

On Exponential Synchronization of Kuramoto Oscillators

Nikhil Chopra and Mark W. Spong

Abstract—In this paper we study the problem of exponential synchronization for one of the most popular models of coupled phase oscillators, the Kuramoto model. We consider the special case of finite oscillators with distinct, bounded natural frequencies. Our first result derives a lower bound on the coupling gain which is necessary for the onset of synchronization. This bound improves the one derived by Jadbabaie et al. [8]. We then calculate a lower bound on the coupling gain that is sufficient to guarantee oscillator synchronization and derive further sufficient conditions to ensure exponential synchronization of the angular frequencies of all oscillators to the mean natural frequency of the group. We also characterize the coupling gain that is sufficient for the oscillator phase differences to approach any desired compact set in finite time.

Index Terms—multi-agent systems, synchronization, Kuramoto oscillators, exponential stabilization, cooperative control.

I. INTRODUCTION AND BACKGROUND

Collective synchronization has long been observed in biological, chemical, physical and social systems. This phenomenon is observed when the individual frequencies of coupled oscillators converge to a common frequency despite differences in the natural frequencies of the individual oscillators. Examples in biology and physics include groups of synchronously flashing fireflies [1], crickets that chirp in unison [20], and Josephson junctions [23]. Collective synchronization was first studied by Wiener [22], who conjectured its involvement in the generation of alpha rhythms in the brain. It was then taken up by Winfree [24] who used it to study circadian rhythms in living organisms. Winfree's model was significantly extended by Kuramoto [10], [11] who developed results for what is now popularly known as the Kuramoto model. An elegant summary of Kuramoto's work, and later attempts to answer the questions that were raised by his formulations, can be found in [17].

Research partially supported by the Office of Naval Research under grant N00014-02-1-0011, N00014-05-1-0186, and by the National Science Foundation under grants ECS-0122412, HS-0233314, and CCR-0209202

N. Chopra is with the Department of Mechanical Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20740 email:nchopra@umd.edu

M.W. Spong is with Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 W. Main St., Urbana, IL 61801 USA email:mspong@uiuc.edu

The Kuramoto model consists of a population of N oscillators whose dynamics are governed by the following equations

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{k=1}^N \sin(\theta_k - \theta_i), \quad i = 1, \dots, N \quad (1)$$

where θ_i is the phase of the i^{th} oscillator, ω_i is its natural frequency, and $K > 0$ is the coupling gain. The problem is then to characterize the coupling gain K so that the oscillators synchronize.

Definition 1.1: The oscillators are said to synchronize if

$$\lim_{t \rightarrow \infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0 \quad \forall i, j = 1, \dots, N$$

Recently, control theoretic methods have been used in [2], [3], [7], [8], [14], [15], [16], [21], [19] to address the synchronization phenomenon. Stability analysis was carried out for a unidirectional ring of oscillators with Kuramoto-type dynamics in [15]. In [14], the authors demonstrated that only phase locking solutions, where the phases differences correspond to $(-\frac{\pi}{2}, \frac{\pi}{2})$, can be locally asymptotically stable, and a condition for guaranteeing local asymptotic stability was derived. Control and graph theoretic methods were used in [8] to analyze Kuramoto oscillators for an arbitrary bidirectional graph topology. The authors derived a coupling gain K_L necessary for the existence of the phase-locked state in the traditional Kuramoto model (all-to-all connectivity). It was also shown that there exists a large enough coupling gain K so that the vector of phase differences locally $(-\frac{\pi}{2}, \frac{\pi}{2})$ converges to a unique constant. Recently, local exponential stability of the phase-locked state in the Kuramoto model was studied in [12].

There are some important questions still associated with the Kuramoto model and they form the motivation for this paper. It has been observed via numerical simulations that for a large enough coupling gain K , the oscillator frequencies exponentially synchronize to the mean natural frequency of the group. Local exponential stability of the Kuramoto model has recently been studied in [12]. However, till date there is no global analysis that shows that the oscillator frequencies in the original Kuramoto model (where the oscillators have different natural frequencies) synchronize exponentially. In this paper we extend our preliminary results in [2] to solve the exponential synchronization problem by studying the nonlinear Kuramoto model (1).

In this paper, we study the case of a finite number N of Kuramoto oscillators with all-to-all connectivity. We assume that the natural frequencies ω_i of the oscillators are arbitrarily chosen from the set of reals. Our contribution in this paper can be summarized as follows:

- In Section II we derive a lower bound on the coupling gain K

that is necessary for the existence of the phase-locked state in the Kuramoto model. This bound improves the result of Jadbabaie et al. [8].

