PUTNAM'S RESOLVING MAPS IN DIMENSION ZERO

MIKE BOYLE

Abstract. A block code from an irreducible shift of finite type can be lifted canonically through resolving maps to a resolving map. There is an application to Markovian maps.

Contents

1. Introduction

Let $\mathcal{S}$ be the set of 1-to-1 almost everywhere factor maps between irreducible Smale spaces. Ian Putnam [16] proved the following theorem: given π in $\mathcal{S}$, there are Smale spaces $\overline{X}$ and $\overline{Y}$, and maps α, β and $\widetilde{\pi}$ in $\mathcal{S}$, such that α and β are u-resolving, $\widetilde{\pi}$ is s-resolving and the following diagram commutes:

$$\begin{tikzcd} \overline{X} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\alpha"'] & \overline{Y} \arrow[d, "\beta"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

Putnam's result was a surprise even in dimension zero, where resolving maps have been studied extensively. In the zero dimensional case (where a Smale space is a shift of finite type and a finitely presented system is a sofic shift), we will reinterpret and extend Putnam's work. Because a lot is known about resolving maps, this adds to our understanding of general factor maps. We hope this work will also be relevant to progress in positive dimension.

We view Putnam's construction in two stages: first construct the map β, then construct the diagram by a fiber product argument. For β, we construct a canonically associated map we call π_+, the futures cover of π. This map is itself a

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consequence of a more general construction, the full futures cover of π (Section 3). The fiber product argument is carried out in Section 4, and associates a canonical lift to π. For these sections, the map π can be any block code from an irreducible SFT to a sofic shift, even an infinite to one code. In the infinite to one case, the lift $\widetilde{\pi}$ is only "eresolving"–resolving in the sense of existence (Sec. 2)–but this is meaningful.

In Section 5, we show that our construction is the unique (up to topological conjugacy) minimal construction – any other such lift must factor through it. This, and the generalization to finitely presented range systems, answer (in the zero dimensional case) the two questions raised by Putnam in [16].

The futures cover arises from an effort to simplify Putnam's construction. Where both the futures cover and the map β constructed by Putnam are defined, they and their associated diagrams are isomorphic (Section 6). The futures cover is also closely related to a canonical cover of Nasu (Section 6). The three constructions (futures cover, Putnam cover, Nasu cover) are all isomorphic where they are all defined (for 1-1 a.e. maps between irreducible SFTs).

An unexpected spinoff of this investigation is some new information about infinite to one maps. A magic word construction very familiar for finite to one maps turns out to be meaningful (7.1) and of some use for infinite to one maps. We also learn that any factor map between irreducible SFTs can after precomposition with a degree one resolving map be made Markovian (7.2).

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2. Background

In this section, we recall and slightly expand some background results and terminology. For further detail, see the texts [8, 11]. We will be concerned with subshifts (X,T) where X is a shift-invariant space of doubly infinite sequences over a finite alphabet, with the usual topology (relative of product of discrete, making X zero dimensional compact metrizable) and T is the restriction of the shift map σ to X. (We use the left shift, $(\sigma x)_i = x_{i+1}$.) The alphabet for (X,T) may be denoted $\mathcal{A}(X)$ or $\mathcal{A}(T)$, and the set of words of length n occurring in points of X is denoted $\mathcal{W}_n(X)$ or $\mathcal{W}_n(T)$. A point x of a subshift (X,S) is *left transitive* (*right transitive*) if for every S-word W there exist negative (positive) integers $i \leq j$ such that $W = x[i,j]$. A point is *doubly transitive* if it both left transitive and right transitive. Points x and x' in a subshift are *left asymptotic* if there exists n in $\mathbb{Z}$ such that $x(-\infty,n] = x'(-\infty,n]$. Systems (X,T) and (X',T') are *topologically conjugate* if there is a homeomorphism $\varphi : X \to X'$ such that $T'\varphi = \varphi T$. An SFT is a system (X,S) such that there exists n such that a point x is in X if and only if $x[i+1,i+n] \in \mathcal{W}_n(S)$ for every $i \in \mathbb{Z}$. An SFT is *1-step* if this length n can be chosen to be 2. An *edge SFT* is a 1-step SFT for which the alphabet is the set of edges in a nondegenerate directed graph $\mathcal{G}$ and EE' is an allowed word of length 2 if the terminal vertex of E equals the initial vertex E'. An SFT is *irreducible* if it has a dense forward orbit. A directed graph is *nondegenerate* if every vertex has at least one incoming edge and at least one outgoing edge. An edge SFT defined by a nondegenerate directed graph $\mathcal{G}$ is irreducible iff there is a path in $\mathcal{G}$ from any vertex to any other. An *irreducible component* of an SFT is a maximal irreducible subsystem. The entropy of an SFT is the maximum of the entropies of its irreducible components.

A block code is a map $\varphi : (X, S) \to (Y, T)$ between subshifts such that for some n there is a map $\varphi : \mathcal{W}_{2n+1}(S) \to \mathcal{A}(T)$ such that for all x in X and $i \in \mathbb{Z}$, $(\varphi x)_i = \varphi(x[i-n, i+n])$. When such an n exists, the block code has *range n*. If $n = 0$, then φ is a 1-block code. Every continuous shift commuting map (or code) between subshifts is given by a block code. If φ is a surjective block code, then it is a *factor map.* A factor map φ from an irreducible SFT onto a subshift Y has a well defined *degree d*: the cardinality of the preimage of every double transitive point in Y is d. In this case we also say that φ is d-to-1 almost everywhere. A surjective code respects entropy if it is finite to one. An irreducible SFT has no proper subsystem of equal entropy. A *sofic shift* is a subshift which is the image of an SFT under a block code. The SFTs are the zero dimensional Smale spaces and the sofic shifts are the zero dimensional finitely presented dynamical systems (see [7, 16]).

Let φ denote a factor map from a subshift (X, S) to a subshift (Y, T). φ is *right resolving* if it is a 1-block code with the following property: if $a_0 \in \mathcal{A}(X)$, with image symbol $b_0 \in \mathcal{A}(Y)$, and if $b_0 b_1$ is a Y-word, then there exists at most one $a_1 \in \mathcal{A}(X)$ such that a_1 maps to b_1 and $a_0 a_1$ is an X word. φ is *right closing*, or *u-resolving*, if for all left asymptotic x and $x' \in X$, $\varphi(x) = \varphi(x')$ implies $x = x'$. (The terminology "right closing" is standard in symbolic dynamics [8, 11]; Putnam's more general terminology "u-resolving" means injective on unstable sets [16], which in this setting means right closing. We will generally favor Putnam's terminology.)

Similarly, φ is *left resolving* if it is a 1-block code with the following property: if $a_1 \in \mathcal{A}(X)$, with image symbol $b_1 \in \mathcal{A}(Y)$, and if $b_0 b_1$ is a Y-word, then there exists at most one $a_0 \in \mathcal{A}(X)$ such that a_0 maps to b_0 and $a_0 a_1$ is an X word. Likewise, φ is *left closing*, or *s-resolving*, if for all right asymptotic x and $x' \in X$, $\varphi(x) = \varphi(x')$ implies $x = x'$. Here "s-resolving" means injective on stable sets [16]. We say a map is resolving if it is left, right, u- or s-resolving. A resolving map is finite to one and therefore its domain and image are subshifts of equal entropy.

