A dimension group for local homeomorphisms and endomorphisms of onesided shifts of finite type

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Abstract: We attach to a local homeomorphism S of a compact zero dimensional metrizable space X two abstract ordered groups, the images group $G(S)$ and the preimages group $\widetilde{G}(S)$. The images group is a dimension group with a distinguished order unit. We use these groups to study commuting local homeomorphisms. When S is a onesided mixing shift of finite type (SFT), $G(S)$ is isomorphic to the direct limit group of a defining matrix, and a onesided SFT commuting with S is determined up to eventual conjugacy (but not conjugacy) by its action on $G(S)$. We obtain complete and computable invariants for the eventual conjugacy classes of onesided SFTs, and show commuting onesided SFTs have a common measure of maximal entropy. This places severe constraints on the existence of commuting presentations of onesided SFTs. There are applications to cellular automata and links to Ruelle's thermodynamic formalism.

1. Introduction This paper stems from the work of Blanchard and Maass [BM] on onesided cellular automata. They consider a local homeomorphism F of a onesided full shift S on N symbols commuting with the shift (i.e., a onesided cellular automaton F which is a local homeomorphism). When F is expansive (in the onesided sense), they show there is a positive integer k such that

(i) k and N are divisible by the same primes, and

(ii) for large enough n, F^n is conjugate to a full shift on k^n symbols.

They also show F and S have the same unique measure of maximal entropy. Their ingenious proofs are combinatorial and intricate.

In this paper we generalize the Blanchard and Maass results to maps commuting with onesided shifts of finite type (SFTs), with proofs which are less intricate (if more abstract). To

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do this we introduce two ordered groups, the images group and the preimages group, which we define for an arbitrary local homeomorphism S of a compact zero dimensional metric space. Roughly speaking, the images group of S, $G(S)$, is defined from the clopen sets by identifying clopen sets C, D if for some positive integer n, the restrictions of S^n to C and to D are injective and have the same image (Definition 4.1). Roughly speaking, the preimages group $\widetilde{G}(S)$ is defined by identifying points whose inverse images eventually fill clopen sets in the same way (Definition 6.1). If F and S are commuting local homeomorphisms, then they induce commuting homomorphisms F^*, S^* from $G(S)$ to $G(S)$, and they induce commuting homomorphisms $\widetilde{F}$, $\widetilde{S}$ from $\widetilde{G}(S)$ to $\widetilde{G}(S)$. Here $\widetilde{F}$ is injective, F^* is surjective, and S^* is bijective.

Now suppose that S is a onesided mixing SFT and F is a (onesidedly) expansive map commuting with S. It was proved independently by Kurka [K] and Nasu [N2] that F is then itself a onesided SFT, and in fact F is mixing (Lemma 2.9). This is the setting for our main results. We show (Theorem 3.7) that F and S must have the same *eventual vertex set*, a topological invariant we introduce to index the eventual total amalgamation. We also show (Theorem 4.5, Theorem 6.7) that $G(S) = G(F)$ and $\widetilde{G}(S) = \widetilde{G}(F)$ (this is equality, not just isomorphism: the groups are determined by identical relations on the clopen sets and points). Moreover, the eventual conjugacy class of F is determined by F^* and also by $\widetilde{F}$ (Theorem 7.4). Note that the eventual conjugacy class of a onesided SFT F is a much finer invariant than the eventual conjugacy class (i.e., the shift equivalence class) of the twosided SFT which is its natural extension. We classify onesided SFTs up to eventual conjugacy, concretely by easily computed matrix relations (Theorem 8.2), and in the mixing case abstractly by the paired actions on the images and preimages groups (Theorem 8.3). These results give drastic and computable constraints on the eventual conjugacy classes of onesided SFTs which can commute with a given onesided mixing SFT: roughly speaking, the eventual total amalgamations must commute (Theorem 7.1) and must generate the same dimension group as a direct limit (Theorem 7.4). We remark that Nasu has already shown [N1; N2, Sec. 8] that there must be matrices A, B over $\mathbf{Z}^+$ such that for all nonnegative $(i,j) \neq (0,0)$, F^iS^j is conjugate to the SFT defined by the matrix A^iB^j. We neither use nor reprove this strong result.

We also prove that commuting mixing onesided SFTs must have the same measure of maximal entropy (Theorem 9.1). (We give three proofs, with dimension groups, the Ruelle operator and expansive subdynamics.) In contrast, Nasu has constructed an example of commuting twosided mixing SFTs [N1,Sec.10, Ex.1] which do not have the same measure of maximal entropy.

Our construction of the images group is very much in the tradition of Krieger's construction of a dimension group for a twosided SFT by certain identifications on unstable sets. Indeed,

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for a mixing SFT S, the images group $G(S)$ is isomorphic to the dimension group attached to the corresponding twosided SFT by Krieger (and the preimages group $\widetilde{G}(S)$ embeds as a subgroup of the dual of $G(S)$). The images and preimages groups were the natural constructions we found for establishing a dimension group framework for onesided SFTs; and it turns out that for any local homeomorphism of a compact zero dimensional metrizable space, the images group is a dimension group (Theorem 10.2). For a onesided mixing SFT, the images group can also be described in terms of a Ruelle operator (Remark 9.4), as pointed out to us by M.Rychlik.