- In Section III we derive sufficient conditions for exponential synchronization of the oscillator frequencies to the mean natural frequency of the group, and then in Section IV a lower bound on the coupling gain is developed to achieve the sufficient conditions. We believe this is the first result that demonstrates almost global exponential synchronization in the Kuramoto model for distinct natural frequencies of the individual oscillators. Furthermore, we characterize the coupling gain that guarantees convergence of the oscillator phase differences to any desired set in $(-\pi, \pi)$, provided certain conditions on the initial phase differences are satisfied.

II. NECESSARY CONDITION FOR SYNCHRONIZATION

As we are interested in the evolution of the phase differences, we write the phase difference dynamics using (1) as

$$\begin{aligned} \dot{\theta}_i - \dot{\theta}_j = & \omega_i - \omega_j + \frac{K}{N} \left\{ -2 \sin(\theta_i - \theta_j) + \sum_{\substack{k=1 \\ k \neq i, j}}^N (\sin(\theta_k - \theta_i)) \right. \\ & \left. + \sin(\theta_j - \theta_k) \right\} \end{aligned} \quad (2)$$

If the oscillators are to synchronize, i.e. $\dot{\theta}_i(t) - \dot{\theta}_j(t) \rightarrow 0$ as $t \rightarrow \infty$, the right hand side of (2) must go to zero. Our first result calculates a lower bound on the coupling gain K necessary for the right-hand side of equation (2) to equal zero. In other words, we calculate a necessary condition for the onset of synchronization in (1).

Theorem 2.1: Consider the system of oscillators described by (1). There exists a coupling gain $K = K_c > 0$ below which the oscillators cannot synchronize.

Proof: The oscillators can only synchronize if the expression in the right hand side of equation (2) has at least one fixed point $\forall i, j = 1, \dots, N$ $i \neq j$. The fixed point equation can be written as

$$\begin{aligned} \omega_j - \omega_i = & \frac{K}{N} \left\{ 2 \sin(\theta_j - \theta_i) + \sum_{\substack{k=1 \\ k \neq i, j}}^N (\sin(\theta_k - \theta_i)) \right. \\ & \left. + \sin(\theta_j - \theta_k) \right\} \end{aligned} \quad (3)$$

Assuming without loss of generality that $\omega_j > \omega_i$, to calculate a lower bound on the coupling gain K satisfying (3), we need to maximize the expression

$$E_{ij} = 2 \sin(\theta_j - \theta_i) + \sum_{\substack{k=1 \\ k \neq i, j}}^N (\sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k)) \quad (4)$$

The first order necessary conditions for maximizing E_{ij} are given by

$$\frac{\partial E_{ij}}{\partial \theta_i} = -2 \cos(\theta_j - \theta_i) - \sum_{\substack{k=1 \\ k \neq i, j}}^N \cos(\theta_k - \theta_i) = 0 \quad (5)$$

$$\frac{\partial E_{ij}}{\partial \theta_j} = 2 \cos(\theta_j - \theta_i) + \sum_{\substack{k=1 \\ k \neq i, j}}^N \cos(\theta_j - \theta_k) = 0 \quad (6)$$

$$\frac{\partial E_{ij}}{\partial \theta_k} = \cos(\theta_k - \theta_i) - \cos(\theta_j - \theta_k) = 0 \quad (7)$$

where $k = 1, \dots, N$, $k \neq i, j$. Equation (7) implies that either $\theta_j = \theta_i + 2p\pi$ or $\theta_k = p\pi + \frac{\theta_i + \theta_j}{2}$, $p = 0, 1, 2, 3, \dots$. The first solution implies that $E_{ij} = 0$. As we are looking for a maximum, we investigate the other solution. Replacing θ_k by the even solution ($p = 0, 2, \dots$) in (5) yields

$$\begin{aligned} 2 \cos(\theta_j - \theta_i) + \sum_{\substack{k=1 \\ k \neq i, j}}^N \cos\left(\frac{\theta_j - \theta_i}{2}\right) &= 0 \\ \Rightarrow 2 \cos(\theta_j - \theta_i) + (N-2) \cos\left(\frac{\theta_j - \theta_i}{2}\right) &= 0 \\ \Rightarrow 4 \cos^2\left(\frac{\theta_j - \theta_i}{2}\right) - 2 + (N-2) \cos\left(\frac{\theta_j - \theta_i}{2}\right) &= 0 \end{aligned}$$

Solving the above quadratic equation we get

$$\cos\left(\frac{\theta_j - \theta_i}{2}\right) = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 32}}{8}$$

As $\cos(x) \leq 1 \forall x \in R$, a well defined solution for all N is given by

$$(\theta_j - \theta_i)_{opt1} = 2 \cos^{-1}\left(\frac{-(N-2) + \sqrt{(N-2)^2 + 32}}{8}\right) \quad (8)$$

