 \otimes \cdots \otimes a_n) >$, $a_i \varepsilon \mathcal{A}$, $a^{\vee} \varepsilon \mathcal{A}^{\vee}$ and $h \varepsilon N\mathbb{Z}E\Sigma_n$. It is a standard result in this situation that the map $\theta_n^*$ is a chain map and is an *approximation to the diagonal map* (i.e. homotopic to the diagonal map) $\Delta : \mathcal{A}^{\vee} \to (\mathcal{A}^{\vee})^{\otimes^n}$. (Here, as well as elsewhere in this section, we use the observation that for any vector space $V$ over $\mathbb{Z}/l$, a vector $v \varepsilon V$ ( a vector $v^{\vee} \varepsilon V^{\vee}$) is determined by its pairing $< v, w >$ with all vectors $w \varepsilon V^{\vee}$ (its pairing $< u, v^{\vee} >$ with all vectors $u \varepsilon V$, respectively.).)

We now define

(4.0.6) $$d : N\mathbb{Z}E\Sigma_n \otimes \mathcal{A}^{\vee} \to N\mathbb{Z}E\Sigma_n \otimes (\mathcal{A}^{\vee})^{\otimes^n}$$

by the formula $d(h, a^{\vee}) = (h, \theta_n^*(h, a^{\vee}))$. This in turn defines a map

(4.0.7) $$d^* : (N\mathbb{Z}E\Sigma_n)^{\vee} \otimes \mathcal{A}^{\otimes^n} \to (N\mathbb{Z}E\Sigma_n)^{\vee} \otimes \mathcal{A}$$

by the formula:

$$< d^*(h^{\vee}, a_1 \otimes \cdots \otimes a_n), h' \otimes a^{\vee} >=< d(h', a^v), h^{\vee} \otimes a_1 \otimes \cdots \otimes a_n >$$

$$=< \theta_n(h', a_1 \otimes \cdots a_n), a^{\vee} > \otimes < h', h^{\vee} >.$$

(Here $h' \varepsilon N\mathbb{Z}E\Sigma n$, $h^{\vee} \varepsilon (N\mathbb{Z}E\Sigma n)^{\vee}$, $a^{\vee} \varepsilon A^{\vee}$, $a_i \varepsilon \mathcal{A}$.) We now let $n = l$ and let $\pi$ denote the cyclic subgroup $\mathbb{Z}/l$ of $\Sigma_l$. One may recall that the action of $\sigma \varepsilon \Sigma_n$ on $NE\Sigma_n$ and of $\sigma^{-1}$ on $\mathcal{A}^{\otimes^n}$ cancel out. Tracing through these actions of $\Sigma_n$ on the maps in the above steps, one concludes that the map $d^*$ induces a map on the quotients:

(4.0.8) $$d^* : (N\mathbb{Z}E\Sigma_n)^{\vee} \underset{Z\pi}{\otimes} \mathcal{A}^{\otimes^n} \to (N\mathbb{Z}E\Sigma_n)^{\vee} \underset{Z\pi}{\otimes} \mathcal{A}$$

Now the cohomology of the complex $(N\mathbb{Z}E\Sigma_n)^{\vee} \underset{N(\mathbb{Z}(\pi))}{\otimes} \mathcal{A}$ identifies with $H^*(B\pi; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ whereas the cohomology of the complex $(N\mathbb{Z}E\Sigma_n)^{\vee} \underset{Z\pi}{\otimes} \mathcal{A}^{\otimes^n}$ identifies with the equivariant cohomology: $H^*(\mathcal{A}^{\otimes^n}, \pi; \mathbb{Z}/l)$. Therefore, the map $d^*$ defines a map

(4.0.9) $$\bar{d}^* : H^*(\mathcal{A}^{\otimes^n}, \pi; \mathbb{Z}/l) \to H^*(B\pi; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$$

One may also observe readily that the $l$-th power map defines a map $H^*(\mathcal{A}) \to H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; \mathbb{Z}/l)$, $a \mapsto a^l$. Let $\{e_i, fe_i | i \geq 0\}$ denote a basis of the $\mathbb{Z}/l$-vector space $H_*(B\Sigma_n; \mathbb{Z}/l)$ dual to the basis $\{w^i, vw^i | i \geq 0\}$ for $H^*(B\Sigma_n; \mathbb{Z}/l)$, i.e. $< e_i, w^j >= 0$, if $i \neq j$ and $= 1$ if $i = j$. Also $< fe_i, w^j >= 0$ for all $i$ , $j$, $< fe_i, vw^j >= 0$ for $i \neq j$ and $= 1$ for $i = j$. Observe that now we have the following computation for a class $x \varepsilon H^q(\mathcal{A})$:

