Transient Heat Conduction in a square pipe, with an outer side of L=W=1 unit length and an inner side of w=L/2=W/2=0.5 unit length.
Case a) Constant temperature T_{out}=1 at outer wall & constant temperature T_{in}=0 at inner wall at 0≤t.
Case b) The outer wall is insulated (i.e., h_{out}=0) and the inner wall is at a constant temperature of T_{in}=0 at 0≤t.
Case c) The outer wall is insulated (i.e., h_{out}=0) and the inner wall is subjected to convection with $T_{in.\infty}$=exp(-t/τ) τ=10, 1000, 100,000 at 0≤t.

```
     side length=L=1
y=W+------------------+
   |                  |
   | square pipe wall |
   |     +---------+    |
   |     |length=w|    |
   |     |         |    |
   |     |         |    | side length=W=1
   |     |  Tin=0  |    |
   |     +---------+    |
   |          |****\    |
   |          |******\  |   * indicates the minimal octant section
y=0+------------------+
  x=0 ........       x=L
  i=0 1 2......      i=nx
```

Instructor: Nam Sun Wang

Transient heat conduction in 2-dimensional plane

$$\rho \cdot c_P \cdot \left(\frac{d}{dt}T\right) = k \cdot \left(\frac{d^2}{dx^2}T + \frac{d^2}{dy^2}T\right) + qdot \qquad qdot = 0$$

Approximate derivatives with the central difference formula

$$\left(\frac{d^2}{dx^2}T\right)_{i,j} = \frac{T_{i,j+1} - 2 \cdot T_{i,j} + T_{i,j-1}}{\Delta x^2} \qquad \left(\frac{d^2}{dy^2}T\right)_{i,j} = \frac{T_{i+1,j} - 2 \cdot T_{i,j} + T_{i-1,j}}{\Delta y^2}$$

$$\left(\frac{d}{dt}T\right)_{i,j} = \alpha \cdot \left(\frac{T_{i,j+1} - 2 \cdot T_{i,j} + T_{i,j-1}}{\Delta x^2} + \frac{T_{i+1,j} - 2 \cdot T_{i,j} + T_{i-1,j}}{\Delta y^2}\right)$$

Assign model parameters

$k := 1 \qquad \alpha := 5 \cdot 10^{-6} \qquad T_{init} := 1 \qquad h_{in} := 10 \qquad T_{in.inf}(t, \tau) := \exp\left(-\frac{t}{\tau}\right) \qquad h_{out} := 0$ i.e., insulated $\qquad T_{out.inf} := 1$ (not used)

$nx := 20 \qquad i := 0 .. nx \qquad L := 1 \qquad \Delta x := \frac{1}{nx} \qquad xv_i := \Delta x \cdot i$

$ny := 20 \qquad j := 0 .. ny \qquad W := 1 \qquad \Delta y := \frac{1}{ny} \qquad yv_j := \Delta y \cdot j \qquad w := \frac{W}{2} = 0.5$

inner channel $l := \frac{L}{2} = 0.5$ left & right of inner channel $nx_{in.left} := nx \cdot \left(\frac{L - l}{2 \cdot L}\right) = 5 \qquad nx_{in.right} := nx \cdot \frac{L + l}{2 \cdot L} = 15$

inner channel $w := \frac{W}{2} = 0.5$ bottom & top of inner channel $ny_{in.bottom} := ny \cdot \left(\frac{W - w}{2 \cdot W}\right) = 5 \qquad ny_{in.top} := ny \cdot \frac{W + w}{2 \cdot W} = 15$

Whole Square Pipe. If we disregard computation efficiency and resort to brute force, we can simply solve for the whole pipe with (nx+1)·(ny+1) nodes (rather than just a 1/8 section thereof), and we collect the heat balance equations at all the nodes in the general matrix form dT(t,T)/dt (rather than the standard vector form). The coding is quite simple -- only three different equations: two for the two boundaries and one for the interior points.