Similarly, it can be verified that the odd solutions ($p = 1, 3, \dots$) lead to the solution

$$(\theta_j - \theta_i)_{opt2} = 2 \cos^{-1}\left(\frac{(N-2) - \sqrt{(N-2)^2 + 32}}{8}\right)$$

The solutions $(\theta_j - \theta_i)_{opt1}, (\theta_j - \theta_i)_{opt2}$ satisfy $\frac{\pi}{2} \leq (\theta_j - \theta_i)_{opt1} \leq \pi$ and $\pi \leq (\theta_j - \theta_i)_{opt2} \leq \frac{3\pi}{2}$ respectively. Therefore, using (4) we have $E_{ij}[(\theta_j - \theta_i)_{opt1}] > E_{ij}[(\theta_j - \theta_i)_{opt2}]$, and hence the candidate optimal value of $\theta_j - \theta_i$ is $(\theta_j - \theta_i)_{opt} = (\theta_j - \theta_i)_{opt1}$. To verify the second order necessary condition, note that the off-diagonal terms of the Hessian of E_{ij} consists of $\sin(\cdot)$ terms and the diagonal terms are simply the negative sum of the off-diagonal terms. As $\frac{\pi}{2} \leq (\theta_j - \theta_i)_{opt1} < \pi$, it follows from the even solution that all phase differences $((\theta_j - \theta_i)_{opt}, (\theta_k - \theta_i)_{opt}, (\theta_j - \theta_k)_{opt})$ modulo 2π belong to the interval $[\frac{\pi}{4}, \pi]$. Thus, $\sin(\theta_j - \theta_i)_{opt}, \sin(\theta_k - \theta_i)_{opt}, \sin(\theta_j - \theta_k)_{opt} \geq 0$. Using Gershgorin's Theorem [6], the Hessian for the function E_{ij} is negative semidefinite and hence E_{ij} is a concave function over the domain $[\frac{\pi}{4}, \pi]$. Thus, the necessary conditions for maximization are

sufficient as well and the optimal value of E_{ij} is given as

$$E_{max} = 2 \sin(\theta_j - \theta_i)_{opt} + 2(N-2) \sin\left(\frac{(\theta_j - \theta_i)_{opt}}{2}\right) \\ \leq 2(N-1)$$

Consequently, using (3) the critical coupling gain required for the onset of synchronization in (2) is given by $K_c = \frac{(\omega_j - \omega_i)N}{E_{max}}, \forall i, j$. If the natural frequencies belong to a compact set, then the critical coupling gain required for onset of synchronization in (1) is given as

$$K_c = \frac{(\omega_{max} - \omega_{min})N}{E_{max}} \quad (9)$$

where $\omega_{max}, \omega_{min}$ are the maximum and minimum frequencies in the set of natural frequencies. ■

The phrase *critical coupling gain required for onset of synchronization* does not imply that at K_c the oscillators synchronize. The critical gain K_c is the gain below which the oscillators cannot synchronize. It is interesting to compare condition (9) with that for the critical gain obtained in [8]. The value for the critical coupling in [8] is given as

$$K_L = \frac{(\omega_{max} - \omega_{min})N}{2(N-1)} \quad (10)$$

As $E_{max} \leq 2(N-1)$, it follows that synchronization is not possible for all coupling gains K satisfying $K_L \leq K < K_c$.

The critical gain condition (9) can be written as

$$K_c = \frac{(\omega_{max} - \omega_{min})N}{2 \sin(\theta_j - \theta_i)_{opt} + 2(N-2) \sin\left(\frac{(\theta_j - \theta_i)_{opt}}{2}\right)} \quad (11)$$

For the case $N = 2$, from (8) we have $(\theta_j - \theta_i)_{opt} = 2 \cos^{-1}\left(\frac{\sqrt{32}}{8}\right) = \frac{\pi}{2}$. Therefore, using (11), the critical coupling is given as $K_c = \omega_{max} - \omega_{min}$. Substituting $N = 2$ in (10) we get that $K_L = \omega_{max} - \omega_{min}$. Consequently, $K_L = K_c$ when there are only two oscillators in the system.

When $N \rightarrow \infty$, the equation (8) yields the solution $(\theta_j - \theta_i)_{opt} = \pi$. Substituting the optimal solution in (11) and taking limit

$$\lim_{N \rightarrow \infty} K_c = \lim_{N \rightarrow \infty} \frac{(\omega_{max} - \omega_{min})}{2 \frac{\sin(\theta_j - \theta_i)_{opt}}{N} + 2\left(1 - \frac{2}{N}\right) \sin\left(\frac{(\theta_j - \theta_i)_{opt}}{2}\right)} \\ = \frac{(\omega_{max} - \omega_{min})}{2}$$

Comparing the above estimate to the critical gain condition proposed by [8],

$$\lim_{N \rightarrow \infty} K_L = \lim_{N \rightarrow \infty} \frac{(\omega_{max} - \omega_{min})}{2\left(1 - \frac{1}{N}\right)} = \frac{(\omega_{max} - \omega_{min})}{2} \\ = \lim_{N \rightarrow \infty} K_c$$

Thus, the critical gain condition proposed in this paper is equivalent to the one proposed in [8] for the infinite oscillator case. The comparison

between the two critical gains can be summarized as

$$K_c = K_L \quad N = 2, N \rightarrow \infty \\ K_c > K_L \quad \text{for all other } N$$

It is interesting to note that for the two oscillator system, $K_L = K_c = \omega_{max} - \omega_{min}$ is also sufficient for oscillator synchronization. We refer the reader to [8] for further details.