$$< \bar{d}^*(x^l), (-)^{\vee} \otimes e_i >=< \bar{\theta}_l^*(e_i, (-)^{\vee}), x^l >=< (-)^{\vee}, \bar{\theta}_l(e_i, x^l) > \text{ and}$$

$$< \bar{d}^*(x^l), (-)^{\vee} \otimes fe_i >=< \bar{\theta}_l^*(fe_i, (-)^{\vee}), x^l >=< (-)^{\vee}, \bar{\theta}_l(fe_i, x^l) >$$

where $(-)^{\vee} \varepsilon H^*(\mathcal{A})^{\vee}$ and $\bar{\theta}$ ($\bar{\theta}_l^*$) is the map induced by $\theta$ ($\theta_l^*$, respectively) on taking homology of the corresponding complexes. Since the map $\theta_l^*$ was observed to be chain homotopic to the diagonal, it follows that $\bar{d}^* = \Delta^*$ where $\Delta$ is the obvious diagonal. Therefore, the coefficient of $w^i$ ($vw^i$) in the expansion of $\bar{d}^*(x^l) \varepsilon H^*(B\Sigma_l; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ identifies with $\bar{\theta}_i(e_i, x^l)$ ($\bar{\theta}_i(fe_i, x^l)$, respectively). This completes the proof of the proposition ☐

The formulae in ( 4.0.3) and ( 4.0.4) are stated in terms of the cohomology of the cyclic groups. This has a reformulation in terms of the cohomology of the symmetric groups $\Sigma_l$ which will be readily comparable to the formula in ( 2.1.2). First one may compute the cohomology of the symmetric group $H^*(B\Sigma_l, \mathbb{Z}/l)$ to be the $\mathbb{Z}/l$-vector space with basis given by $\{y^i, xy^i | i \geq 0\}$ where $y$ is a class in $H^{2l-2}(B\Sigma_l; \mathbb{Z}/l)$ and $x$ is a class in $H^{2l-3}(B\Sigma_l; \mathbb{Z}/l)$. In fact $y = \prod_{i=1}^{l-1} iw = (l-1)!w^{l-1} = -w^{l-1}$ where $w$ is the class in $H^2(B\pi; \mathbb{Z}/l)$ considered in ( 4.0.4). Now $x = -vw^{l-2}$. Then the cohomology operation $Q^r$ and $\beta Q^r$ expressed in terms of the equivariant cohomology with respect to the symmetric group replacing the equivariant cohomology with respect to the cyclic group $\mathbb{Z}/l$ has the following form:

(4.0.10) $$\Delta^*(w^l) = \Sigma_{r \geq 0} Q^r(w) y^{d-r} + \beta P^r(w) x y^{d-r-1}, \quad w \varepsilon H^{2d}_{et}( \quad , \mathbb{Z}/l(d))$$

This uniquely defines the cohomology operations as they are stable with respect to suspension and hence extend uniquely to cohomology classes with odd degree.

**Proof of the main theorem**.

Here the observations in 4.0.1 and 4.0.2 are important. In addition, one needs to observe that weight-suspension in mod-$l$ étale cohomology is defined by multiplication by the Bott element $B$: since $B$ is a unit, the weight suspension is an isomorphism in mod-$l$ étale cohomology.

Therefore, the main difference of the above formula with the one in 3.2 is that the classes $y^i$ and $xy^i$ have no weight, or equivalently have weight 0. Observe that the operations $Q^r$ and $\beta Q^r$ defined above are maps:

$Q^r : H^{2q}_{et}(X, \mu_l(q)) \rightarrow H^{2q+2r(l-1)}_{et}(X, \mu_l(q.l))$ and

$\beta Q^r : H^{2q}_{et}(X, \mu_l(q)) \rightarrow H^{2q+2r(l-1)+1}_{et}(X, \mu_l(q.l))$

Since the operations above raise a cohomology class in $H^{2q}_{et}(X, \mu_l(q))$ to the $l$-th power, and the classes $y^i$ and $xy^i$ have zero-weight, the Tate-twist $q.l$ appears in the target of these operations.

Therefore, we obtain the relation between the classical operations and the operations on étale cohomology induced by the motivic operations as follows:

$$Q^r = B^{(q-r).(l-1)}.P^r, \quad \beta Q^r = B^{(q-r).(l-1)}.\beta P^r$$

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Department of Mathematics, University of British Columbia, Vancouver, Canada
*E-mail address*: pbrosnan@math.ubc.ca

Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA.
*Current address*: Max Planck Institut für Mathematik, P.O. Box 7280, Bonn 53072 GERMANY
*E-mail address*: joshua@math.ohio-state.edu, joshua@mpim-bonn.mpg.de