We say φ is *right eresolving* if the right resolving condition holds, but with the uniqueness condition replaced by existence. That is, in the definition of right resolving, we replace "at most one" with "at least one". We say φ is u-eresolving if given x in X and $y \in Y$ such that $\varphi(x)$ is left asymptotic to y, there exists at least one $\overline{x}$ in X such that $\overline{x}$ is left asymptotic to x and $\varphi(\overline{x}) = y$. For block codes, u-eresolving has been called *right continuing* [6, Sec. 5]. The terms *left eresolving*, *left continuing* and s-eresolving are defined analogously.

In the next proposition, we record some known facts relating these notions. Obviously the left/s analogues likewise hold.

Proposition 2.1. *Let φ be a block code from a subshift (X, S) onto a subshift (Y, T).*

1. *If φ is right resolving, then φ is u-resolving (i.e. right closing).*
2. *If φ is u-resolving, then there is a topological conjugacy ψ and a right resolving φ' such that φ' is ψ followed by φ.*
3. *If φ is right eresolving, then φ is u-eresolving (i.e. right continuing).*
4. *If φ is u-eresolving, then there is a topological conjugacy ψ and a right eresolving φ' such that φ' is ψ followed by φ.*

(5) *If (X,S) and (Y,T) are irreducible SFTs of equal entropy, then φ is u-resolving if and only if it is u-eresolving; if in addition these SFTs are 1-step and φ is 1-block, then φ is right resolving if and only if it is right eresolving.*

(6) *If φ is u-resolving and (X,S) is an irreducible SFT, then (Y,T) is an SFT if and only if φ is u-eresolving.*

Proof. (1) and (3) are easy observations. (2) is proved in [8, Proposition 4.33]. (4) is proved in [6, Proposition 5.1]. (5) is partly the result [11, Proposition 8.2.2], and the rest of (5) and (6) are proved by a similar appeal to the Perron-Frobenius theory element [11, Theorem 4.4.7]. □

Remark 2.2. Resolving maps from irreducible SFTs onto strictly sofic shifts are important in the theory of sofic shifts, and such maps cannot be eresolving. On the other hand, eresolving maps between irreducible SFTs of unequal entropy are relevant to understanding Markovian codes [6]. In this paper we will encounter both types of map in the same setting, so for clarity we have introduced the new term eresolving. Caveat: the term "resolving" has sometimes been used to mean u-resolving; or, eresolving; or, both resolving and eresolving.

Lemma 2.3. *Suppose $\varphi : (X,S) \to (Y,T)$ is a u-resolving block code such that (X,S) is SFT, (Y,T) is irreducible SFT, and (X_0,S_0) is an irreducible component of (X,S) of maximum entropy.*

Then any point of (X,S) which is left asymptotic to X_0 is contained in X_0.

Proof. This is [8, Lemma 5.1.4] in that case that φ is right resolving. The conclusion then follows from Proposition 2.1(2). □

"Magic words" are an important technical tool in the theory of codes from SFTs. Let $\varphi : (X,S) \to (Y,T)$ be a surjective one-block code. Above every Y-word $W = W[1,n]$ of length $|W| = n$ there is a set of X-words $W' = W'[1,n]$ of length n which by the coding rule are sent to W, i.e. $\varphi W' = W$. Given $1 \le i \le n$, set $d(W,i) = |\{W'_i : \varphi W' = W\}|$. We define the *resolving degree* $\delta(\varphi)$ of φ to be the minimum of $d(W,i)$ over all Y-words W and $1 \le i \le |W|$. W is a *magic word* if, for some i, $d(W,i) = \delta(\varphi)$. A *magic symbol* for φ is a magic word of length 1, that is, an element μ of $\mathcal{A}(Y)$ such that whenever W is a Y-word and $W_i = \mu$, then $d(W,i) = \delta(\varphi)$. (In other words, if μ occurs at a position in W, then "above" W we see every symbol which maps to μ, and this set of symbols has cardinality $\delta(\varphi)$.) For an irreducible SFT (X,S) and finite-to-one block code $\varphi : (X,S) \to (Y,T)$, the proof that φ has a well defined degree proceeds by showing every doubly transitive point has exactly $\delta(\varphi)$ preimages [8, 11].

We have defined magic words and symbols exactly as in Kitchens' text [8, p.102 and p.113], except that we do not restrict to the case that φ is finite to one. In particular, our number $\delta(\varphi)$ is the number d in [8, p.101]. Two block codes α, β are *topologically conjugate* if there are topological conjugacies ψ_1, ψ_2 such that $\psi_1\beta = \alpha\psi_2$. We introduce the term "resolving degree" because it turns out to be a conjugacy invariant of infinite to one maps as well as finite to one maps (7.1). Likewise, the next lemma will be useful to us in the general (not just finite to one) case.

Lemma 2.4. *A surjective block code φ is topologically conjugate to a 1-block code with a magic symbol. If φ is u-resolving, then in addition this code can be required to be right resolving.*

Proof. The first statement is proved in [8, Proposition 4.3.2]. The restriction in the statement to finite to one codes is not needed in the proof. For the second statement, after passing to a topologically conjugate code, we may assume φ is right resolving. The recoding construction of [8, Proposition 4.3.2] is easily adjusted to respect the right resolving property. □

The next lemma says that a u-resolving map from an irreducible SFT onto a sofic shift is (despite Proposition 2.1(6)) u-eresolving with respect to left transitive points in the image.

Lemma 2.5. *Suppose (X, S) is SFT and $\varphi : (X, S) \to (Y, T)$ is a u-resolving block code. Suppose y is left transitive in Y, and x is a point in X such that φx is left asymptotic to y.*

Then there exists a point x' in X such that x' is left asymptotic to x and $\varphi x' = y$. Also, if φ is right resolving and $(\varphi x)(-\infty, 0] = y(-\infty, 0]$, then x' can be chosen such that $x'(-\infty, 0] = x(-\infty, 0]$.