III. SUFFICIENT CONDITIONS FOR SYNCHRONIZATION

In this section we develop sufficient conditions for synchronization of Kuramoto oscillators and characterize the rate of convergence.

Theorem 3.1: Consider the system of oscillators described by (1). If there exists $T \geq 0$ such that $\forall t \geq T, |\theta_i - \theta_j| < \frac{\pi}{2} - \epsilon \quad \forall i, j$, where $0 < \epsilon < \frac{\pi}{2}$, then the oscillator frequencies $\dot{\theta}_i$ synchronize exponentially to the mean frequency $\Omega = \frac{1}{N} \sum_{i=1}^N \omega_i$ and satisfy $|\dot{\theta}_i - \Omega| \leq \sigma_T e^{-K \sin(\epsilon)(t-T)}, \quad \sigma_T > 0 \quad \forall i = 1, \dots, N$.

Proof: Differentiating the Kuramoto model (1) we have

$$\ddot{\theta}_i = \frac{K}{N} \sum_{j=1}^N \cos(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i), \quad i = 1, \dots, N \quad (12)$$

Consider the positive function, $S = \frac{1}{2} \dot{\theta}^T \dot{\theta}$ where $\dot{\theta} = [\dot{\theta}_1 \dots \dot{\theta}_N]^T$. Differentiating S along trajectories of (1) and rearranging terms we have that,

$$\dot{S} = -\frac{K}{N} \sum_{j=1}^N \sum_{i=1}^N \cos(\theta_i - \theta_j)(\dot{\theta}_i - \dot{\theta}_j)^2 = -\frac{K}{N} \dot{\theta}^T L \dot{\theta} \quad (13)$$

where, $L_{ii} = \sum_{k=1, k \neq i}^N \cos(\theta_k - \theta_i), L_{ij} = -\cos(\theta_i - \theta_j) \quad \forall i, j = 1, \dots, N \quad i \neq j$. If all the phase differences are contained in the set $|\theta_i - \theta_j| < \frac{\pi}{2}, L_{ii} > 0, \quad i = 1, \dots, N$. Using Gershgorin's Theorem [6], all eigenvalues of the matrix L are located in the union of the N discs

$$\cup_{i=1}^N \left\{ z \in \mathbb{C} : \left| z - \sum_{\substack{k=1 \\ k \neq i}}^N \cos(\theta_k - \theta_i) \right| \leq \sum_{\substack{k=1 \\ k \neq i}}^N \cos(\theta_k - \theta_i) \right\}$$

Thus all eigenvalues are located in the right half plane, and hence the matrix L is positive semidefinite. The row sums of the matrix L are zero, which implies that the matrix L has a zero eigenvalue with 1_N as the corresponding N dimensional eigenvector of ones. The matrix L is also popularly known as the weighted Laplacian [4] in the graph theory literature.

It is to be noted that (see [5], [14]) $\sum_{i=1}^N \dot{\theta}_i = \sum_{i=1}^N \omega_i$. Therefore, define $\Omega = \frac{1}{N} \sum_{i=1}^N \dot{\theta}_i = \frac{1}{N} \sum_{i=1}^N \omega_i$ which implies that Ω is an invariant quantity. Following [13], the vector $\dot{\theta}$ can be written as

$$\dot{\theta} = \Omega 1_N + \delta \quad (14)$$

where $\mathbf{1}_N$ is the N dimensional vector of ones associated with the zero eigenvalue of the matrix L , δ satisfies $\sum_{i=1}^N \delta_i = 0$ (as $\sum_{i=1}^N \dot{\theta}_i = N\Omega$). The vector δ is orthogonal to $\mathbf{1}_N$ and was referred to as the group disagreement vector in [13]. Substituting (14) in (13) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\Omega \mathbf{1}_N + \delta)^T (\Omega \mathbf{1}_N + \delta) &= -\frac{K}{N} (\Omega \mathbf{1}_N + \delta)^T L (\Omega \mathbf{1}_N + \delta) \\ \Rightarrow \frac{1}{2} \frac{d}{dt} (\Omega \mathbf{1}_N^T \Omega \mathbf{1}_N + \delta^T \Omega \mathbf{1}_N + \Omega \mathbf{1}_N^T \delta + \delta^T \delta) &= -\frac{K}{N} (\Omega \mathbf{1}_N^T L \Omega \mathbf{1}_N \\ + \delta^T L \Omega \mathbf{1}_N + \Omega \mathbf{1}_N^T L \delta + \delta^T L \delta) \end{aligned}$$