The paper is organized to move directly to the main results in the SFT setting. After this, in section 10, we prove that the images group is always a dimension group. In Section 11, we compute the images group for a nontrivial example, the cellular automaton which presents multiplication by 2 in the 6-shift. Here the images group is a simple dimension group of infinite rank with a unique trace.

2. Definitions, background, basic facts Let X be a compact, metric zero-dimensional space. Let d denote a metric on X generating the topology. Let $S : X \to X$ be a continuous, onto map. We consider (X, S) as a dynamical system, not necessarily invertible. The map S is *open* if for any open set O the image SO is open. S is *locally injective* if for any point $x \in X$ there is some neighborhood O of x such that S restricted to O is injective. S is a local homeomorphism iff S is open and locally injective. A stronger notion than local injectivity is expansiveness.

Definition 2.1 S is *expansive* iff there is some $\delta > 0$ such that for $x, y \in X$ with $x \neq y$ there is some $n \geq 0$ such that $d(S^n x, S^n y) > \delta$.

In contrast to this, we say a homeomorphism S is twosided expansive if there is some $\delta > 0$ such that for $x, y \in X$ with $x \neq y$ there is some $n \in \mathbf{Z}$ such that $d(S^n x, S^n y) > \delta$. (This is usually called expansive. We adopt Definition 2.1 for the current paper because we are concerned primarily with noninjective maps.)

The expansive dynamical systems (X, S) are exactly those isomorphic to the *onesided subshifts*, that is, for some $N \geq 1$, X is a closed subset of $\{1,\dots, N\}^{\mathbf{N}}$ (with the product topology of the discrete topology on $\{1,\dots, N\}$), S is the left shift map and $SX = X$. Since there are no forward expansive homeomorphisms on non-finite sets, the shift map S is not injective, provided X is non-finite [Schwartzman, Thm 3.9 in BL].

A subclass of the onesided subshifts are the onesided shifts of finite type. A onesided subshift (X, S) with $X \subset \{1,\dots, N\}^{\mathbf{N}}$ is a shift of finite type (SFT) iff there is a finite set of blocks, say B such that $x \in X$ iff no subblock of x is an element in B. As in the twosided case, SFTs can be described by a square integer valued matrix with non negative entries.

The (onesided!) SFTs were completely classified by Williams [W], as follows. Let A be an $\alpha \times \alpha$ nonnegative integral matrix. If the ith and jth columns of A are equal, we can produce an $(\alpha\text{-}1) \times (\alpha\text{-}1)$ matrix A' by amalgamating these columns: add row i of A to row j, then erase row i and column i. Continue this process of amalgamating to produce smaller square matrices, until a matrix A_t (the "total amalgamation" of A) is produced which has all columns distinct. Williams proved that A_t (modulo conjugation by a permutation matrix) is independent of the particular sequence of column amalgamations, and that square nonnegative integral matrices define topologically conjugate onesided SFTs if and only if their total amalgamations are equal (modulo conjugation by a permutation matrix). For example,

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

define nonconjugate onesided SFTs, although they define conjugate twosided SFTs. It is sometimes useful to amalgamate all nondistinct columns of A in one operation. For this we define (as in [BFK]) the total one-step amalgamation A_1 of A, which is unique up to conjugation by a permutation matrix. Let R be a matrix whose columns are exactly the distinct columns of A. Then there is a unique 0-1 matrix S, with every column sum equal to 1, such that $A = RS$. Define $A_1 = SR$. Applying successive total one-step amalgamations will again produce the total amalgamation. For an example, consider

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} = RS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \ SR = A_1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ A_t = (3).$$

Standing convention: All subshifts are onesided unless explicitly indicated to be twosided. Maps are not assumed to be injective unless explicitly indicated to be homeomorphisms.

Theorem 2.2[P] Let (X, S) be a subshift. Then S is a SFT iff S is open.

Thus all subshifts which are not of finite type are examples where the map S is locally injective but not a local homeomorphism.

Definition 2.3 An *endomorphism* F of (X, S) is a continuous, onto map $F : X \to X$ such that $SF = FS$. We denote the set of endomorphisms of (X, S) by $\mathrm{end}(S)$.

Kurka's proof [K, Theorem 2] gives the following result (see also [BM, 3.2]).

Theorem 2.4 [K] Let S be an irreducible SFT and $F \in \text{end}(S)$. Then F is locally injective iff F is a local homeomorphism.

An immediate consequence of Theorems 2.2 and 2.4 is the following characterization, due independently to Kurka and Nasu, of endomorphisms of irreducible SFTs which are also SFT.

Theorem 2.5 [K, N2] Let S be an irreducible SFT and $F \in \text{end}(S)$. Then F is a SFT iff F is expansive.