Proof. After passing to a topologically conjugate map, we can assume without loss of generality that φ is right resolving with a magic symbol μ. Choose $n < 0$ such that $y_n = \mu$. Because μ is a magic symbol, for all $k > 0$ there exists an X-word $a_n \cdots a_{n+k}$ such that $\varphi(a_n \cdots a_{n+k}) = y_n \cdots y_{n+k}$ and $a_n = x_n$. Because φ is right resolving, the word $a_n \cdots a_{n+k}$ is uniquely determined by a_n and y. Define x' by setting $x'_i = x_i$ if $i \leq n$ and $x'_i = a_i$ if $i > n$. □

3. The futures cover of a map

To begin, let $\pi : (X, S) \to (Y, T)$ be a 1-block code from a 1-step SFT onto a sofic shift. We will associate to π a surjective right resolving 1-block code $\pi_{++} : (Y_{++}, T_{++}) \to (Y, T)$, where (Y_{++}, T_{++}) is a 1-step SFT. Given $y \in Y$ and $m \in \mathbb{Z}$, define the following subset of $\mathcal{A}(S)$:

$$U_m(y) \; = \; \{x_m : x \in X \text{ and } \pi(x_j) = y_j \text{ if } j \leq m\} \; . \tag{3.1}$$

Define the alphabet of T_{++} to be

$$\mathcal{A}(T_{++}) \; = \; \{U_0(y) : y \in Y\} = \; \{U_m(y) : y \in Y \text{ and } m \in \mathbb{Z}\} \; .$$

Define the allowed T_{++}-words of length two (and thus determine the 1-step SFT T_{++}) as follows: UU' is allowed if $U \in \mathcal{A}(T_{++})$ and there exists j in $\mathcal{A}(T)$ such that

$$U' \; = \; \{i \in \mathcal{A}(S) : \pi(i) = j \text{ and } i \text{ follows some element of } U\} \; .$$

Note, given U, j and U' as above, we do have U' in $\mathcal{A}(T_{++})$, since there exists a y in Y with $U_0(y) = U$ and $y_1 = j$, and then $U' = U_1(y)$. Similarly, we see that the set of all Y_{++}-words is

$$\{U_0(y)U_1(y) \cdots U_n(y) : n \geq 0, y \in Y\} \; .$$

The one-block code $\pi_{++} : (Y_{++}, T_{++}) \to (Y, T)$ is defined by sending a Y_{++}-symbol U to the Y-symbol which is $\pi(i)$ for every i in U. The map π_{++} is clearly well-defined, surjective and right resolving.

Finally, we associate to π_{++} the shift-commuting Borel injection $\tau_{++} : Y \hookrightarrow Y_{++}$ defined by setting $(\tau_{++}y)_i = U_i(y)$. The map τ_{++} is a section to π ($\pi\tau_{++}$ is the identity map on Y).

Definition 3.2. The code π_{++} (and the SFT (Y_{++}, T_{++})) is called the *full futures cover* of π, and τ_{++} is called its canonical section.

The next result adapts arguments of Krieger [10].

Proposition 3.3. *Suppose $\pi : (X,S) \to (Y,T)$ and $\pi' : (X',S') \to (Y',T')$ are 1-block codes from 1-step SFTs onto sofic shifts. Let $\eta : (X,S) \to (X',S')$ and $\kappa : (Y,T) \to (Y',T')$ be topological conjugacies such that $\kappa\pi = \pi'\eta$. Let the full futures covers and sections for π and π' be as in Definition 3.2. Then there is a unique topological conjugacy $\widetilde{\kappa}$ such that the following diagram commutes:*

$$\begin{tikzcd}
Y_{++} \arrow[rrr, dashed, "\widetilde{\kappa}"] \arrow[dd, shift right, "\pi_{++}"'] & & & Y'_{++} \arrow[dd, shift left, "\pi'_{++}"] \\
& X \arrow[r, "\eta"] \arrow[dl, "\pi"] & X' \arrow[dr, "\pi'"'] & \\
Y \arrow[uu, shift right, "\tau_{++}"'] \arrow[rrr, "\kappa"'] & & & Y' \arrow[uu, shift left, "\tau'_{++}"]
\end{tikzcd}$$

Proof. The images of the sections τ_{++} and τ'_{++} are dense, and between these images $\widetilde{\kappa}$ is determined. Thus the uniqueness claim is clear. Let N be a positive integer such that the block codes η, κ, η^{-1} and κ^{-1} have range at most N.

Given $y \in Y$ and $y' = \kappa y \in Y'$, we claim that $U_0(y')$ is the set of symbols $(\eta x)_0$ such that $x \in X$ and x satisfies the following two conditions:

(1) $x_i \in U_i(y)$, $-3N \le i \le -N$,
(2) $x[-3N \cdots N]$ is an X-word which determines $(\kappa\pi x)[-2N, 0] = y'[-2N, 0]$.

To prove the claim, first suppose $x' \in X'$ and $(\pi' x')(-\infty, 0] = y'(-\infty, 0]$. Set $x = \eta^{-1}x'$. Then $(x')_0 = (\eta x)_0$, and $(\kappa\pi x)(-\infty, 0] = y'(-\infty, 0]$. This shows $x[-3N, N]$ satisfies (2). Because $y'(-\infty, 0]$ determines $(\kappa^{-1}y')(-\infty, -N]$, and $\kappa^{-1}y' = \pi x$, we have $x_i \in U_i(y)$ for $i \le -N$, and x satisfies (1).

Conversely, suppose $x \in X$ satisfying (1) and (2). Because $x_{-3N} \in U_{-3N}(y)$, there exists $\overline{x}$ in X such that $(\pi\overline{x})_i = y_i$ if $i \le -3N$, and $\overline{x}_{-3N} = x_{-3N}$. Because X is 1-step SFT, without loss of generality we choose x such that in addition to (1) and (2) we have $\overline{x}_i = x_i$, $i \le -3N$.

Now $(\kappa\pi x)(-\infty, -2N] = y'(-\infty, -2N]$ (because $(\pi x)_i = y_i$, $i \le -N$), and $(\kappa\pi x)[-2N, 0] = y'[-2N, 0]$ (by (2)). Since $(\kappa\pi x)(-\infty, 0] = y'(-\infty, 0]$, we have $(\eta x)_0 \in U_0(y')$. This completes the proof of the claim.

The claim shows that for any y in Y, the Y_{++}-word $(\tau y)[-3N, N]$ determines the Y'_{++} symbol $\tau y'[0]$. Because the image of τ is dense in Y_{++}, this provides a function K from Y_{++}-words of length $4N+1$ to $\mathcal{A}(Y'_{++})$. Define the block code $\widetilde{\kappa}$ on Y_{++} by setting $(\widetilde{\kappa}w)_0 = K(w[-3N, N])$. Clearly now $\widetilde{\kappa}\tau = \tau'\kappa$. It follows easily that $\widetilde{\kappa}$ maps Y_{++} onto Y'_{++}, and $\kappa\pi_{++} = \pi'_{++}\widetilde{\kappa}$.

In the same way, construct a block code $\widetilde{\kappa^{-1}}$ from Y'_{++} to Y_{++}. The compositions of $\widetilde{\kappa}$ and $\widetilde{\kappa^{-1}}$ give the identity map on a dense set, hence everywhere. Therefore $\widetilde{\kappa}$ is a conjugacy. $\square$

Given a block code $\pi : X \to Y$ from an SFT onto a sofic system, let φ be a conjugacy from X to a higher block presentation, $(\varphi x)_0 = x[-k \cdots k]$, where k is sufficiently large that the composition $\pi\varphi^{-1}$ is a 1-block map from a 1-step SFT, so its full futures cover $(\pi\varphi^{-1})_{++}$ has been defined. We also say that $(\pi\varphi^{-1})_{++}$ is *a full futures cover* of π, and (suppressing k) denote it as π_{++}. More generally, we make the following definition.