Using invariance of Ω ($\frac{d}{dt} \Omega \mathbf{1}_N^T \Omega \mathbf{1}_N = 0$), orthogonality of the disagreement vector δ ($\delta^T \mathbf{1}_N = 0$), and the fact that $\mathbf{1}_N$ is an eigenvector associated with the zero eigenvalue of L ($\mathbf{1}_N^T L = 0$) we have

$$\frac{d(\delta^T \delta)}{dt} = -\frac{2K}{N} \delta^T L \delta \quad (15)$$

In order to get a bound on the convergence rate of δ , it is easy to verify that the smallest non-zero eigenvalue of the matrix L (using the fact that the smallest nonzero eigenvalue of the $N \times N$ Laplacian of a complete graph is N , and $\min\{\cos(\theta_i - \theta_j) : |\theta_i - \theta_j| \leq \frac{\pi}{2} - \epsilon\} = \cos(\frac{\pi}{2} - \epsilon) = \sin(\epsilon)$), is lower bounded by $N \sin(\epsilon)$. Using this fact in (15) yields

$$\begin{aligned} \frac{d(\delta^T \delta)}{dt} &\leq -2\frac{K}{N} \lambda_{\min}(L) \delta^T \delta \leq -2K \sin(\epsilon) \delta^T \delta \\ \Rightarrow \|\delta\| &\leq \sigma_T e^{-K \sin(\epsilon)(t-T)} \\ \Rightarrow |\dot{\theta}_i - \Omega| = |\delta_i| &\leq \sigma_T e^{-K \sin(\epsilon)(t-T)} \quad \forall i = 1, \dots, N \quad (16) \end{aligned}$$

where the notation $\|\cdot\|$ denotes the Euclidean norm of δ . Thus the exponential convergence rate for synchronization is no worse than $K \sin(\epsilon)$. ■

It also follows from the above result that the order parameter [17] exponentially converges to a constant. The underlying assumption in the previous result is that $\exists T \geq 0$ s.t. $|\theta_i - \theta_j| < \frac{\pi}{2} - \epsilon \quad \forall i, j \quad \forall t \geq T$. In the next section we use the coupling gain K to steer the phase differences to this desired set in finite time.

IV. CONTROLLING THE PHASE DIFFERENCES

In this section, a lower bound on the coupling gain K is developed which is sufficient to trap the oscillator phase differences within any desired arbitrary compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$, and hence by Theorem 3.1 guarantee oscillator synchronization. We consider two possible cases

- 1) The initial phase of all oscillators lies within the set described by $\mathcal{D} = \{\theta_i, \theta_j \mid |\theta_i - \theta_j| \leq \frac{\pi}{2} - \epsilon\}; \quad \forall i, j$ where $\epsilon < \frac{\pi}{2}$ is an arbitrary positive number.

- 2) Let $\theta_p = [\theta_1 - \theta_2 \dots \theta_1 - \theta_N \dots \theta_{N-1} - \theta_N]^T$, be the $\frac{N(N-1)}{2} \times 1$ vector of phase differences. The initial phase differences satisfy the condition $\|\theta_p\| \leq \pi - \gamma$ where γ is an arbitrary small strictly positive number.

The first assumption allows for the case when all the initial phase differences lie within a compact set in $(-\frac{\pi}{2}, \frac{\pi}{2})$. The second condition relaxes this constraint but imposes a norm constraint on the phase differences. However, it is evident the above cases do not span $(-\pi, \pi)$ and we will address this in our future work. For both of the above scenarios, we show that there exists a large enough gain to trap the phase differences within any desired arbitrary compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Starting with the first case, we develop a lower bound on the coupling gain K , denoted by K_{inv} , that makes the set \mathcal{D} positively invariant for all oscillators, i.e. $\theta_i(0) - \theta_j(0) \in \mathcal{D} \Rightarrow \theta_i(t) - \theta_j(t) \in \mathcal{D} \quad \forall t > 0$. The phase difference dynamics, as described by (2), can be rewritten as

$$\begin{aligned} \dot{\theta}_i - \dot{\theta}_j &= K \left\{ \frac{\omega_i - \omega_j}{K} - \sin(\theta_i - \theta_j) + \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^N (\sin(\theta_i - \theta_j)) \right. \\ &\left. + \sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k) \right\} \quad (17) \end{aligned}$$