Case a) Constant temperature T_{out}=1 at outer wall & constant temperature T_{in}=0 at inner wall at 0≤t.

$$d2Tdx2(T,i,j) := \frac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} \qquad d2Tdy2(T,i,j) := \frac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2}$$

```
dTdt(t,T) := | "initialize & outer wall"
             | dTdt[nx,ny] ← 0
             | "interior points"
             | for i ∈ 1 .. nx − 1
             |   for j ∈ 1 .. ny − 1
             |     dTdt[i,j] ← α·(d2Tdx2(T,i,j) + d2Tdy2(T,i,j))
             | "inner pipe conduit & inner wall"
             | for i ∈ nx_in.left .. nx_in.right
             |   for j ∈ ny_in.bottom .. ny_in.top
             |     dTdt[i,j] ← 0
             | dTdt
```

$dT_{i,j}/dt=0$ implements $T_{i,j}$=constant

Integrate dT(t,T)/dt /w Euler's method (i.e., explicit method)

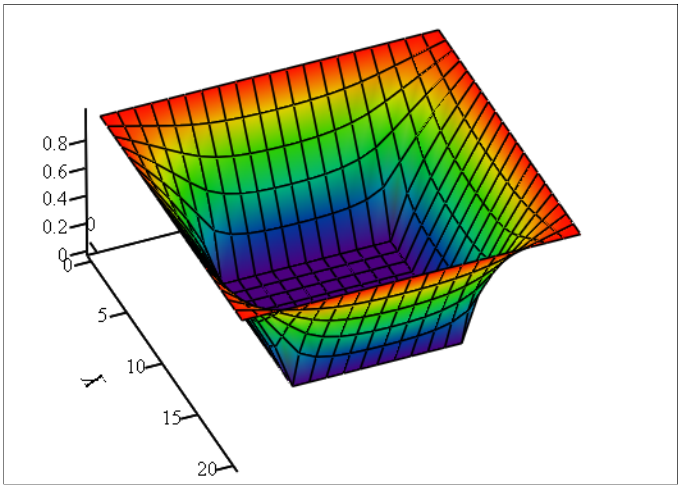
critical Fourier No & Δt $\quad Fo = \frac{\alpha\cdot\Delta t}{\Delta x^2} < \frac{1}{4} \qquad \Delta t_{crit} := \frac{1}{4}\cdot\frac{\Delta x^2}{\alpha} = 125$ ... be sue to use a Δt that is less than this critical value.

I.C. T(x,0)=1

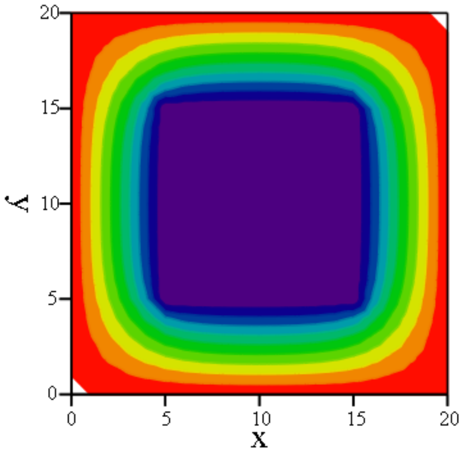
```
T_init.matrix := | "initialize"
                 | T[nx,ny] ← 0
                 | "interior points & outer wall"
                 | for i ∈ 0 .. nx
                 |   for j ∈ 0 .. ny
                 |     T[i,j] ← T_init
                 | "inner pipe conduit & inner wall"
                 | for i ∈ nx_in.left .. nx_in.right
                 |   for j ∈ ny_in.bottom .. ny_in.top
                 |     T[i,j] ← 0
                 | T
```