Definition 3.4. If p is any u-resolving map from an SFT onto Y and there exists a conjugacy φ such that $p = \pi\varphi$, where π is a full futures cover as defined above, then we say that p is *a full futures cover of* π.

By Proposition 3.3, if p and p' are two different full futures covers of π, then there is a conjugacy φ such that $p' = p\varphi$. Thus given π we have a distinguished class of u-resolving maps from SFTs onto Y.

Next we consider the special case which is our main interest.

Definition 3.5. Suppose π is a one block code from a 1-step irreducible SFT (X, S) onto a sofic shift (Y, T). Let (Y_+, T_+) be the unique maximum entropy irreducible component of the SFT (Y_{++}, T_{++}), and let π_+ be the restriction of π_{++} to this component. We call π_+ (or (Y_+, T_+)) the *futures cover of* π. The restriction of τ_{++} to the left transitive points of Y is denoted τ_+.

The next proposition shows the definition makes sense and lists some basic properties of the futures cover.

Proposition 3.6. *Suppose $\pi : (X, S) \to (Y, T)$ is a 1-block code from an irreducible 1-step SFT onto a sofic shift.*

1. *The full futures cover of π has a unique irreducible component of maximal entropy (denoted Y_+).*
2. *The futures cover π_+ is surjective right resolving with degree 1, and $\tau_+ : Y \hookrightarrow Y_+$.*
3. *Y_+ is a 1-step SFT and the set of allowed Y_+-words is*
$$\{U_0(y)U_1(y) \cdots U_n(y) : n \geq 0, y \textit{ is left transitive in } Y\}$$
4. *After removal of the section maps τ_{++} and τ'_{++}, Proposition 3.3 remains true if each $++$ is replaced by $+$.*

Proof. The claim (1) and the rest of (2) do not depend on the conjugacy class of π_{++}; so, on account of Proposition 3.3 and Lemma 2.4, to prove them we may assume that π has a magic symbol, μ. Let $\widetilde{\mu}$ denote $\pi^{-1}\{\mu\}$. If $\pi_{++}x = y$ and $y_n = \mu$, then x_n must equal $\widetilde{\mu}$. Because π_{++} is right resolving, for any n in $\mathbb{Z}$ the conditions $y = \pi_{++}x$ and $x_n = \widetilde{\mu}$ determine $x[n, \infty)$. Consequently, if y is left transitive in Y, then y has a unique preimage under π_{++}. Because π_{++} must map any irreducible component of Y_{++} of maximum entropy onto Y, it follows that there is only one such component. We let Y_+ denote this component and we let π_+ denote the restriction of π_{++} to Y_+.

The claim (2) is now clear.

The image of τ_+ is a dense subset of Y_+, and we have already seen that the set of all Y_{++}-words is
$$\{U_0(y)U_1(y) \cdots U_n(y) : n \geq 0, y \in Y\} \ ,$$
i.e., it is the set of words $(\tau_{++}y)[0, n]$, $y \in Y$. This proves (3).

The existence claim of (4) follows from (1) and Proposition 3.3. The uniqueness claim holds because $\widetilde{\kappa}$ is uniquely determined on a dense set, the image of τ_+. $\square$

As with the full futures cover, we make a definition which gives us a canonical conjugacy class.

Definition 3.7. Given a block code π from an irreducible SFT onto a sofic shift, we say that a map p is *a futures cover of* π if it is conjugate to the restriction of a full futures cover of π to its unique irreducible component of maximum entropy.

4. A resolving lift of a block code

To begin, let π be a one block code from an irreducible 1-step SFT (X, S) onto a sofic shift (Y, T). Let π_+ be the futures cover $Y_+ \to Y$ of Definition 3.5. We will apply a fiber product construction to produce (in the zero dimensional case) a generalization of Putnam's resolving lift.

The *fiber product* (a.k.a. *fibered product*, or *pull-back*) of $\pi : (X, S) \to (Y, T)$ and $\pi_+ : (Y_+, T_+) \to (Y, T)$ is the subsystem of $(X \times Y_+, S \times T_+)$ with domain $F = \{(x, w) : \pi(x) = \pi_+(w)\}$. The coordinate projections $p_1 : (x, w) \mapsto x$ and $p_2 : (x, w) \mapsto w$ give a commuting diagram of surjective maps

$$\begin{tikzcd} & F \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "\pi"'] & & Y_+ \arrow[dl, "\pi_+"] \\ & Y & \end{tikzcd}$$

We view F as a subshift, with alphabet

$$\begin{aligned} \mathcal{A}(F) &= \{(x_j, w_j) : (x, w) \in F\} \\ &\subset \{(i, U) \in \mathcal{A}(X) \times A(Y_+) : \pi(i) = \pi_+(U)\}\ . \end{aligned}$$

The subshift F is a 1-step SFT, because X and Y_+ are 1-step SFTs, and π and π_+ are one-block codes. Because π_+ is finite to one, so is p_1; so an irreducible component of F has maximum entropy (which is equal to that of X) if and only its image under p_1 is all of X. If there were two such components, then there would be a left transitive point x in X with two preimages in F, so the left transitive point πx would have two preimages under π_+ in Y_+, which is impossible. Thus F has a unique irreducible component of maximum entropy in F, which we denote F_0; and the restriction ρ_1 of p_1 to F_0 has degree one. Let ρ_2 similarly denote the restriction of p_2 to F_0. We have a commuting diagram of surjective maps

$$\begin{tikzcd} & F_0 \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ X \arrow[dr, "\pi"'] & & Y_+ \arrow[dl, "\pi_+"] \\ & Y & \end{tikzcd}$$

Because π_+ is right resolving, so is p_1, and therefore so is ρ_1.

Lemma 4.1. $\mathcal{A}(F_0) = \{(i, U) \in \mathcal{A}(X) \times \mathcal{A}(Y_+) : i \in U\}$.

Proof. First, suppose $(i, U) \in \mathcal{A}(F_0)$. Then (i, U) must occur as z_0 in some point z of F_0 which maps (via $\pi\rho_1 = \pi_+\rho_2$) to a left transitive point y in Y. Because $(\rho_1 z)_0 = i$ and $\pi(\rho_1 z) = y$, we have $i \in U_0(y)$. Because $(\rho_2 z)_0 = U$ and $\pi_+(\rho_2 z) = y$ and y has a unique preimage under π_+, which is the sequence $(U_m(y))_{m\in\mathbb{Z}}$, we have $U = U_0(y)$. Thus $i \in U$.