Consider the term under the summation in (17). This can be rewritten using standard trigonometric manipulations as $\frac{1}{N} \sin(\theta_i - \theta_j) C_k$ where $C_k = \left(1 - \frac{\cos(\theta_k - \frac{(\theta_i + \theta_j)}{2})}{\cos(\frac{(\theta_i - \theta_j)}{2})}\right)$. It is easy to see that $\forall (\theta_i - \theta_j) \in \mathcal{D}, |C_k| < 1$. Using the transformed summation, (17) can be rewritten as

$$\begin{aligned} \dot{\theta}_i - \dot{\theta}_j &= K \left\{ \frac{\omega_i - \omega_j}{K} - \sin(\theta_i - \theta_j) + \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^N C_k \sin(\theta_i - \theta_j) \right\} \\ &= K \left\{ \frac{\omega_i - \omega_j}{K} - \sin(\theta_i - \theta_j) \left(1 - \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^N C_k\right) \right\} \quad (18) \end{aligned}$$

We are now in a position to state the first result of this section.

Theorem 4.1: Consider the system dynamics as described by (18). Let all initial phase differences at $t=0$ be contained in the compact set $\mathcal{D} = \{\theta_i, \theta_j \mid |\theta_i - \theta_j| \leq \frac{\pi}{2} - \epsilon \quad \forall i, j = 1, \dots, N\}$ where $0 < \epsilon < \frac{\pi}{2}$. Then there exists a coupling gain $K_{inv} > 0$ such that $(\theta_i(t) - \theta_j(t)) \in \mathcal{D} \quad \forall t > 0$.

Proof: Let a positive definite Lyapunov function for the dynamic system governed by (18) be given as $V = \frac{1}{2K} (\theta_i - \theta_j)^2$. The

derivative of V along trajectories of the system (18) is given as

$$\begin{aligned}\dot{V} &= \frac{1}{K}(\theta_i - \theta_j)(\dot{\theta}_i - \dot{\theta}_j) \\ &= (\theta_i - \theta_j) \left(\frac{\omega_i - \omega_j}{K} - \sin(\theta_i - \theta_j) \left(1 - \frac{1}{N} \sum_{\substack{k=1 \\ k \neq i}}^N C_k \right) \right) \\ &\leq |\theta_i - \theta_j| \left| \frac{\omega_i - \omega_j}{K} \right| - (\theta_i - \theta_j) \sin(\theta_i - \theta_j) \left(1 - \sum_{\substack{k=1 \\ k \neq i}}^N \frac{C_k}{N} \right) \\ &\leq |\theta_i - \theta_j| \left| \frac{\omega_i - \omega_j}{K} \right| - (\theta_i - \theta_j) \sin(\theta_i - \theta_j) \left(1 - \frac{N-2}{N} \right)\end{aligned}$$

where we have used the fact that $|C_k| < 1$. Thus the derivative can be written as

$$\dot{V} \leq |\theta_i - \theta_j| \left| \frac{\omega_i - \omega_j}{K} \right| - (\theta_i - \theta_j) \sin(\theta_i - \theta_j) \frac{2}{N}$$

It is to be noted that the function $\sin(\theta_i - \theta_j)(\theta_i - \theta_j)$ is always nonnegative in the considered domain. Therefore, if $K > \frac{N|\omega_i - \omega_j|}{2 \cos(\epsilon)}$, the derivative of the Lyapunov function is negative at $|\theta_i - \theta_j| = \frac{\pi}{2} - \epsilon$, and thus the phase difference $\theta_i - \theta_j$ cannot leave the set \mathcal{D} . Finally, if $K = K_{inv} > \frac{N|\omega_{max} - \omega_{min}|}{2 \cos(\epsilon)}$, all phase differences $\theta_i - \theta_j \quad \forall i = 1, \dots, N$ are positively invariant with respect to the compact set \mathcal{D} . ■

As the phase differences are trapped within the set \mathcal{D} by appropriately choosing the coupling gain, using Theorem 3.1 the oscillators are guaranteed to synchronize.

Consider the second case when the initial phase differences satisfy the condition $\|\theta_p(0)\| \leq \pi - \gamma$. Our next result shows that for a large enough coupling gain K , the vector θ_p is ultimately bounded.

Theorem 4.2: Consider the system dynamics as described by (17). Let the initial vector of phase differences be contained in the set $\|\theta_p\| \leq \pi - \gamma$. Then there exists a coupling gain $K(s) > 0$, for some $s > 0$ such that the phase differences enters the set described by $\|\theta_p\| \leq s$ in finite time.