$\Delta t := 100 \qquad nt := 50 \qquad p := 0..nt \qquad T := 0 \qquad T_0 := T_{init.matrix} \qquad T_{p+1} := T_p + dTdt\left(p\cdot\Delta t, T_p\right)\cdot\Delta t$

nt = FRAME

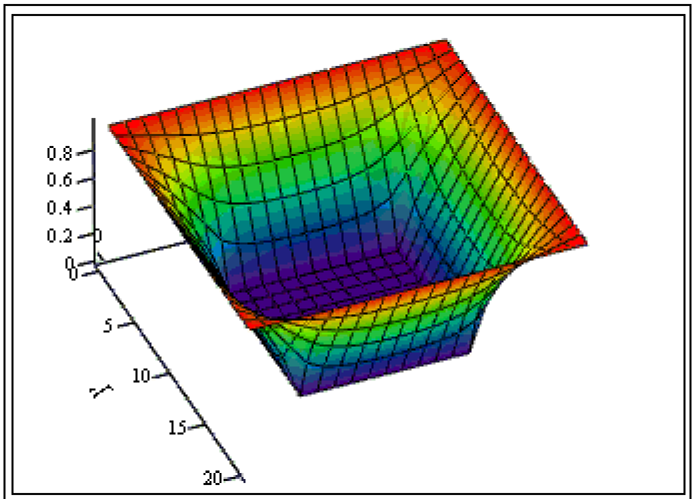


T_{nt}



T_{nt}

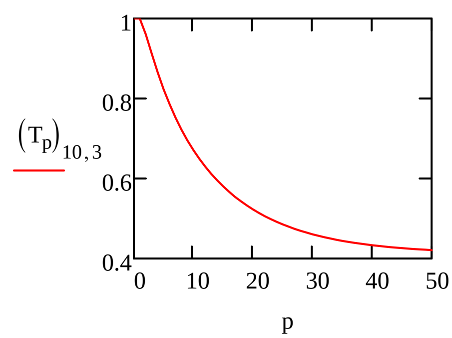
Click on the following to play an animation saved from the above plot. To creat animation, uncomment to "nt:=FRAME"; |Tools|Animation|Record|, "For FRAME from 0 to 50" (which is "nt"); mark the region to be animated by dragging a rectangle; click on "Animate" button, followed by "Save as" button. Copy|paste the .avi file.



T_{nt}

conduct2D-square-pipe-transient.avi

Reach steady-state in about $t := \Delta t \cdot nt = 5000$



Case b) Insulated outer wall & constant temperature T_{in}=0 at inner wall at 0≤t.

$$q'' = -k\cdot T'(x=0,y) = h_{out}\cdot\left(T_{out.inf} - T(x=0,y)\right) \longrightarrow \quad -k\cdot\frac{T_{i+1,j} - T_{i-1,j}}{2\cdot\Delta x} = h_{out}\cdot\left(T_{out.inf} - T_{i,j}\right)$$

B.C. at left edge $\quad i = 0 \quad T_{i-1,j} = T_{i+1,j} - 2\cdot\Delta x\cdot\frac{h_{out}}{k}\cdot\left(T_{i,j} - T_{out.inf}\right)$

$$\left(\frac{d^2}{dx^2}T\right)_{i,j} = \frac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} = \frac{2\cdot T_{i+1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{out}}{k}\cdot T_{out.inf}}{\Delta x^2}$$

B.C. at right edge $\quad i = nx \quad T_{i+1,j} = T_{i-1,j} - 2\cdot\Delta x\cdot\frac{h_{out}}{k}\cdot\left(T_{i,j} - T_{out.inf}\right)$

$$\left(\frac{d^2}{dx^2}T\right)_{i,j} = \frac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} = \frac{2\cdot T_{i-1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{out}}{k}\cdot T_{out.inf}}{\Delta x^2}$$

$$d2Tdx2_{out}(T,i,j) := \begin{cases} \dfrac{2\cdot T_{i+1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{out}}{k}\cdot T_{out.inf}}{\Delta x^2} & \text{if } i = 0 \\ \dfrac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} & \text{if } (0 < i)\cdot(i < nx) \\ \dfrac{2\cdot T_{i-1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{out}}{k}\cdot T_{out.inf}}{\Delta x^2} & \text{if } i = nx \end{cases}$$

B.C. at bottom edge $\quad j = 0 \quad T_{i,j-1} = T_{i,j+1} - 2\cdot\Delta y\cdot\frac{h_{out}}{k}\cdot\left(T_{i,j} - T_{out.inf}\right)$

$$\left(\frac{d^2}{dy^2}T\right)_{i,j} = \frac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2} = \frac{2\cdot T_{i,j+1} - 2\cdot\left(1 + \Delta y\cdot\frac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\frac{h_{out}}{k}\cdot T_{out.inf}}{\Delta y^2}$$