Next, suppose $U \in \mathcal{A}(Y_+)$ and $i \in U$. Then there exists a left transitive y in Y and an x in X such that $\pi(x_j) = y_j$ for $j \leq 0$, and $U_0(y) = U$ and $x_0 = i$. Thus $(x, y)_i = (x_i, U_i(y))$ for $i \leq 0$, and for some $n \leq 0$ the sequence $(x, y)(-\infty, n]$ is a sequence from F_0. By Lemma 2.3, there are no paths exiting from F_0, so $(i, U) \in \mathcal{A}(F_0)$. □

Proposition 4.2. *Let (X, S) be an irreducible 1-step SFT. Let $\pi : (X, S) \to (Y, T)$ be a surjective one-block code with a magic symbol. In the associated fiber product diagram*

$$\begin{tikzcd} & F_0 \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ X \arrow[dr, "\pi"'] & & Y_+ \arrow[dl, "\pi_+"] \\ & Y & \end{tikzcd}$$

all maps are surjective one-block; ρ_1 and π_+ are right resolving; and ρ_2 is left eresolving. If π is finite to one, then ρ_2 is left resolving.

Proof. The claims before "ρ_2 is left eresolving" are already established. Now suppose $(i_1, U_1) \in \mathcal{A}(F_0)$ and U_0U_1 is a word of length two in Y_+. To show ρ_2 is left eresolving, we must show there exists $(\overline{i_0}, \overline{U_0})$ in $\mathcal{A}(F_0)$ which can precede (i_1, U_1) and which maps to U_0. Because $(i_1, U_1) \in \mathcal{A}(F_0)$, we have $i_1 \in U_1$; because U_1 follows U_0, we may choose i_0 in U_0 such that i_1 follows i_0 in X. Because $i_0 \in U_0$, we have $(i_0, U_0) \in \mathcal{A}(F_0)$. Let $(\overline{i_0}, \overline{U_0}) = (i_0, U_0)$. We have shown ρ_2 is left eresolving.

The final claim follows from Proposition 2.1(5): a map between equal entropy irreducible SFTs is left resolving if and only if it is left eresolving. □

We can now assemble a presentation-invariant version of our zero dimensional generalization of Putnam's theorem.

Theorem 4.3. *Suppose $\pi : (X, S) \to (Y, T)$ is a block code from an irreducible SFT onto a sofic shift. Then there are surjective block codes α, β, γ from irreducible SFTs which give a commutative diagram*

$$\begin{tikzcd} {} \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & {} \arrow[d, "\beta"] \\ {} \arrow[r, "\pi"] & {} \end{tikzcd}$$

such that α and β are 1-1 a.e. u-resolving and γ is s-eresolving. If π is finite to one, then γ is also s-resolving.

Such a diagram will be produced if β is a version of the futures cover π_+ for π and α, γ are the restrictions to the unique maximal entropy component of the *projections from the fiber product of π and β. In this case, the constructed diagram is canonically associated to π: if $(\alpha, \beta, \gamma, \pi)$ and $(\alpha', \beta', \gamma', \pi')$ are two such diagrams and subshift conjugacies η, κ effect a conjugacy of π and π', then there will be a unique pair of subshift conjugacies $\widetilde{\eta}, \widetilde{\kappa}$ such that the following diagram commutes:*

$$\begin{tikzcd}
& & \cdot \arrow[r, "\gamma'"] \arrow[d, "\alpha'"] & \cdot \arrow[d, "\beta'"] \\
\cdot \arrow[r, "\gamma"] \arrow[d, "\alpha"] \arrow[urr, dashed, "\widetilde{\eta}"] & \cdot \arrow[d, "\beta"] \arrow[urr, dashed, "\widetilde{\kappa}"] & \cdot \arrow[r, "\pi'"] & \cdot \\
\cdot \arrow[r, "\pi"] \arrow[urr, "\eta"] & \cdot \arrow[urr, "\kappa"] & &
\end{tikzcd}$$

Proof. The conclusions of Theorem 4.3 are invariant under replacement of π by a topologically conjugate map. By Lemma 2.4, the map π is topologically conjugate to a map which satisfies the conditions of Proposition 4.2, and this proves the existence claims of Theorem 4.3. That the construction is canonical then follows from Proposition 3.6(4), because the fiber product construction is canonical for a given pair π, π_+. $\square$

Remark 4.4. A (mostly) weaker version of Putnam's result in zero dimension was stated some time ago in [9, before Theorem A]. Given π bounded to one between irreducible SFTs, [9] provides a factorization $\pi\alpha = \beta\widetilde{\pi}$, with all maps between irreducible SFTs, with $\widetilde{\pi}$ left resolving and β right resolving. However, here the map α need not be in any way resolving, and the maps α, β need not have degree one.

The overlooked resolving possibilities for α probably had to do, at least in part, with the nature of fiber products. In Proposition 4.2, if F_0 is replaced by the full fiber product $F(\pi, \pi_+)$, then the projection $F(\pi, \pi_+) \to Y_+$ will be u-resolving (s-resolving) if and only if π is already u-resolving (s-resolving). Thus it is surprising (at least, without the guidance of Putnam's theorem) that the restriction to F_0 has any resolving property independent of π, and this is perhaps part of the reason that Putnam's theorem had no zero dimensional precursor.

5. The canonical mapping property

In this section we will restrict attention to maps in the category $\mathcal{C}$ of surjective finite to one block codes from irreducible SFTs to sofic shifts. For c_1, c_2 in $\mathcal{C}$, the composition c_1c_2 is left/right/u/s -resolving if and only if both c_1 and c_2 are [5, Propositions 4.10 and 4.11].

Definition 5.1. Given $\pi \in \mathcal{C}$, we define $\mathcal{D}(\pi)$ to be the category of commutative diagrams $D = (a, b, c, \pi)$ of the form

$$\begin{tikzcd}
\cdot \arrow[r, "c"] \arrow[d, "a"'] & \cdot \arrow[d, "b"] \\
\cdot \arrow[r, "\pi"] & \cdot
\end{tikzcd}$$

which satisfy the following conditions:

(1) a, b and c are in $\mathcal{C}$;
(2) a and b are u-resolving;
(3) c is s-resolving.

Note that the domains of a, π and c are irreducible SFTs.

For $D = (a, b, c, \pi)$ and $D' = (a', b', c', \pi')$ in $\mathcal{D}(\pi)$, a morphism $D' \to D$ is a pair (φ, ψ) such that $\{\varphi, \psi\} \subset \mathcal{C}$, and the following diagram commutes (i.e., D' factors through D).

$$\begin{tikzcd} {} \arrow[r, "c'"] \arrow[d, "\varphi"'] \arrow[dd, bend right=60, "a'"'] & {} \arrow[d, "\psi"] \arrow[dd, bend left=60, "b'"] \\ {} \arrow[r, "c"] \arrow[d, "a"] & {} \arrow[d, "b"'] \\ {} \arrow[r, "\pi"] & {} \end{tikzcd}$$

We may write such a morphism as $(\varphi, \psi) : D' \to D$. Note that φ and ψ here are forced to be u-resolving. Diagrams D, D' are isomorphic in this category (they are "conjugate over π") if there are topological conjugacies φ, ψ such that $(\varphi, \psi) : D' \to D$. The *conjugacy class* of D is the set of all D' in $\mathcal{D}(\pi)$ which are conjugate over π to D. A diagram D is *minimal* in a subset $\mathcal{E}$ of $\mathcal{D}(\pi)$ if $D \in \mathcal{E}$ and $D' \to D$ for every D' in $\mathcal{E}$. For technical reasons we will be particularly concerned with the following subcategory of $\mathcal{D}(\pi)$.