Remarks 2.6 1) The multiplication by 2 in the 6-shift (see §9), gives a non-trivial example (i.e. not the identity map) for a local homeomorphism of a SFT which is not expansive. [BM]

2) Right-closing endomorphisms for onesided full shifts were studied by [BM]. An endomorphism of a subshift is right-closing iff it is locally injective.

3) An important problem, due to Nasu [N1, Question 2.1, p.33], is to determine whether a twosided expansive automorphism of a twosided irreducible SFT is always itself a twosided SFT. (A counterexample for the reducible case has been obtained by D.Fiebig)

4) Theorem 2.4 and Theorem 2.5 are false in the reducible case. For that consider the onesided reducible SFT (X, S), where $X = \{ x \in \{0, 1, 2\}^{\mathbf{N}} \mid x_i = 2 \Rightarrow x_{i+1} = 2 \}$ and define an endomorphism F by $F(x)_i = (x_{i+1} + x_{i+2}) \bmod 2$ if $x_{i+2} \in \{0, 1\}$ and $F(x)_i = x_{i+1}$ else. F is expansive, since the sequence $((F^i x)_1, (F^i x)_2)$ $i \geq 0$, determines x. F is not a onesided SFT, since the number of preimages is not locally constant: In the set $\{ x \in X \mid x_i = 0,\ 1 \leq i \leq n \}$ the point 0^∞ has the four preimages, namely 0^∞, 10^∞, 01^∞ and 1^∞ and the point 0^n2^∞ has only two preimages, namely 00^n2^∞ and 10^n2^∞. Thus F is not a local homeomorphism, though it is locally injective.

Example 2.7 for a non-open, and thus non locally injective, endomorphism of the onesided 3-shift. Let $S = \{0, 1, 2\}^{\mathbf{N}}$ and define F by $F(x)_i = (x_i + x_{i+1}) \bmod 2$ if $x_i, x_{i+1} \in \{0, 1\}$ and $F(x)_i = x_i$ else. The set $F(\{ x \in X \mid x_1 = 0 \})$ is not open, since its complement is not closed: the points 0^n12^∞ belong to its complement (the unique preimage of 0^n12^∞ is $1^{n+1}2^\infty$), but they converge to the point $0^\infty \in F(\{ x \in X \mid x_1 = 0 \})$. □

The following simple observation will be a helpful topological tool whenever we are dealing with local homeomorphisms.

Lemma 2.8 Let S be a local homeomorphism. Let $\mathcal{R}$ be any clopen partition of X. Then there is a clopen partition $\mathcal{P}$ of X which refines $\mathcal{R}$ such that S is injective on each atom of $\mathcal{P}$ and the S-images of two atoms are either disjoint or equal. Thus the distinct sets $SP, P \in \mathcal{P}$, form a partition of X into clopen sets. In particular, for every clopen set C of X there is a clopen set D such that S is injective on D and $SD = C$.

Proof. Replacing $\mathcal{R}$ by a refinement we may assume that S is injective on each atom of $\mathcal{R}$. Since S is open the images $SR, R \in \mathcal{R}$ are clopen. $\{ SR \mid R \in \mathcal{R} \}$ is a cover for X, since S is onto. Let $\mathcal{Q}$ be any clopen partition finer than the cover $\{ SR \mid R \in \mathcal{R} \}$. Then $\mathcal{P} = S^{-1}\mathcal{Q} \vee \mathcal{R}$ is a clopen partition finer than $\mathcal{R}$. Let $P = S^{-1}Q \cap R$ for some $Q \in \mathcal{Q}$, $R \in \mathcal{R}$. Since $\mathcal{Q}$ is finer than the cover $\{ SR \mid R \in \mathcal{R} \}$, we have that $SP = Q$.

Thus there is a subcollection of atoms $P_1,P_2,...,P_n$ of $\mathcal{P}$ such that their S-images are disjoint and cover X. For a clopen set C choose $D = S^{-1}C \cap \cup_i P_i$. This proves the lemma. □

Blanchard and Maass [BM] proved the following in the case that S is a full shift with combinatorial arguments.

Lemma 2.9 Let S be a mixing SFT and $F \in \mathrm{end}(S)$ an expansive endomorphism. Then F is a mixing SFT.

Proof F is a SFT by Theorem 2.5. Let H be a directed graph such that F is given as the shift on the onesided infinite paths of edges of H. We shall show that H has only one irreducible component. Then, since we may replace F by F^p for any $p \geq 1$, F has to be mixing. First we show that S induces a permutation π of the set of irreducible components of H. Consider an irreducible component A of H. Take a onesided infinite path x which stays in A and whose forward orbit is dense in A. Let B denote the irreducible component the path $S(x)$ eventually lies in. Since S is a block map, any path which is eventually in A is mapped under S to a path eventually in the same component as $S(x)$, i.e. in B. Thus $\pi(A) = B$ is welldefined. The map π is surjective, and thus a permutation, since given a component B choose an infinite path x in B, then a point in $S^{-1}(x)$ eventually belongs to some component, say A, thus $\pi(A) = B$. Replacing S by a suitable power we may assume that π is the identity. Let A be a minimal component of H (no outgoing edges) and fix an edge a of A. Let b be any other edge of H. Since S is mixing, there is $m > 0$ and a point x such that $x_0 = a$ and $(S^m x)_0 = b$. Since S^m is a blockmap, there is k such that $S^m\{z \mid z[0,k] = x[0,k] \}$ is contained in $\{ z \mid z_0 = b \}$. Since $x_0 = a \in A$ and A is minimal, we may choose z with $z[0,k] = x[0,k]$ and such that z is F-periodic. Thus $S^m z$ sees infinitely often b, now $\pi = \mathrm{id}$ implies that b is an edge in the component A. □