B.C. at top edge $\quad j = ny \quad T_{i,j+1} = T_{i,j-1} - 2\cdot\Delta y\cdot\frac{h_{out}}{k}\cdot\left(T_{i,j} - T_{out.inf}\right)$

$$\left(\frac{d^2}{dy^2}T\right)_{i,j} = \frac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2} = \frac{2\cdot T_{i,j-1} - 2\cdot\left(1 + \Delta y\cdot\frac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\frac{h_{out}}{k}\cdot T_{out.inf}}{\Delta y^2}$$

$$d2Tdy2_{out}(T,i,j) := \begin{cases} \dfrac{2\cdot T_{i,j+1} - 2\cdot\left(1 + \Delta y\cdot\dfrac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\dfrac{h_{out}}{k}\cdot T_{out.inf}}{\Delta y^2} & \text{if } j = 0 \\ \dfrac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2} & \text{if } (0 < j)\cdot(j < ny) \\ \dfrac{2\cdot T_{i,j-1} - 2\cdot\left(1 + \Delta y\cdot\dfrac{h_{out}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\dfrac{h_{out}}{k}\cdot T_{out.inf}}{\Delta y^2} & \text{if } j = ny \end{cases}$$

```
dTdt(t,T) := | "initialize"
             | dTdt[nx,ny] ← 0
             | "interior points & outer wall"
             | for i ∈ 0 .. nx
             |   for j ∈ 0 .. ny
             |     dTdt[i,j] ← α·(d2Tdx2_out(T,i,j) + d2Tdy2_out(T,i,j))
             | "inner pipe conduit & inner wall"
             | for i ∈ nx_in.left .. nx_in.right
             |   for j ∈ ny_in.bottom .. ny_in.top
             |     dTdt[i,j] ← 0
             | dTdt
```

Integrate dT(t,T)/dt /w Euler's method (i.e., explicit method). Change nt until max(T) is <0.01. Then go back to see when T_{max}=0.01 is reached.

$\Delta t := 50$ $nt := 1000$ $p := 0..nt$ $T := 0$ $T_0 := T_{init.matrix}$ $T_{p+1} := T_p + dTdt(p\cdot\Delta t, T_p)\cdot\Delta t$

The highest temperature is at the corners. $nt := 830$ $\max(T_{nt}) = 0.01$ $(T_{nt})_{0,0} = 0.01$

$\Delta t\cdot nt = 41500$ For constant temperature at the inner wall, it takes ~41,500 time units to reach T_{max}=0.01

$(T_{nt})^{\langle 0\rangle T} =$

	0	1	2	3	4	5	6
0	0.01	0.01	$9.886\cdot10^{-3}$	$9.196\cdot10^{-3}$	$8.366\cdot10^{-3}$	$7.508\cdot10^{-3}$	...

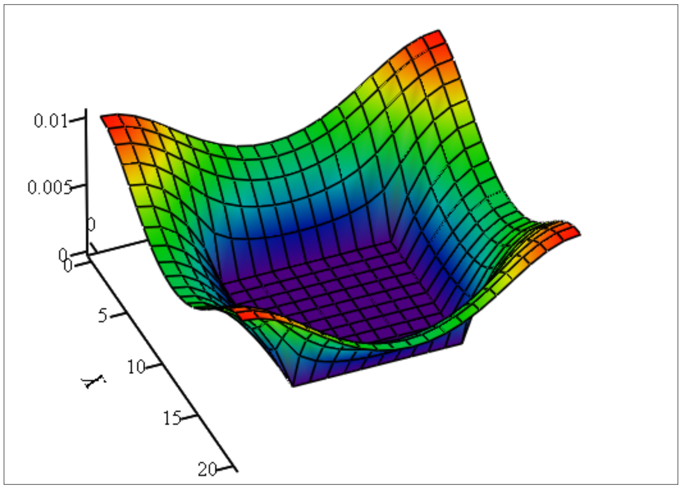
Check (be sure a different step size does not affect the results):