Definition 5.2. $\mathcal{D}^{(1)}(\pi)$ is the full subcategory of $\mathcal{D}(\pi)$ whose objects are the diagrams $D = (a, b, c, \pi)$ in $\mathcal{D}(\pi)$ such that the SFT $F(b, \pi)$ (the fiber product of b and π) has a unique irreducible component of maximum entropy.

The condition of Definition (5.2) is satisfied whenever the degree of b is 1. A "full" subcategory of $\mathcal{D}(\pi)$ is simply one whose morphisms are all the $\mathcal{D}(\pi)$ morphisms between its objects. When a fiber product $F(b, \pi)$ contains a unique irreducible component of maximum entropy, we will denote that component as $F_0(b, \pi)$.

The next proposition shows that much of the structure of a diagram $D = (a, b, c, \pi)$ in $\mathcal{D}(\pi)$ is dictated by the pair (b, π).

Proposition 5.3. *Suppose D is the following diagram in $\mathcal{D}(\pi)$.*

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[d, "a"'] & W \arrow[d, "b"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

Then there is an irreducible component V' of maximum entropy in $F(\pi, b)$ and a constant to one map $\varphi : V \to V'$, such that the following diagram D'

$$\begin{tikzcd} V' \arrow[r, "p_2"] \arrow[d, "p_1"'] & W \arrow[d, "b"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is in $\mathcal{D}(\pi)$, and $(\varphi, Id) : D \to D'$. Here p_1 and p_2 denote the restrictions to V' of the usual projections $(x, w) \mapsto x$, $(x, w) \mapsto w$ from $F(\pi, b)$ to X and W.

Proof. For $v \in V$, define $\varphi(v) = (av, cv)$. Then φ maps V into $F(\pi, b)$. Because V is irreducible, the image of φ must be contained in an irreducible component V' of $F(\pi, b)$. Then $h(V) = h(\varphi V) \le h(F(\pi, b)) = h(X) = h(V)$, so V' has full entropy in $F(\pi, b)$ and $\pi V = V'$.

Because $p_1\varphi = a$ is u-resolving, so are p_1 and φ; because $p_2\varphi = c$ is s-resolving, so are p_2 and φ. The resolving properties for p_1 and p_2 show that $D' = (p_2, p_1, b, \pi)$ is in $\mathcal{D}(\pi)$. The map φ is constant to one because it is a u-resolving and s-resolving map between irreducible SFTs [8, Proposition 4.3.4]. Clearly $(\varphi, \mathrm{Id}) : D \to D'$. $\square$

Given $\pi \in \mathcal{C}$, let D_π be the element of $\mathcal{D}(\pi)$ constructed as for Proposition 4.2, i.e. D_π is the diagram

$$\begin{tikzcd} F_0 \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] & Y_+ \arrow[d, "\pi_+"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

where $F_0 = F_0(\pi, \pi_+)$, all maps are 1-block codes, and the three SFTs F_0, Y_+ and X are irreducible 1-step. We will characterize the conjugacy class of D_π with a mapping property (answering a question in [16]).

Theorem 5.4. *Suppose* $\pi \in \mathcal{C}$.

(1) *If* $D \in \mathcal{D}^{(1)}(\pi)$*, then there exists a unique morphism* $(\varphi_1, \varphi_2) : D \to D_\pi$.
(2) D *is minimal in* $\mathcal{D}^{(1)}(\pi)$ *if and only if* D *is conjugate over* π *to* D_π.

Question 5.5. Is 5.4(1) true with $\mathcal{D}(\pi)$ in place of $\mathcal{D}^{(1)}(\pi)$?

Proof. To prove (1), suppose we have $D \in \mathcal{D}(\pi)$ as below.

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[d, "a"'] & W \arrow[d, "b"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

Without loss of generality, we suppose the SFTs are 1-step and the maps in the diagram are 1-block codes.

By Proposition 5.3, without loss of generality we may assume $V = F_0(\pi, b)$, with $(a, c) = (p_1, p_2)$ where p_1, p_2 are the projection maps from $F_0(\pi, b)$ to X and W. Again without loss of generality, by Proposition 2.1(2), we may suppose that b is right resolving. Because a then arises from the fiber product construction, a is also right resolving. The map c is at this point still only s-resolving (i.e. left closing).

We will find $(\varphi, \psi) : D \to D_\pi$. We first give the rule for ψ. For w left transitive in W, define ψw by

$$(\psi w)_m = U_m(bw) \ , m \in \mathbb{Z} \ ,$$

where $U_m(y)$ was defined in (3.1). This is a shift-commuting rule and the left transitive points are dense; so, to show the rule determines a block code, it suffices to show the map $w \mapsto U_0(bw)$ is uniformly continuous on left transitive points w.

Because $c : V \to W$ is s-eresolving (by Proposition 2.1(5)), we may choose $N \geq 0$ such that if $v \in V$ and $(cv)_i = (w')_i$ for $i \geq -N$, then there exists $\widetilde{v}$ in V such that $c\widetilde{v} = w'$ and $\widetilde{v}_i = v_i$ for $i \geq 0$ [6, Proposition 5.1]. Now, suppose $\overline{w}$ and w are left transitive points in W, $\overline{w}[-N, 0] = w[-N, 0]$ and $i \in U_0(bw)$. We will show $i \in U_0(b\overline{w})$. For a point (biinfinite sequence) z, we will use the notations $z_- = z(-\infty, 0]$ and $z_+ = z[0, +\infty)$.

Because $i \in U_0(bw)$, we may choose x in X such that $x_0 = i$ and $(\pi x)_- = (bw)_-$. Because bw is left transitive and b is right resolving and $(bw)_- = (\pi x)_-$, by Lemma

2.5 there is a w' in W such that $(w')_- = w_-$ and $bw' = \pi x$. Therefore (x, w') is a point in the fiber product $F(\pi, b)$. Because w' is left transitive, the point (x, w') must be in the irreducible component of maximum entropy, i.e. $(x, w') \in V$. Define w'' in W by $w''(-\infty, 0] = \overline{w}(-\infty, 0]$ and $w''[-N, \infty) = w'[-N, \infty)$. Note, the overlapping definition is consistent. Now

$$(c(x, w'))[-N, \infty) = w''[-N, \infty) \ ,$$

so there exists $\widetilde{v} = (\widetilde{x}, \widetilde{w})$ such that $c\widetilde{v} = w''$ and $\widetilde{v}[0, \infty) = (x, w')[0, \infty)$. Therefore $(\pi\widetilde{x})(-\infty, 0] = (b\overline{w})(-\infty, 0]$, $\widetilde{x}_0 \in U_0(b\overline{w})$ and $\widetilde{x}_0 = x_0$. We have shown that our rule gives a block code $\psi : W \to Y_+$.