3. Eventual vertex set and total amalgamations. We introduce the "eventual vertex set" of a SFT S. In the full shift case this will lead to a quick abstract proof that expansive endomorphisms of a full shift are eventually conjugate to full shifts, which was shown by Blanchard and Maass with purely combinatorial arguments. The main ingredients are that the eventual vertex set is a natural index set for the total amalgamations of A^n for large enough n and that expansive endomorphisms F of a mixing SFT S share their eventual vertex set with S.

Definition 3.1 Let (X, S) be a SFT. Define an equivalence relation on X by $x \approx y$ (or $x \approx_S y$ if necessary to indicate the map) iff for all clopen sets C there is some non-negative integer N such that for all $n \geq N$ it holds that card $\{S^{-n}x \cap C\}$ = card $\{S^{-n}y \cap C\}$. The *eventual vertex set of* S, V(S), is the partition of X into $\approx$-equivalence classes. An *eventual vertex* is an atom of V(S).

Proposition 3.2 Let $g : X \to Y$ be a conjugacy of the SFTs (X,S) and (Y,F). Then g induces a bijection of the eventual vertex set of S and the eventual vertex set of F.

Proof. Let $x \approx_S y$ and fix a clopen set $D \subset Y$. Then $g^{-1}D \subset X$ is clopen and thus for all large enough n, we have card$\{S^{-n}x \cap g^{-1}D\}$ = card$\{S^{-n}y \cap g^{-1}D\}$. Since $S = g^{-1}Fg$ and g is injective, this implies card$\{F^{-n}gx \cap D\}$ = card$\{F^{-n}gy \cap D\}$ for all large enough n. Thus $g(x) \approx_F g(y)$. By symmetry, g induces a bijection $\pi : V(S) \to V(F)$ by $\pi(V) := g(V)$, $V \in V(S)$. □

We consider a SFT $S = S_A$, given by an $\alpha \times \alpha$ - matrix A with non-negative integer entries. Let W(k) denote the set of points starting in state k. We show that every eventual vertex is a union of certain W(k)'s.

Proposition 3.3 $x \approx_S y$ iff card $\{ S^{-\alpha}x \cap W(k) \}$ = card $\{ S^{-\alpha}y \cap W(k) \}$ for $1 \leq k \leq \alpha$. Thus, an eventual vertex consists of the union of those sets W(k) corresponding to equal columns in A^α and the number of eventual vertices is the number of distinct columns in A^α.

Proof. Since S is a SFT, the equivalence class of x is determined by card $\{ S^{-n}x \cap W(k) \}$, $1 \leq k \leq \alpha$, $n \geq 1$. The sets W(i) partition X, thus $x \in W(i)$ for a unique i. Let $v \in \mathbf{Z}^\alpha$ be the ith standard basis column vector. Then for $1 \leq k \leq \alpha$ and $n \geq \alpha$ we have card $\{ S^{-n}x \cap W(k) \} = (A^n)_{k,i} = (A^n v)_k = (A^{n-\alpha}A^\alpha v)_k$. Since $A^{n-\alpha}$ is injective on $A^\alpha\mathbf{Z}^\alpha$ this completes the proof. □

For a square matrix A, let $N(A)$ denote the least positive integer n such that rank $A^n =$ rank A^{n+1}.

Note, if A is $\alpha \times \alpha$, then $N(A) \le \alpha$. Also, $N(A)$ is the least positive integer n such that the columns of A^n are all in the eventual image of A. We shall see that the eventual vertex set can be used as an index set for the total amalgamation matrices for all A^n with $n \ge N(A)$.

Lemma 3.4 Suppose $n \ge N(A)$. Then the total amalgamation of A^n is the total 1-step amalgamation of A^n.

Proof. Let B be a total 1-step amalgamation of A^n, i.e. $A^n = RS$ and $SR = B$, where R is the $\alpha \times \beta$-matrix consisting of the β distinct columns of A^n, B is $\beta \times \beta$ and S is $\beta \times \alpha$ subdivision matrix. Assume B has a pair of equal columns. Then RB is $\alpha \times \beta$ and has strictly fewer than β distinct columns.We have $A^{2n} = RSRS = RBS$ with the subdivision matrix S, thus A^{2n} has strictly fewer than β distinct columns. Thus A^n is not injective on the space generated by the columns of A^n, contradicting the fact that A^n has all columns in the eventual image of A. $\square$

Definition 3.5 The *standard amalgamation* of A^n is the total 1-step amalgamation $C(n)$ of the form $A^n = RS$ and $SR = C(n)$ where R contains the distinct columns of A^n in the same order as in A^n (this determines S).