$\Delta t := 100$ $nt := 500$ $p := 0..nt$ $T := 0$ $T_0 := T_{init.matrix}$ $T_{p+1} := T_p + dTdt(p\cdot\Delta t, T_p)\cdot\Delta t$

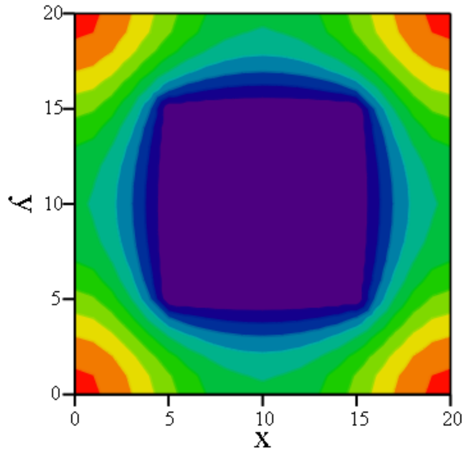
The highest temperature is at the corners. $nt := 415$ $\max(T_{nt}) = 0.01$ $(T_{nt})_{0,0} = 0.01$

$(T_{nt})^{\langle 0\rangle T} =$

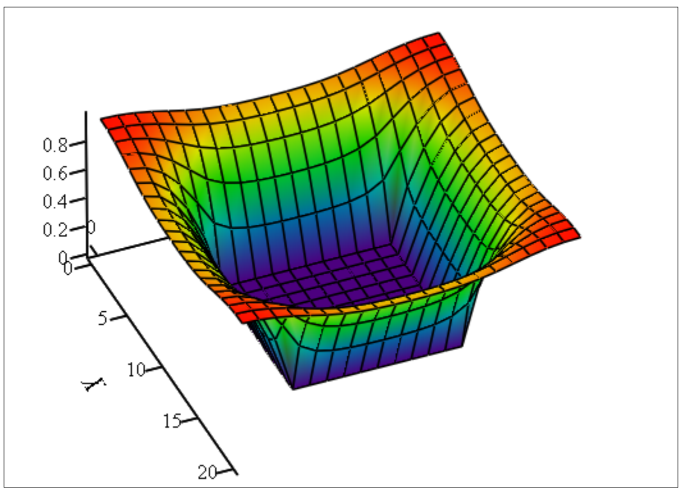
	0	1	2	3	4	5	6
0	0.01	0.01	$9.736\cdot10^{-3}$	$9.057\cdot10^{-3}$	$8.24\cdot10^{-3}$	$7.394\cdot10^{-3}$	...



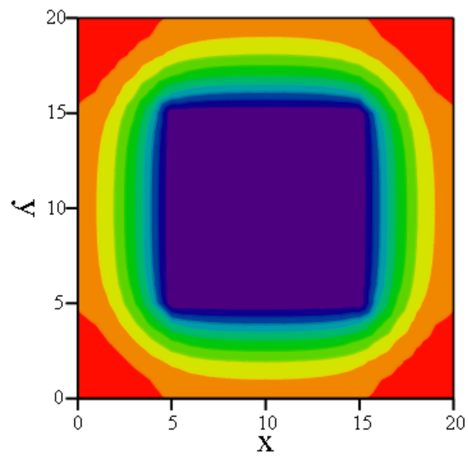
T_{nt}



T_{nt}



T_{20}



T_{20}

Case c) Convection boundary condition at the inner wall

$$q'' = -k\cdot T'\left(x = \frac{W}{4}, y\right) = h\cdot\left(T\left(x = \frac{W}{4}, y\right) - T_{in.inf}\right) \longrightarrow -k\cdot\frac{T_{i+1,j} - T_{i-1,j}}{2\cdot\Delta x} = h_{out}\cdot\left(T_{i,j} - T_{in.inf}\right)$$

B.C. at left edge $i = nx_{in.left}$ $T_{i+1,j} = T_{i-1,j} - 2\cdot\Delta x\cdot\frac{h_{in}}{k}\cdot\left(T_{i,j} - T_{in.inf}\right)$

$$\left(\frac{d^2}{dx^2}T\right)_{i,j} = \frac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} = \frac{2\cdot T_{i-1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{in}}{k}\cdot T_{in.inf}}{\Delta x^2}$$

