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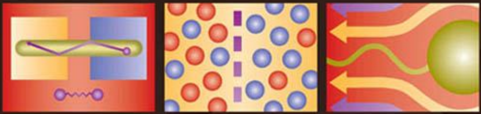

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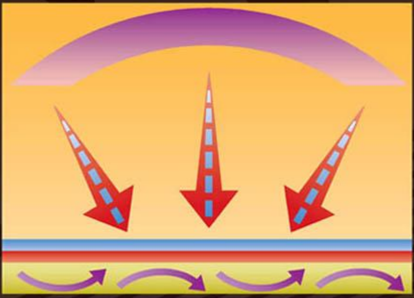
FUNDAMENTALS OF HEAT and MASS TRANSFER

SEVENTH EDITION



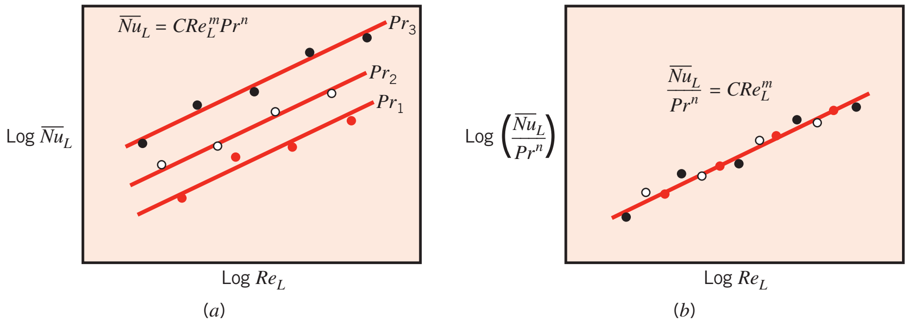


FIGURE 7.2 Dimensionless representation of convection heat transfer measurements.

This influence may be handled in one of two ways. In one method, Equation 7.1 is used with all properties evaluated at a mean boundary layer temperature T_f, termed the *film temperature*.

$$T_f \equiv \frac{T_s + T_\infty}{2} \tag{7.2}$$

The alternate method is to evaluate all properties at T_∞ and to multiply the right-hand side of Equation 7.1 by an additional parameter to account for the property variations. The parameter is commonly of the form $(Pr_\infty/Pr_s)^r$ or $(\mu_\infty/\mu_s)^r$, where the subscripts ∞ and s designate evaluation of the properties at the free stream and surface temperatures, respectively. Both methods are used in the results that follow.

Finally, we note that experiments may also be performed to obtain convection mass transfer correlations. However, under conditions for which the heat and mass transfer analogy (Section 6.7.1) may be applied, the mass transfer correlation assumes the same form as the corresponding heat transfer correlation. Accordingly, we anticipate correlations of the form

$$\overline{Sh}_L = C\, Re_L^m\, Sc^n \tag{7.3}$$

where, for a given geometry and flow condition, the values of C, m, and n are the same as those appearing in Equation 7.1.

7.2 The Flat Plate in Parallel Flow

Despite its simplicity, parallel flow over a flat plate (Figure 7.3) occurs in numerous engineering applications. As discussed in Section 6.3, laminar boundary layer development begins at the leading edge ($x = 0$) and transition to turbulence may occur at a downstream location (x_c) for which a critical Reynolds number $Re_{x,c}$ is achieved. We begin by considering conditions in the laminar boundary layer. Specifically, we will analytically determine the velocity, temperature, and concentration distributions that are shown qualitatively in Figures 6.1, 6.2, and 6.3, respectively. From knowledge of these distributions, we will then determine expressions for the local and average friction coefficients, Nusselt numbers, and Sherwood numbers.