For $n \ge \alpha$ all columns of A^n are in the eventual image of A. Thus, by Lemma 3.4, the standard amalgamation $C(n)$ of A^n is a total amalgamation of A^n. Observe also that for all $k \ge 0$ the matrix A^kR contains the distinct columns of $A^{\alpha+k}$ in the same order as in $A^{\alpha+k}$. Thus $C(\alpha+k) = S(A^kR)$ is the standard amalgamation of $A^{\alpha+k}$ and the size of $C(\alpha+k)$ is the same as that of $C(\alpha)$. The following dynamical interpretation of the amalgamations $C(n)$ implies that the common size of all the $C(n)$, $n \ge \alpha$, is the number of eventual vertices of S.

Note that for a given presentation S_A of a SFT S, the natural ordering of the state space induces a linear ordering on the eventual vertices: define $V \le V`$ iff $\min \{k \mid W(k) \subset V\} \le \min \{k \mid W(k) \subset V`\}$ (in other words: the first column in A^α that “belongs” to V is to the left of the first one that belongs to $V`$).

Lemma 3.6 Let $V(S) = \{ V_1, \ldots, V_d \}$ be the eventual vertex set of the SFT $S = S_A$, indexed such that $V_i \le V_j$ whenever $i \le j$. Let $C(n)$ denote the standard amalgamation of A^n. Then for all $n \ge N(A)$ it holds that

$$C(n)_{i,k} = \text{card } \{ S^{-n}x \cap V_i \} \text{ for all } 1 \le i, k \le d \text{ and any } x \in V_k.$$

Proof. The size of $C(\alpha)$ is the number of distinct columns in A^α, which is d by Proposition 3.3. So let $1 \le i, k \le d$ and $x \in V_k$. Then x starts in state k', say. Thus,

$$\text{card}\,\{ S^{-n}x \cap V_i \} = \sum_{\{ 1 \le j \le \alpha \,|\, j'\text{th column of } A^\alpha = i'\text{th column of } A^\alpha \}} A^n_{j,k'} = C(n)_{i,k}. \qquad \square$$

Thus, the number of eventual vertices determines the size of the total amalgamation matrices of A^n for all large enough n. We show that an expansive endomorphism F, which is a mixing SFT by Lemma 2.9, shares its eventual vertex set with S, and thus eventually all total amalgamation matrices of F^n have the same size as those of S^n.

Theorem 3.7 Let S be a mixing SFT and F an expansive endomorphism of S. Then F is a mixing SFT with eventual vertex set $V(F) = V(S)$.

Proof Let $x \approx_F y$. Then there is some $i \in \mathbf{N}$ such that for all sets $W(k) = \{ x \in X \mid x$ starts in state k (w.r.t. S) $\}$ we have $\text{card}\,\{ F^{-i}x \cap W(k) \} = \text{card}\,\{ F^{-i}y \cap W(k) \}$. Thus, since S is a SFT, for each clopen set D we have $\text{card}\,\{ S^{-n}F^{-i}x \cap D \} = \text{card}\,\{ S^{-n}F^{-i}y \cap D \}$ for all large enough n. Fix a clopen set C. Since F is an expansive endomorphism of S and S is a SFT, F is a local homeomorphism by Theorem 2.4. Thus, by Lemma 2.8 there is a clopen set D such that F^i is injective on D and $F^iD = C$. Then for all large enough n we have that $\text{card}\,\{ S^{-n}x \cap C \} = \text{card}\,\{ S^{-n}F^{-i}x \cap D \} = \text{card}\,\{ S^{-n}F^{-i}y \cap D \} = \text{card}\,\{ S^{-n}y \cap C \}$, and thus $x \approx_S y$. Since F is also a mixing SFT, by Lemma 2.9, we get by symmetry that $x \approx_F y$ iff $x \approx_S y$, i.e. $V(S) = V(F)$. $\square$

The SFTs eventually conjugate to onesided full shifts are those whose eventual vertex set is a singleton. We now recover as an immediate corollary the result of Blanchard and Maass.

Corollary 3.8 [BM] An expansive endomorphism of a full shift is eventually conjugate to some full shift.

Example 5.6 shows that expansive endomorphisms of full shifts need not be conjugate to full shifts. Blanchard and Maass showed that there are restrictions on which full shifts occur here. In the next section we introduce the concept of the images group, which will give us a classification of the expansive endomorphisms of full shifts up to eventual conjugacy, and a good deal more.