Proof: A positive definite function of the phase differences is given as $V_p = \frac{1}{2} \|\theta_p\|^2$. Differentiating V_p along solutions of (17) yields

$$\begin{aligned}\dot{V}_p &= \sum_{i < j} (\dot{\theta}_i - \dot{\theta}_j)(\theta_i - \theta_j) \\ &= \sum_{i < j} \left((\omega_i - \omega_j)(\theta_i - \theta_j) - K \sin(\theta_i - \theta_j)(\theta_i - \theta_j) \right) \\ &\quad + \frac{K}{N} \sum_{\substack{k=1 \\ k \neq i, j}}^N \left(\sin(\theta_i - \theta_j) + \sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k) \right) (\theta_i - \theta_j)\end{aligned}$$

The term under the double summation can be simplified as

$$\begin{aligned}(N-2) \sum_{i < j} \sin(\theta_i - \theta_j)(\theta_i - \theta_j) + \sum_{i < j} \sum_{\substack{k=1 \\ k \neq i, j}}^N \left(\sin(\theta_k - \theta_i) \right. \\ \left. + \sin(\theta_j - \theta_k) \right) (\theta_i - \theta_j)\end{aligned}\tag{19}$$

Consider any two oscillators u, v , with $v < u$, among the N oscillators. Corresponding to these two oscillators, the first term in the above equation contains the expression $(N-2) \sin(\theta_v - \theta_u)(\theta_v - \theta_u)$. Now consider the second term in (19). As both the oscillators are interconnected to $N-2$ other oscillators, the second term contains the two sums $\sum_{r=1, r \neq u, v} \sin(\theta_u - \theta_v)(\theta_v - \theta_r)$ and $\sum_{r=1, r \neq u, v} \sin(\theta_v - \theta_u)(\theta_u - \theta_r)$. Adding the two sums results in the expression $(N-2) \sin(\theta_u - \theta_v)(\theta_v - \theta_u)$. From the above discussion it follows that

$$\sum_{i < j} \sum_{\substack{k=1 \\ k \neq i, j}}^N \left(\sin(\theta_i - \theta_j) + \sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k) \right) (\theta_i - \theta_j) = 0$$

The derivative of V_p is then given as

$$\dot{V}_p = \sum_{i < j} \left((\omega_i - \omega_j)(\theta_i - \theta_j) - K \sin(\theta_i - \theta_j)(\theta_i - \theta_j) \right)$$

Using the fact that at time $t=0$, $\text{sinc}(\theta_i - \theta_j) = \frac{\sin(\theta_i - \theta_j)}{\theta_i - \theta_j} \leq \alpha \quad \forall i, j$ where $0 < \alpha \leq 1$ (as $\|\theta_p\| \leq \pi - \gamma$ and sinc is a decreasing even function in $(0, \pi)$), the derivative reduces to

$$\dot{V}_p \leq \sum_{i < j} \left((\omega_i - \omega_j)(\theta_i - \theta_j) - K\alpha(\theta_i - \theta_j)^2 \right)$$

Let $\omega_d = [\omega_1 - \omega_2 \dots \omega_1 - \omega_N \dots \omega_{N-1} - \omega_N]^T$ denote the vector of the difference of natural frequencies. Then using the Cauchy-Schwarz inequality,

$$\dot{V}_p \leq \|\omega_d\| \|\theta_p\| - K\alpha \|\theta_p\|^2 \leq \|\theta_p\| (\|\omega_d\| - K\alpha \|\theta_p\|)$$

Choose $K(s) = \frac{\|\omega_d\|}{\alpha s} + 2m$, $m > 0$. The derivative of the Lyapunov function reduces to

$$\begin{aligned}\dot{V}_p &\leq \|\theta_p\| (\|\omega_d\| - \left(\frac{\|\omega_d\|}{\alpha s} + 2m \right) \alpha \|\theta_p\|) \\ &\leq -2m\alpha \|\theta_p\|^2 \quad \text{if } \|\theta_p\| \geq s \\ &\Rightarrow \dot{V}_p \leq -4m\alpha V_p\end{aligned}$$

From the above inequality it is evident that V_p , and hence θ_p , decays at an exponential rate. Therefore, the phase differences enter the set $\|\theta_p\| \leq s$ in finite time. ■

It is to be noted that the vector θ_p is always decreasing until it enters the desired set. To ensure synchronization, choose $s < \frac{\pi}{2}$ and the coupling gain $K(s)$ appropriately, then $\exists T \geq 0$ s.t. $|\theta_i - \theta_j| < \frac{\pi}{2} \quad \forall i, j \quad \forall t \geq T$. Consequently, oscillator synchronization follows from Theorem 3.1.