We summarize our situation with the following commuting diagram

$$\begin{tikzcd}
V \arrow[rrr, "c"] \arrow[dr, dotted] \arrow[dd, "a"'] & & & W \arrow[dl, "\psi"'] \arrow[dd, "b"] \\
& F_0 \arrow[r, "\rho_2"] \arrow[dl, "\rho_1"'] & Y_+ \arrow[dr, "\pi_+"] & \\
X \arrow[rrr, "\pi"'] & & & Y
\end{tikzcd}$$

in which F_0 denotes $F_0(\pi, \pi_+)$ and $V = F_0(\pi, b)$. The dotted arrow $V \to F_0$ is the map φ we need but have not yet defined. The maps a, ψc factor through π, π_+ and therefore can be factored through $F(\pi, \pi_+)$ by some map $\varphi : V \to F(\pi, \pi_+)$. The image of φ must be F_0; φ defines the dotted arrow. Now $(\varphi, \psi) : D \to D_\pi$. There can be only one morphism $(\varphi, \psi) : D \to D_\pi$ because it is uniquely determined on the dense set of transitive points, because π_+ and ρ_1 are one to one on transitive points. This proves (1).

On to (2). It is obvious that a diagram conjugate over π to a minimal element of $\mathcal{D}^{(1)}(\pi)$ (such as D_π) will also be a minimal element of $\mathcal{D}^{(1)}(\pi)$. Now suppose $D = (a, b, c, \pi)$ is a minimal element of $\mathcal{D}^{(1)}(\pi)$. We have $(\varphi_1, \psi_1) : D_\pi \to D$, which forces a and b to have degree 1. We also have $(\varphi_2, \psi_2) : D \to D_\pi$, so $(\varphi_1\varphi_2, \psi_1\psi_2) : D \to D$. By uniqueness of the morphism, $\varphi_2\varphi_1$ and $\psi_2\psi_1$ must be identity maps, as must $\varphi_1\varphi_2$ and $\psi_1\psi_2$. Thus D and D_π are conjugate over π. $\square$

Remark 5.6. Suppose $\pi \in \mathcal{C}$. The characterization of the conjugacy class of π_+ by a minimal mapping property is analogous to the characterization of the right Fisher cover of a sofic system by a minimal mapping property [4]. Similarly, the relation of the futures cover of π to the full futures cover is analogous to the relation of the Fisher cover to the Krieger cover [10].

6. The maps of Nasu and Putnam

Let $\pi : X \to Y$ be a block code from an irreducible SFT onto a sofic shift. Without loss of generality (after recoding if necessary), we take the code to be one-block and the SFT to be 1-step. In Proposition 4.2, we associated to π a certain diagram $D_\pi = (\rho_1, \rho_2, \pi, \pi_+)$. In this section we will explain the relationship of D_π to the construction of Putnam, and the relation of π_+ to a canonical cover defined by Nasu.

Putnam. Given a surjective 1-1 a.e. morphism π between irreducible Smale spaces, Putnam constructed a commuting diagram $\widetilde{D} = (\alpha, \widetilde{\pi}, \beta, \pi)$,

$$\begin{tikzcd} \overline{X} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\alpha"'] & \overline{Y} \arrow[d, "\beta"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

with all of the maps 1-1 a.e. surjective morphisms π between irreducible Smale spaces, with α and β u-resolving, and with $\widetilde{\pi}$ s-resolving.

Theorem 4.3 and Putnam's theorem overlap in the case that π is a surjective, 1-1 a.e. block code between irreducible SFTs. We claim in this case that the two diagrams D and $\widetilde{D}$ are conjugate over π. To show this, we first find a conjugacy of subshifts $\psi : Y_+ \to \overline{Y}$ such that $\pi_+ = \beta\psi$. Without loss of generality, we may presume π is a one block map and X is a 1-step SFT.

Putnam constructed $\overline{Y}$ as the completion in a certain metric δ_Y of the set $\pi(W)$ of points left asymptotic to the orbit of a chosen periodic point y_0, where y_0 has a unique preimage x_0 under π, and W is the set of points in X left asymptotic to the orbit of x_0. Our desired map ψ must agree with the natural injection ι from $(\pi_+)^{-1}(\pi(W))$ to $\pi(W)$. We claim that ι is uniformly continuous (with respect in the domain to the restriction of a metric giving the usual topology on Y_+, and with respect in the range to the metric δ_Y).

For brevity, we will not define all the terms and facts we use from the subtle construction of [16]. Let y and y' be in $\pi(W)$, with $z = \pi_+^{-1}(y)$ and $z' = \pi_+^{-1}(y')$ in Y_+. Now suppose that $N > 0$ and $z[-N, N] = z'[-N, N]$. By definition of Y_+, we have $y[-N, N] = y'[-N, N]$ and moreover there exists a bijection $\nu : \pi^{-1}(y) \to \pi^{-1}(y')$ such that $(\nu w)[-N, N] = w[-N, N]$, for all w in $\pi^{-1}(y)$. Define a point y'' in Y by $y''(-\infty, \infty) = y(-\infty, -N]y'[-N + 1, \infty)$. Then for large enough N, the path (y, y'', y') is a rectangular path from y to y'' ([16, Definition 2.13]), whose length is the sum $\mathrm{dist}(y, y'') + \mathrm{dist}(y'', y')$, which is at most 2^{-N+1}, and which is an upper bound for $\delta_Y^0(y, y')$ ([16, Definition 2.14]). Then for a number r with $0 < r < 1$ and the uniform constant $C = \max \delta_Y^0$, we get

$$\delta_Y(y, y') \leq \sum_{k=0}^{N} r^k 2^{-N+k} + \sum_{k=N+1}^{\infty} Cr^k \ .$$

This quantity goes to zero with N.

Thus ι is uniformly continuous, and extends to a continuous map $\psi : Y_+ \to \overline{Y}$. This map commutes with the shift on a dense set, hence everywhere, and is thus a block code. There is also by Theorem 5.4 a block code $\psi' : \overline{Y} \to Y_+$ such that $\beta = \pi_+\psi'$. Clearly ψ and ψ' are inverse to each other, so ψ is a conjugacy, and β is a version of π_+. Thus by Proposition 5.3, the map α is the map ρ_1 preceded by some constant to one factor map; because the degree of α is one, the constant to one map is a conjugacy of subshifts. This completes the proof.

Nasu.

In [12], given a one-block code $\pi : X \to Y$ from an edge SFT onto a sofic system, Masakazu Nasu associated to π a map he called the *induced right resolving cover* of π. (This cover appears as an important tool in [9].) This cover is a one-block code we will denote $\eta_+ : N_+ \to Y$. Here N_+ is another edge SFT.