4. The images group Let (X, S) be a dynamical system such that S is a local homeomorphism. We shall associate an ordered group to S, the images group $G(S)$. Any endomorphism F of S which is a local homeomorphism will induce a surjective homomorphism $F^*: G(S) \to G(S)$. In the SFT case, we identify the images group as the dimension group. Furthermore we exhibit a natural factoring from the images group of an endomorphism onto that of the underlying SFT. In § 11 we give a concrete example of an images group which is not one of a SFT.

We briefly review the definition of ordered groups. By an ordered group G we mean a countable abelian group G together with a subset G_+ such that $G_+ + G_+$ is contained in G_+, $G = G_+ + (- G_+)$ and $G_+ \cap (- G_+) = \{0\}$. We call G_+ the positive elements of G and use notation $b \geq a$ if $(b - a) \in G_+$. An ordered-group homomorphism is a group homomorphism $G \to H$ which sends G_+ into H_+.

We define the images group $G(S)$ for a surjective local homeomorphism $S : X \to X$ of a compact metric zero dimensional space X. Let $CO(X)$ denote the set of clopen (closed and open) sets in X. The abelian group with free generators the nonempty clopen sets is the set $\{ \Sigma n_iC_i \mid C_i$ clopen nonempty, n_i integers, only finitely many nonzero $\}$ with addition. We denote this group by $\langle CO(X) \rangle$. We define the following relations R on $\langle CO(X) \rangle$. Let C, D, $C_i \in CO(X)$.

(1) If C is the disjoint union of the sets C_i then $C \approx \Sigma C_i$.

(2) If S^n is injective on C, S^n is injective on D and $S^nC = S^nD$ then $C \approx D$.

Definition 4.1 Let $\langle R \rangle$ denote the subgroup of $\langle CO(X) \rangle$ generated by the relations R. The *images group* for S is the quotient group $G(S) = \langle CO(X) \rangle / \langle R \rangle$, i.e. the abelian group with free generators the nonempty clopen sets, with generating relations R. We denote the equivalence class of Σn_iC_i by $[\Sigma n_iC_i]$. $G(S)$ is an ordered group, where $[\Sigma n_iC_i] \geq 0$ by definition if the equivalence class $[\Sigma n_iC_i]$ contains an element with all coefficients non-negative. We denote the positive elements of $G(S)$ by $G(S)_+$.

Let R' be the set of relations defined by (1). Then $\langle CO(X) \rangle / \langle R' \rangle$ is isomorphic to the group $C(X,\mathbf{Z})$ of $\mathbf{Z}$-valued continuous functions on X. Since $\langle R' \rangle$ is a subgroup of $\langle R \rangle$, there is an onto group homomorphism $C(X,\mathbf{Z}) \to G(S)$.

We shall see in § 10 that $G(S)$ is a dimension group.

Now let $F \in \text{end}(S)$ be a local homeomorphism. We shall define an induced group homomorphism $F^* : G(S) \to G(S)$. First we define a group homomorphism $F' : \langle CO(X) \rangle \to G(S)$ as follows. For a clopen set C let C_i be a partition of C such that F is injective on each C_i. Since F is open, the images $F(C_i)$ are again clopen , thus we may define $F'(C) = [\Sigma_i F(C_i)]$. This is well defined : Let D_j be another partition of C such that F is injective on each D_j. Then, by (1), $\Sigma\ FC_i \approx \Sigma\ F(C_i \cap D_j) \approx \Sigma\ FD_j$, thus $[\Sigma_i F(C_i)] = [\Sigma_j F(D_j)]$. Now extend F' by linearity to a map $\langle CO(X)\rangle \to G(S)$.

Theorem 4.2 An onto ordered-group homomorphism $F^* : G(S) \to G(S)$ induced by F is defined by $F^*[\Sigma\, n_iC_i] = F'(\Sigma\, n_iC_i)$. If $F, U \in \mathrm{end}(S)$ are local homeomorphisms then $(FU)^* = F^*U^* : G(S) \to G(S)$. In particular $F^*S^* = S^*F^*$.

Proof. First we have to show that $\langle R\rangle \subset \ker F'$, where $\langle R\rangle$ is generated by the relations (1) and (2) above. Assume C is clopen and C_i a clopen partition of C, i.e. $C \approx \Sigma\, C_i$ by (1). Then $F'(C) = F'(\Sigma\, C_i)$, since we may use a partition of C into sets on which F is injective and which refines the clopen partition given by the C_i.

Now, let C and D be clopen sets such that S^n is injective on C and on D and $S^nC = S^nD$. Since S is open, the set $S^nC = S^nD$ is clopen. Partition S^nC into clopen sets $P_1,\dots P_k$ on which F is injective. Let $C_i := S^{-n}P_i \cap C$, $D_i := S^{-n}P_i \cap D$ for $1 \le i \le k$. Then FS^n is injective on each C_i and D_i. Since $FS^n = S^nF$, this implies that F is injective on each C_i and D_i, and that S^n is injective on FC_i and FD_i, which in turn, because of $S^nFC_i = FS^nC_i = FP_i = FS^nD_i = S^nFD_i$ implies by (2) that $[FC_i] = [FD_i]$ for all i. Thus $F'(C) = [\Sigma\, FC_i] = [\Sigma\, FD_i] = F'(D)$.