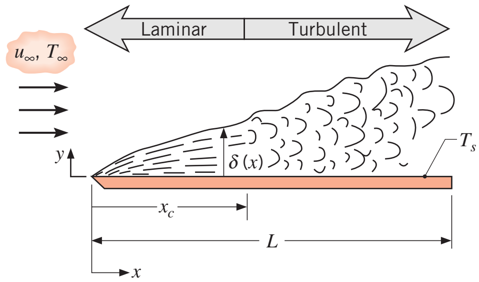


FIGURE 7.3 The flat plate in parallel flow.

7.2.1 Laminar Flow over an Isothermal Plate: A Similarity Solution

The major convection parameters may be obtained by solving the appropriate form of the boundary layer equations. Assuming *steady, incompressible, laminar* flow with *constant uid properties* and *negligible viscous dissipation* and recognizing that $dp/dx = 0$, the boundary layer equations (6.27, 6.28, 6.29, and 6.30) reduce to

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7.4}$$

Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{7.5}$$

Energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{7.6}$$

Species:

$$u\frac{\partial \rho_A}{\partial x} + v\frac{\partial \rho_A}{\partial y} = D_{AB}\frac{\partial^2 \rho_A}{\partial y^2} \tag{7.7}$$

Solution of these equations is simplified by the fact that for constant properties, conditions in the velocity (hydrodynamic) boundary layer are independent of temperature and species concentration. Hence we may begin by solving the hydrodynamic problem, Equations 7.4 and 7.5, to the exclusion of Equations 7.6 and 7.7. Once the hydrodynamic problem has been solved, solutions to Equations 7.6 and 7.7, which depend on u and v, may be obtained.

Hydrodynamic Solution The hydrodynamic solution follows the method of Blasius [1, 2]. The first step is to define a stream function $\psi(x, y)$, such that

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x} \tag{7.8}$$

Equation 7.4 is then automatically satisfied and hence is no longer needed. New dependent and independent variables, f and η, respectively, are then defined such that

$$f(\eta) \equiv \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}} \tag{7.9}$$

$$\eta \equiv y \sqrt{u_\infty / \nu x} \tag{7.10}$$

As we will find, use of these variables simplifies matters by reducing the partial differential equation, Equation 7.5, to an ordinary differential equation.

The Blasius solution is termed a *similarity solution,* and η is a *similarity variable.* This terminology is used because, despite growth of the boundary layer with distance x from the leading edge, the velocity profile u/u_∞ remains *geometrically similar.* This similarity is of the functional form

$$\frac{u}{u_\infty} = \phi\left(\frac{y}{\delta}\right)$$

where δ is the boundary layer thickness. We will find from the Blasius solution that δ varies as $(\nu x/u_\infty)^{1/2}$; thus, it follows that

$$\frac{u}{u_\infty} = \phi(\eta) \tag{7.11}$$

Hence the velocity profile is uniquely determined by the similarity variable η, which depends on both x and y.

From Equations 7.8 through 7.10 we obtain

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \sqrt{\frac{\nu x}{u_\infty}} \frac{df}{d\eta} \sqrt{\frac{u_\infty}{\nu x}}\, u_\infty = \frac{df}{d\eta} \tag{7.12}$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\left(u_\infty \sqrt{\frac{\nu x}{u_\infty}} \frac{\partial f}{\partial x} + \frac{u_\infty}{2}\sqrt{\frac{\nu}{u_\infty x}} f\right)$$

$$v = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left(\eta \frac{df}{d\eta} - f\right) \tag{7.13}$$

By differentiating the velocity components, it may also be shown that

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\eta \frac{d^2 f}{d\eta^2} \tag{7.14}$$

$$\frac{\partial u}{\partial y} = u_\infty \sqrt{\frac{u_\infty}{\nu x}} \frac{d^2 f}{d\eta^2} \tag{7.15}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_\infty^2}{\nu x}\frac{d^3 f}{d\eta^3} \tag{7.16}$$