B.C. at right edge $i = nx_{in.right}$ $T_{i-1,j} = T_{i+1,j} - 2\cdot\Delta x\cdot\frac{h_{in}}{k}\cdot\left(T_{i,j} - T_{in.inf}\right)$

$$\left(\frac{d^2}{dx^2}T\right)_{i,j} = \frac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} = \frac{2\cdot T_{i+1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{in}}{k}\cdot T_{in.inf}}{\Delta x^2}$$

$$d2Tdx2_{in}(T,i,j,t,\tau) := \begin{cases} \dfrac{2\cdot T_{i-1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{in}}{k}\cdot T_{in.inf}(t,\tau)}{\Delta x^2} & \text{if } i = nx_{in.left} \\ \dfrac{T_{i+1,j} - 2\cdot T_{i,j} + T_{i-1,j}}{\Delta x^2} & \text{if } \left(nx_{in.left} < i\right)\cdot\left(i < nx_{in.right}\right) \\ \dfrac{2\cdot T_{i+1,j} - 2\cdot\left(1 + \Delta x\cdot\frac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta x\cdot\frac{h_{in}}{k}\cdot T_{in.inf}(t,\tau)}{\Delta x^2} & \text{if } i = nx_{in.right} \end{cases}$$

⟵ Should be 0 or any other arbitrary number because we are not concerned about the inner channel.

B.C. at bottom edge $j = ny_{in.bottom}$ $T_{i,j+1} = T_{i,j-1} - 2\cdot\Delta y\cdot\frac{h_{in}}{k}\cdot\left(T_{i,j} - T_{in.inf}\right)$

$$\left(\frac{d^2}{dy^2}T\right)_{i,j} = \frac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2} = \frac{2\cdot T_{i,j-1} - 2\cdot\left(1 + \Delta y\cdot\frac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\frac{h_{in}}{k}\cdot T_{in.inf}}{\Delta y^2}$$

B.C. at top edge $j = ny_{in.top}$ $T_{i,j-1} = T_{i,j+1} - 2\cdot\Delta y\cdot\frac{h_{in}}{k}\cdot\left(T_{i,j} - T_{out.inf}\right)$

$$\left(\frac{d^2}{dy^2}T\right)_{i,j} = \frac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2} = \frac{2\cdot T_{i,j+1} - 2\cdot\left(1 + \Delta y\cdot\frac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\frac{h_{in}}{k}\cdot T_{in.inf}}{\Delta y^2}$$

$$\mathrm{d2Tdy2}_{in}(T,i,j,t,\tau) := \begin{cases} \dfrac{2\cdot T_{i,j-1} - 2\cdot\left(1 + \Delta y\cdot\dfrac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\dfrac{h_{in}}{k}\cdot T_{in.inf}(t,\tau)}{\Delta y^2} & \text{if } j = ny_{in.bottom} \\ \dfrac{T_{i,j+1} - 2\cdot T_{i,j} + T_{i,j-1}}{\Delta y^2} & \text{if } (ny_{in.bottom} < j)\cdot(j < ny_{in.top}) \\ \dfrac{2\cdot T_{i,j+1} - 2\cdot\left(1 + \Delta y\cdot\dfrac{h_{in}}{k}\right)\cdot T_{i,j} + 2\cdot\Delta y\cdot\dfrac{h_{in}}{k}\cdot T_{in.inf}(t,\tau)}{\Delta y^2} & \text{if } j = ny_{in.top} \end{cases}$$

⟵ Should be 0 or any other arbitrary number because we are not concerned about the inner channel.