Clearly, F^* sends positive elements to positive elements.

Now we show that F^* is onto. It suffices to show that for any clopen set C there is a clopen set D with $F'(D) = [C]$. Choose a clopen set D such that F is injective on D and $F(D) = C$ (Lemma 2.8). Then $F'(D) = [F(D)] = [C]$ by definition.

Finally the relation $(FU)^* = F^*U^*$ follows from the definition of the induced actions, since for a clopen set C with small enough diameter we have $(FU)^*[C] = [FU(C)] = [F(UC)] = F^*[UC] = F^*U^*[C]$. □

Theorem 4.3 Let S be a local homeomorphism. Then $S^* : G(S) \to G(S)$ is a group isomorphism.

Proof. We define an inverse group homomorphism T^* for S^*. By Lemma 2.8 there is a clopen subset Y such that $SY = X$ and S is injective on Y. For $C \in CO(X)$ let $T'(C) = [S^{-1}C \cap Y]$. Extend T' by linearity to a group homomorphism $T' : \langle CO(X)\rangle \to G(S)$. We check that T' induces a homomorphism $T^* : G(S) \to G(S)$, i.e. that $\langle R\rangle \subset \ker T'$. If a clopen set C is a disjoint union of clopen sets C_i, then $T'(\Sigma C_i) = [\Sigma(S^{-1}C_i \cap Y)] = [S^{-1}C \cap Y] = T'(C)$. If C and D are clopen sets such that S^n is injective on C and on D and $S^nC = S^nD$ then $T'(C) = [S^{-1}C \cap Y] = [S^{-1}D \cap Y] = T'(D)$ since S^{n+1} is injective on $S^{-1}C \cap Y$ and $S^{-1}D \cap Y$ with common image S^nC.

Now let C be a clopen set such that S is injective on C. Then $T^*S^*[C] = T^*[SC] = [S^-$

${}^1SC \cap Y] = [C]$ since S is injective on C and $S^{-1}SC \cap Y$ with common image SC. Thus $T^*S^* = id$. But also $S^*T^*[C] = S^*[S^{-1}C \cap Y] = [S(S^{-1}C \cap Y)] = [C]$. Thus $S^*T^* = id$, i.e. T^* is the inverse of S^*. □

In contrast to Theorem 4.3, for an endomorphism F of S, which is a local homeomorphism, $F^* : G(S) \to G(S)$ is not injective in general, as we shall see in section 11.

Lemma 4.4 Let (X,S) and (Y,F) be local homeomorphisms and $g : X \to Y$ be a conjugacy of S and F. Then g induces a group isomorphism $g^* : G(S) \to G(F)$ such that $S^* = g^{*-1} F^* g^*$. If S is a SFT then also $g^* [V]_S = [\pi V]_F$ for all eventual vertices V, where $\pi : V(S) \to V(F)$ denotes the bijection between the eventual vertex sets induced by g.

Proof. If a clopen set $C \subset X$ is partitioned into clopen sets, say C_i, then $gC \subset Y$ is partitioned into clopen sets gC_i. If $C, D \subset X$ are clopen and S^n is injective on C and on D and $S^nC = S^nD$, then F^n is injective on gC and on gD and $F^ngC = F^ngD$, since g is a homeomorphism. Thus, we can define $g^* : G(S) \to G(F)$ by $g^*[\Sigma n_C C]_S := [\Sigma n_C g(C)]_F$. In the same way, $(g^{-1})^* : G(F) \to G(S)$ is a welldefined group homomorphism and its inverse is g^*. g^* conjugates S^* and F^*. If S is a SFT then $g^*[V]_S = [gV]_F = [\pi V]_F$. □

We restrict to the case that S is a SFT and F is a locally injective endomorphism of S. Thus, F is a local homeomorphism by Theorem 2.4 and F^* and S^* are both defined. We shall see that in this case the images group coincides with the dimension group.

Recall that for an integer valued $\alpha \times \alpha$ matrix A the dimension triple $(G(A), G(A)_+, A^*)$ consists of the dimension group $G(A)$, which is the direct limit group of the family $(\mathbf{Z}^\alpha, A)$ where A is considered as a map $\mathbf{Z}^\alpha \to \mathbf{Z}^\alpha$ acting on row vectors, the positive elements $G(A)_+$, which consists of the equivalence classes $[(v,k)] \in G(A)$ with $v \geq 0$, and the induced group isomorphism $A^*[(v,k)] = [(vA,k)]$ [LM].

Theorem 4.5 Let S be a onesided irreducible SFT given by an $\alpha \times \alpha$ matrix A with nonnegative integer entries. Let $F \in end(S)$ be a locally injective endomorphism of S.

a) $(G(S), G(S)_+, S^*)$ is isomorphic to the dimension triple of A. In particular, $G(S)$ is an abelian group with rank $G(S)$ = rank A^α.

b) $F^* : G(S) \to G(S)$ is a group isomorphism commuting with S^*.

c) $\langle R_F \rangle$ is a subgroup of $\langle R_S \rangle$ and thus the identity map $CO(X) \to CO(X)$ induces an onto ordered-group homomorphism $G(F) \to G(S)$.

d) If S is mixing and F is expansive then $G(S) = G(F)$.