Substituting these expressions into Equation 7.5, we then obtain

$$2\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0 \tag{7.17}$$

Hence the hydrodynamic boundary layer problem is reduced to one of solving a nonlinear, third-order ordinary differential equation. The appropriate boundary conditions are

$$u(x, 0) = v(x, 0) = 0 \quad \text{and} \quad u(x, \infty) = u_\infty$$

or, in terms of the similarity variables,

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = f(0) = 0 \quad \text{and} \quad \left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1 \tag{7.18}$$

The solution to Equation 7.17, subject to the conditions of Equations 7.18, may be obtained by a series expansion [2] or by numerical integration [3]. Selected results are presented in Table 7.1, from which useful information may be extracted. The x-component velocity distribution from the third column of the table is plotted in Figure 7.4*a*. We also note that, to a good approximation, $(u/u_\infty) = 0.99$ for $\eta = 5.0$. Defining the boundary layer thickness δ as that value of y for which $(u/u_\infty) = 0.99$, it follows from Equation 7.10 that

$$\delta = \frac{5.0}{\sqrt{u_\infty/\nu x}} = \frac{5x}{\sqrt{Re_x}} \tag{7.19}$$

TABLE 7.1 Flat plate laminar boundary layer functions [3]

$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.4	0.027	0.133	0.331
0.8	0.106	0.265	0.327
1.2	0.238	0.394	0.317
1.6	0.420	0.517	0.297
2.0	0.650	0.630	0.267
2.4	0.922	0.729	0.228
2.8	1.231	0.812	0.184
3.2	1.569	0.876	0.139
3.6	1.930	0.923	0.098
4.0	2.306	0.956	0.064
4.4	2.692	0.976	0.039
4.8	3.085	0.988	0.022
5.2	3.482	0.994	0.011
5.6	3.880	0.997	0.005
6.0	4.280	0.999	0.002
6.4	4.679	1.000	0.001
6.8	5.079	1.000	0.000

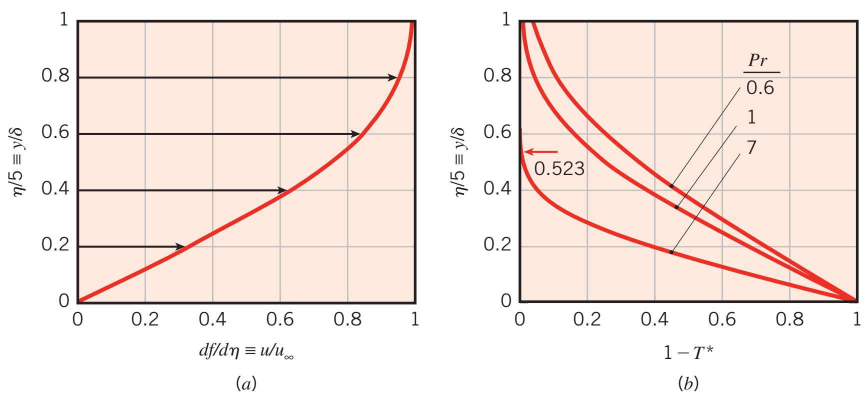


FIGURE 7.4 Similarity solution for laminar flow over an isothermal plate. (a) The x-component of the velocity. (b) Temperature distributions for $Pr = 0.6$, 1, and 7.

From Equation 7.19 it is clear that δ increases with increasing x and ν but decreases with increasing u_∞ (the larger the free stream velocity, the *thinner* the boundary layer). In addition, from Equation 7.15 the wall shear stress may be expressed as

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu u_\infty \sqrt{u_\infty/\nu x}\, \frac{d^2 f}{d\eta^2}\bigg|_{\eta=0}$$

Hence from Table 7.1

$$\tau_s = 0.332 u_\infty \sqrt{\rho \mu u_\infty / x}$$

The *local* friction coefficient is then

$$C_{f,x} \equiv \frac{\tau_{s,x}}{\rho u_\infty^2/2} = 0.664\, Re_x^{-1/2} \tag{7.20}$$