$\mathrm{dTdt}(t,T,\tau) :=$

```
"initialize"
dTdt[nx,ny ← 0
"interior points & outer wall"
for i ∈ 0 .. nx
   for j ∈ 0 .. ny
      dTdt[i,j ← α·(d2Tdx2out(T,i,j) + d2Tdy2out(T,i,j))
"inner pipe conduit & inner wall"
for i ∈ nxin.left .. nxin.right
   for j ∈ nyin.bottom .. nyin.top
      dTdt[i,j ← α·(d2Tdx2in(T,i,j,t,τ) + d2Tdy2in(T,i,j,t,τ))
dTdt
```

Integrate dT/dt(t,T) /w Euler's method (i.e., explicit method) $\Delta t := 100$

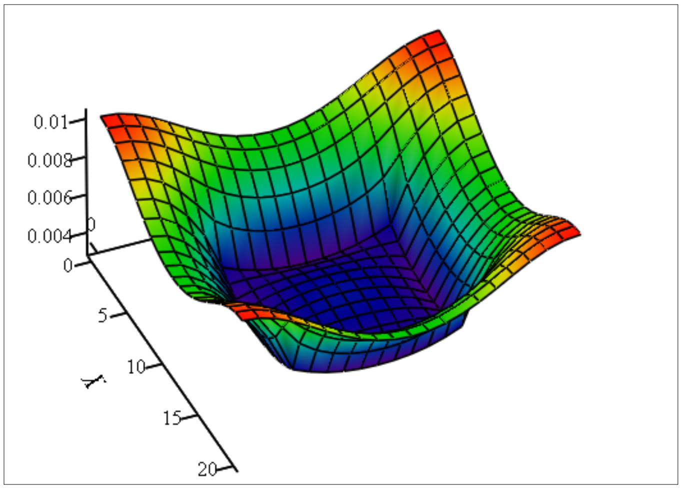
I.C. T(x,0)=1 $T_{init.matrix_{i,j}} := T_{init}$ Change nt until max(T) is <0.01. Then go back to see when T_{max}=0.01 is reached.

Iteration on time t Reset Start with initial condition, iterate with Euler's method

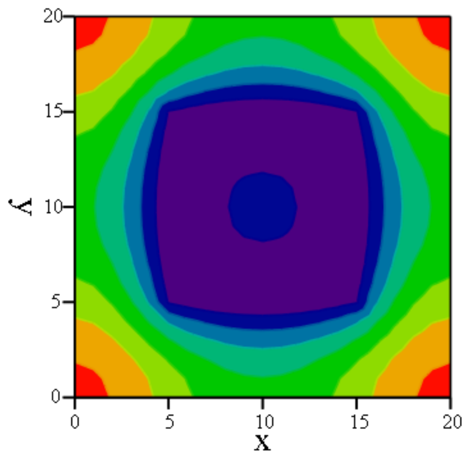
$\tau := 10$ $nt := 1000$ $p := 0..nt$ $T := 0$ $T_0 := T_{init.matrix}$ $T_{p+1} := T_p + dTdt(p\cdot\Delta t, T_p, \tau)\cdot\Delta t$

The highest temperature is at the corners. $nt := 710$ $\max(T_{nt}) = 0.01$ $(T_{nt})_{0,0} = 0.01$

$\Delta t\cdot nt = 71000$ For τ=10, it takes 71,000 time units to reach T_{max}=0.01