V. CONCLUSIONS

In this paper we studied the phenomenon of synchronization in the traditional Kuramoto model with an arbitrary but finite number of oscillators. A necessary condition in the form of a lower bound on the coupling gain $K = K_c$ was established for the existence of a phase-locked state in the Kuramoto model. This bound is tighter than the result provided by Jadbabaie et al. [8]. Sufficient conditions for exponential synchronization of the oscillator frequencies to the mean natural frequency of the group were also developed. To the best of the authors' knowledge this is the first result that demonstrates almost global exponential synchronization in the finite oscillator Kuramoto model with different natural frequencies, thereby complementing the recent local stability results in [12]. A lower bound on the coupling gain was also developed to guarantee oscillator synchronization. Furthermore, the coupling gain was characterized so as to trap the oscillator phase differences within any desired compact set in $(-\pi, \pi)$, provided the initial oscillator phase differences satisfy certain conditions. Future work involves studying synchronization for probabilistic distribution of the natural frequencies and incorporating time delays in communication.

VI. ACKNOWLEDGEMENTS

The authors are grateful to the anonymous reviewers and to Professor Steven Strogatz and Professor Bruce Francis for their detailed and helpful comments.

REFERENCES

- [1] J. Buck, "Synchronous rhythmic flashing of fireflies. II," *Quarterly Review of Biology*, Vol. 63, No. 3, pp. 265-289, 1988.
- [2] N. Chopra and M.W. Spong, "On synchronization of Kuramoto oscillators," *IEEE Conference on Decision and Control*, pp. 3916-3922, 2005.
- [3] N. Chopra and M.W. Spong, "Passivity-based control of multi-agent systems," In *Advances in Robot Control: From Everyday Physics to Human-Like Movements*, Sadao Kawamura and Mikhail Svinin, Eds., pp. 107-134, Springer Verlag, 2006.
- [4] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer Graduate Texts in Mathematics, No. 207, New York, 2001.
- [5] J.L. van Hemmen and W.F. Wreszinski, "Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators," *Journal of Statistical Physics*, Vol. 72, pp. 145-166, 1993.
- [6] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
- [7] A. Jadbabaie, J. Lin, and A.S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, Vol. 48, No. 6, pp. 988-1001, 2003.
- [8] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the Kuramoto model of coupled nonlinear oscillators," *Proceedings of the American Control Conference*, pp. 4296-4301, 2004.
- [9] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, New Jersey, 2002.
- [10] Y. Kuramoto, "Self-entrainment of a population of coupled non-linear oscillators," In *International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics*, Vol. 39, pp. 420-422, Springer, New York, 1975.
- [11] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer, Berlin, 1984.
- [12] R.E. Mirollo and S.H. Strogatz, "The spectrum of locked state for the Kuramoto model of coupled oscillators," *Physica D: Nonlinear Phenomena*, Vol. 205, Issue 1-4, pp. 249-266, 2005.
- [13] R. Olfati-Saber and R.M. Murray, "Consensus problems in networks of dynamic agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, Vol. 49, No. 9, pp. 1520-1533, 2004.
- [14] J.A. Rogge and D. Aeyels, "Existence of partial entrainment and stability of phase locking behavior of coupled oscillators," *Progress of Theoretical Physics*, Vol. 112, No. 6, pp. 921-942, 2004.
- [15] J.A. Rogge and D. Aeyels, "Stability of phase locking in a ring of unidirectionally coupled oscillators," *Journal of Physics A*, Vol. 37, pp. 11135-11148, 2004.
- [16] R. Sepulchre, D. Paley, and N.E. Leonard, "Collection motion and oscillator synchronization," *Cooperative Control, Lecture Notes in Control and Information Sciences*, Vol. 309, V. Kumar, N.E. Leonard, and A.S. Morse, Eds., pp. 189-206, Springer Verlag, 2004.
- [17] S.H. Strogatz, "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D*, Vol. 143, Issue 1-4, pp. 1-20, 2000.
- [18] S.H. Strogatz, *SYNC: The Emerging Science of Spontaneous Order*, Hyperion Press, New York, 2003.
- [19] B.I. Triplett, D.J. Klein and K.A. Morgansen, "Discrete time Kuramoto models with delay," In *Networked Embedded Sensing and Control, Lecture Notes in Control and Information Sciences*, pp. 9-23, Springer, 2005.
- [20] T.J. Walker, "Acoustic synchrony: two mechanisms in the snowy tree cricket," *Science* Vol. 166, No. 3907, 891-894, 1969.
- [21] W. Wang W. and J.J.E Slotine, "On Partial Contraction Analysis for Coupled Nonlinear Oscillators," *Biological Cybernetics* Vol. 92, No. 1, pp. 38-53, 2005.
- [22] N. Wiener, *Nonlinear Problems in Random Theory*, MIT Press, Cambridge, Ma, 1958.
- [23] K. Wiesenfeld, P. Colet, and S.H. Strogatz, "Frequency locking in Josephson arrays: Connection with the Kuramoto model," *Physical Review E*, Vol. 57, No. 2, pp. 1563-1569, 1998.
- [24] A.T. Winfree, *The Geometry of Biological Time*, Springer, New York, 1980.