Heat Transfer Solution From knowledge of conditions in the velocity boundary layer, the energy equation may now be solved. We begin by introducing the dimensionless temperature $T^* \equiv [(T - T_s)/(T_\infty - T_s)]$ and assume a similarity solution of the form $T^* = T^*(\eta)$. Making the necessary substitutions, Equation 7.6 reduces to

$$\frac{d^2 T^*}{d\eta^2} + \frac{Pr}{2} f \frac{dT^*}{d\eta} = 0 \tag{7.21}$$

Note the dependence of the thermal solution on hydrodynamic conditions through appearance of the variable f in Equation 7.21. The appropriate boundary conditions are

$$T^*(0) = 0 \quad \text{and} \quad T^*(\infty) = 1 \tag{7.22}$$

Subject to the conditions of Equation 7.22, Equation 7.21 may be solved by numerical integration for different values of the Prandtl number; representative temperature distributions for $Pr = 0.6$, 1, and 7 are shown in Figure 7.4b. Thermal effects penetrate farther into the velocity boundary layer with decreasing Prandtl number and transcend the velocity boundary layer for $Pr < 1$. One important consequence of this solution is that, for

$Pr \gtrsim 0.6$, results for the surface temperature gradient $dT^*/d\eta|_{\eta=0}$ may be correlated by the following relation:

$$\left.\frac{dT^*}{d\eta}\right|_{\eta=0} = 0.332\, Pr^{1/3}$$

Expressing the local convection coefficient as

$$h_x = \frac{q_s''}{T_s - T_\infty} = -\frac{T_\infty - T_s}{T_s - T_\infty}\, k \left.\frac{\partial T^*}{\partial y}\right|_{y=0}$$

$$h_x = k\left(\frac{u_\infty}{\nu x}\right)^{1/2} \left.\frac{dT^*}{d\eta}\right|_{\eta=0}$$

it follows that the *local* Nusselt number is of the form

$$Nu_x \equiv \frac{h_x x}{k} = 0.332\, Re_x^{1/2}\, Pr^{1/3} \qquad Pr \gtrsim 0.6 \tag{7.23}$$

From the solution to Equation 7.21, it also follows that, for $Pr \gtrsim 0.6$, the ratio of the velocity to thermal boundary layer thickness is

$$\frac{\delta}{\delta_t} \approx Pr^{1/3} \tag{7.24}$$

where δ is given by Equation 7.19. For example, for $Pr = 7$, $\delta/\delta_t = 1.91$ ($\delta_t/\delta = 0.523$), as shown in Figure 7.4*b*.

Mass Transfer Solution The species boundary layer equation, Equation 7.7, is of the same form as the energy boundary layer equation, Equation 7.6, with D_{AB} replacing α. Introducing a normalized species density $\rho_A^* = [(\rho_A - \rho_{A,s})/(\rho_{A,\infty} - \rho_{A,s})]$ and noting that, for a fixed surface species concentration

$$\rho_A^*(0) = 0 \quad \text{and} \quad \rho_A^*(\infty) = 1 \tag{7.25}$$

we also see that the species boundary conditions are of the same form as the temperature boundary conditions given in Equation 7.22. Therefore, as discussed in Section 6.7.1, the heat and mass transfer analogy may be applied since the differential equation and boundary conditions for the species concentration are of the same form as for temperature. Hence, with reference to Equation 7.23,

$$Sh_x \equiv \frac{h_{m,x} x}{D_{AB}} = 0.332\, Re_x^{1/2}\, Sc^{1/3} \qquad Sc \gtrsim 0.6 \tag{7.26}$$

By analogy to Equation 7.24, it also follows that the ratio of boundary layer thicknesses is

$$\frac{\delta}{\delta_c} \approx Sc^{1/3} \tag{7.27}$$

The foregoing results may be used to compute important *laminar* boundary layer parameters for $0 < x < x_c$, where x_c is the distance from the leading edge at which transition