T_{nt}

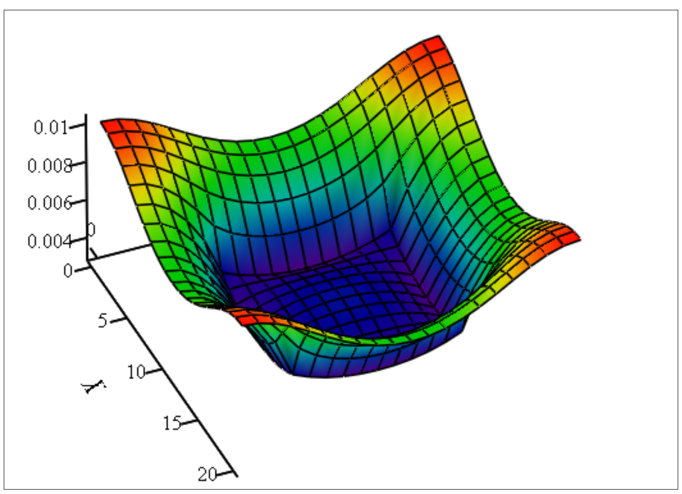


T_{nt}

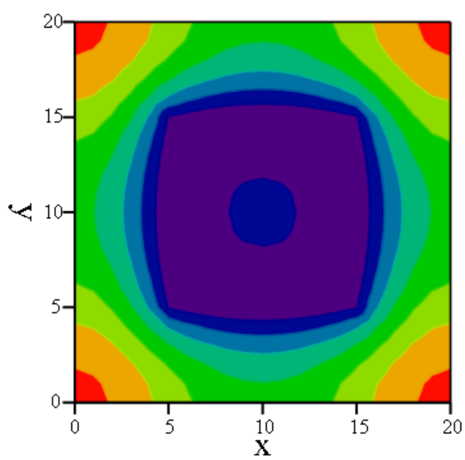
$\tau := 1000$ $nt := 1000$ $p := 0..nt$ $T := 0$ $T_0 := T_{init.matrix}$ $T_{p+1} := T_p + dTdt(p\cdot\Delta t, T_p, \tau)\cdot\Delta t$

The highest temperature is at the corners. $nt := 720$ $\max(T_{nt}) = 0.01$ $(T_{nt})_{0,0} = 0.01$

$\Delta t\cdot nt = 72000$ For τ=1000, it takes 72,000 time units to reach T_{max}=0.01



T_{nt}

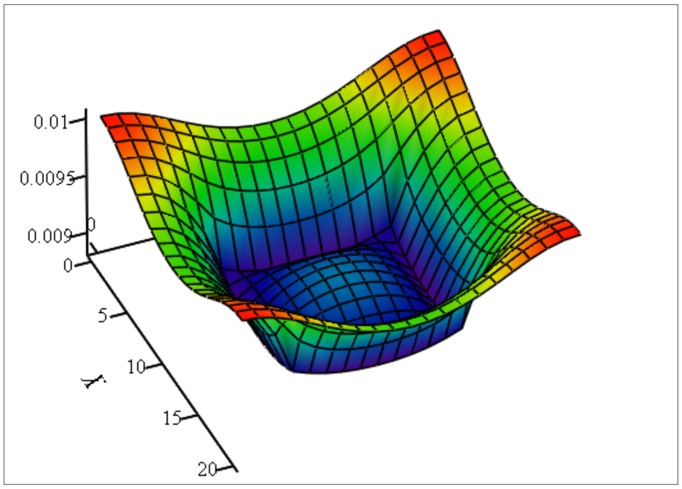


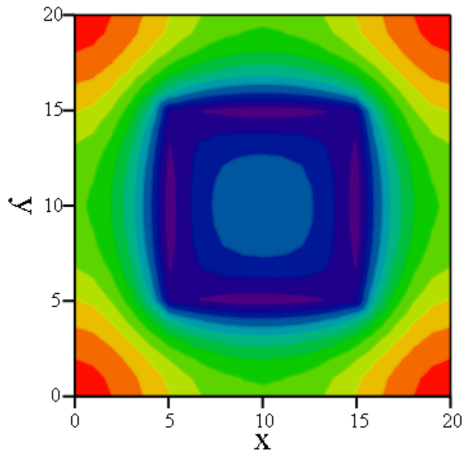
T_{nt}

$\tau := 100000$ $nt := 5000$ $p := 0..nt$ $T := 0$ $T_0 := T_{init.matrix}$ $T_{p+1} := T_p + dTdt(p\cdot\Delta t, T_p, \tau)\cdot\Delta t$

The highest temperature is at the corners. $nt := 4800$ $\max(T_{nt}) = 0.01$ $(T_{nt})_{0,0} = 0.01$

$\Delta t\cdot nt = 480000$ For τ=100,000, it takes 480,000 time units to reach T_{max}=0.01





T_{nt} T_{nt}

Although it takes different amount of time (71K, 72K, & 480K time units for τ=10, 1000, & 100,00, respectively) to reach T_{max}=0.1, the temperature profiles are all similar. For small values of τ that is of the order of Δt, there is very little difference in the transient behavior. For Δt<<τ, the time constant in $T_{in.inf}$ adds another layer of dynamics and time-lag to the temperature response.