The given map π is presented by a labeled graph $\mathcal{G}$, where the unlabeled graph defines the edge SFT and the label (by symbols from $\mathcal{A}(Y)$) defines the one-block code. Nasu defined his cover by constructing another labeled graph, $\mathcal{G}_+$. For a subset V of vertices of $\mathcal{G}$, and a Y-word x, let $S_+(V,x)$ denote the set of terminal vertices $S_+(V,x)$ of the paths labeled x that go from some vertex in V. A *right compatible set* of $\mathcal{G}$ is a set $S_+(V,x)$ where V is a singleton. Nasu defined the vertex set of $\mathcal{G}_+$ to be the set of all maximal right compatible sets and their nonempty sucessors. For each vertex S in $\mathcal{G}_+$, a unique edge labeled a goes from W to $S_+(W,a)$ if $S_+(W,a)$ is nonempty. This defines the edge set of $\mathcal{G}_+$, and thereby the edge SFT N_+ and the one block map $\eta_+ : N_+ \to Y$. Clearly η_+ is surjective and right resolving. Nasu showed [12, Theorem 3.1] that η_+ is canonically associated to π, in the sense that any conjugacy from π to another one-block code $\pi' : X' \to Y'$ (such a conjugacy by definition is given by conjugacies $X \to X'$, $Y \to Y'$ which intertwine π and π') lifts to an isomorphism of η_+ and η'_+:

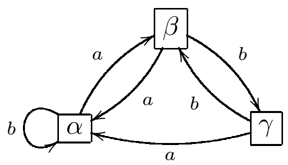
$$\begin{tikzcd}
N_+ \arrow[rrr, dotted] \arrow[dd, "\eta_+"'] & & & X'_+ \arrow[dd, "\eta'_+"] \\
& X \arrow[r] \arrow[dl, "\pi"'] & X' \arrow[dr, "\pi'"] & \\
Y \arrow[rrr] & & & Y'
\end{tikzcd}$$

Nasu's cover η_+ (like the futures cover π_+) is well defined even if the domain SFT is not irreducible or the map π is not finite to one. If X is irreducible, then both N_+ and Y_+ will be irreducible, and the degrees of π_+ and η_+ are well defined. Here the map π_+ will always have degree one, but η_+ might have greater degree. For example, if π is itself right resolving, then clearly (as noted by Nasu [12, p.569]) η_+ will equal π.

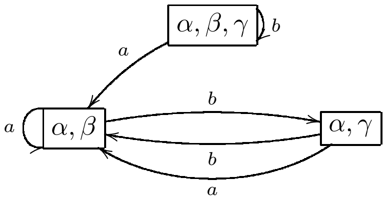
Now restrict to the case that X is irreducible and the degree of π is one. In this case, there is a conjugacy ψ between the domains of π_+ and η_+ such that $\eta_+ = \pi_+\psi$. To see this we simply note that in the degree one case, the symbols $U_0(y)$ in $\mathcal{A}(Y_+)$ are maximal compatible sets from the unique preimage symbol of a magic symbol for π. Thus the identity map embeds Y_+ into N_+, and because the irreducible SFT N_+ has no proper subsystem of equal entropy, this embedding is surjective. In particular, our diagram D_π, produced using $F_0(\pi,\pi_+)$ will be conjugate over π to the diagram produced in the same way using $F_0(\pi,\eta_+)$. Each of the diagrams is another version of Putnam's diagram in dimension zero.

It is naturally to ask whether Nasu's right resolving cover must factor through the futures cover. This is not always the case, as we see in the next example.

Example 6.1. Let π be the map given in [2, Example 1 on p. 491]. This map π can be defined by the following labeled graph $\mathcal{G}$.



Forgetting the labels in the graph gives an unlabeled graph, with adjacency matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, which defines an irreducible edge SFT (X,S). The edge labeling defines a 1-block code π from (X,S) to a shift (Y,T), which is the full 2-shift on symbols a,b. A labeled graph $\mathcal{G}_{++}$ defining a full futures cover (Y_{++},T_{++}) of π as an edge SFT, and the 1-block code π_{++}, can be computed as follows. Vertices of $\mathcal{G}_{++}$ are sets of vertices of $\mathcal{G}$. Begin with the vertex $v=\{\alpha,\beta,\gamma\}$. The set of $\mathcal{G}$-paths labeled a starting from v end in the set $\{\alpha,\beta\}$. Add this set as another vertex and draw an edge labeled a from v to v'. Continue in this fashion until reaching a graph such that no new vertices can be added. Reduce to the maximal subgraph such that each vertex has an incoming edge and an outgoing edge. This produces the following $\mathcal{G}_{++}$.



The futures cover (Y_+,T_+) is obtained by restricting this graph to the irreducible component of maximum entropy, which as an unlabeled graph has adjacency matrix $\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}$. Because π is right resolving, the associated right resolving cover η_+ of Nasu is equal to π. Consequently, in this example there is no factor map φ such that $\eta=\pi_+\varphi$, for the following reason. There is a fixed point x such that $\pi(x)$ is the fixed point b^∞, and φ could only map x to a fixed point y such that $\pi_+(y)=b^\infty$; but there is no such fixed point y in the domain of π_+.

7. Resolving degree and Markovian maps

In this section we point out that the resolving degree is an invariant of topology conjugacy for block codes from irreducible SFTs. (This is well known when the code is finite to one; then resolving degree equals degree.) We also give an application of Theorem 4.3 to Markovian codes.

Proposition 7.1. *Suppose π and π' are topologically conjugate maps from irreducible SFTs. Then they have the same resolving degree.*

Proof. Suppose that π is a one block code with a magic symbol μ. Let η_+ and π_+ be the Nasu and futures covers associated to π. Form the fiber product diagram

$$\begin{tikzcd} & F_0 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ N_+ \arrow[dr, "\eta_+"'] & & Y_+ \arrow[dl, "\pi_+"] \\ & Y & \end{tikzcd}$$

Here F_0 denotes the unique irreducible component of maximum entropy in the fiber product (it is unique because π_+ has degree 1). The projections p_1 and p_2 are surjective. Clearly $\{\mu\}$ is a magic symbol for the map p_2 and the cardinality of its preimage, which is the degree of p_2, is the cardinality of $\pi^{-1}(\mu)$. Because η_+ and

π_+ are canonically associated to π, so is the fiber product diagram and in particular the degree of p_2. Therefore this number will be the same for π and π'. □

A surjective block code between irreducible SFTs is *Markovian* if any (equivalently every) fully supported ergodic Markov measure on the range space lifts to a fully supported ergodic Markov measure on the domain [6, 15]. From the current paper, we find that any block code between irreducible SFTs is in a certain sense not far from Markovian.

Proposition 7.2. *Suppose $\varphi : (X, S) \to (Y, T)$ is a surjective block code between irreducible SFTs. Then there is a u-resolving degree 1 block code α from an irreducible SFT onto (X, S) such that $\varphi\alpha$ is Markovian.*

Proof. By Theorem 4.3, there is an α of the desired form such that $\varphi\alpha$ equals an s-eresolving code followed by a u-resolving code, both between irreducible SFTs. Such codes (and their compositions) are Markovian [6]. □

Question 7.3. It is an open problem since [6] to provide a decision procedure to determine whether a surjective block code between irreducible SFTs is Markovian. Do the constructs around the futures cover provide clues?

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Mike Boyle, Department of Mathematics, University of Maryland, College Park, MD 20742-4015, U.S.A.

E-mail address: mmb@math.umd.edu

URL: www.math.umd.edu/∼mmb