Proof. a) Define a group homomorphism ϕ : <CO(X)> $\to$ direct limit $(\mathbf{Z}^\alpha, A)$ by mapping a cylinder set $\{ x \in X \mid x_1 = a_1, \ldots, x_k = a_k \}$ to the equivalence class of $(e_i, k) \in \mathbf{Z}^\alpha \times \mathbf{Z}$, where e_i is the i´th standard basis vector in $\mathbf{Z}^\alpha$ and i is the terminal vertex of the edge a_k. It is straightforward to verify that this induces a group isomorphism G(S) $\to$ direct limit $(\mathbf{Z}^\alpha, A)$ which intertwines the actions and preserves positivity of elements.

b) Since S is an irreducible SFT, F is a local homeomorphism, Theorem 2.4. By a), G(S) is a group of finite rank [BMT], thus, since F* is onto by Theorem 4.2, F* is also injective.

c) Let $C, D \in$ CO(X) such that F^n is injective on C and on D and such that $F^nC = F^nD$. Then $(F^n)^*[C]_S = [F^nC]_S = [F^nD]_S = (F^n)^*[D]_S$ and thus, b) implies $[C]_S = [D]_S$. This proves that $\langle R_F \rangle$ is a subgroup of $\langle R_S \rangle$, and thus c).

d) By Lemma 2.9, F is also a MSFT, so we get by symmetry from c) $\langle R_F \rangle = \langle R_S \rangle$ and thus G(F) = G(S). □

5. Expansive endomorphisms of full shifts. Blanchard and Maass proved with combinatorial arguments that an expansive endomorphism of a full N-shift is shift equivalent to some k-shift such that k and N are divisible by the same set of primes [BM, 3.13-14]. In fact, their proofs yield not only shift equivalence but eventual conjugacy at the onesided level. In this section, we recover their result by considering the induced action on the images group. More generally, we characterize the actions on the images group of the N-shift by locally injective endomorphisms.

Let X be compact, metric zero-dimensional and $S : X \to X$ a continuous, onto map. We say that S is transitive if for every nonempty open set U, $U = S^{-1}U$ implies that U is dense. (Equivalently, S has a dense forward orbit.) We say that S is *constant-to-1* if for some $n \in \mathbf{N}$ it holds that card $S^{-1}x = n$ for all $x \in X$. The following theorem generalizes the result of [BM] that expansive endomorphisms of full shifts are constant-to-1.

Theorem 5.1 Let S be constant-to-1 and transitive. If $F \in$ end(S) is a local homeomorphism then F is constant-to-1.

Proof. Let S be everywhere n-to-1 and let M be the maximum number of preimages of a point under F. Let $E = \{ x \in X \mid \mathrm{card} F^{-1}x = M \}$. E is clopen, since it is a union of sets FP, for a clopen partition *P* as in Lemma 2.8. Since S is everywhere n-to-1, card $S^{-1}F^{-1}x = Mn$ for every $x \in E$. Thus, by maximality of M and since SF = FS, for any $y \in X$ with $Sy = x$ it holds $\mathrm{card}F^{-1}y = M$. Thus, $S^{-1}E = E$. Since E is nonempty clopen, it follows that $E = X$. □

By [Theorem 2.4], a locally injective endomorphism of a SFT is a local homeomorphism, thus application of the theorem yields:

Corollary 5.2 Let S be a constant-to-1 transitive SFT and $F \in \text{end}(S)$ locally injective. Then F is constant-to-1. In particular, if S is a full shift and $F^n = S$ for some $n \geq 1$ then F is a constant-to-1 SFT.

In the remaining part of this section S denotes the full N-shift. Write $N = (p_1)^{r(1)}(p_2)^{r(2)}...(p_m)^{r(m)}$, where p_i are distinct primes and $r(i) \geq 1$. We first describe how a locally injective endomorphism of S acts on $G(S)$.

Theorem 5.3 a) Let F be a locally injective endomorphism of the full N-shift S. Then for some integer k which has at most the primes of N (i.e. $k = (p_1)^{s(1)}(p_2)^{s(2)}...(p_m)^{s(m)}$ by $s(i) \geq 0$), F is constant k-to-1 and F^* is multiplication with k on $G(S)$.

b) If F is expansive then k has all primes of N, i.e. $s(i) \geq 1$ for all i, and F is eventually conjugate to the full k-shift.

Proof. a) F is constant k-to-1 for some $k \geq 1$ by the Corollary 5.2.

Now split X into a clopen partition, say $\mathcal{P}$, where F is injective on each atom of P and the images of two distinct atoms under F are either disjoint or equal (Lemma 2.8). Thus for each atom P of $\mathcal{P}$ we have that the number of atoms in $F^{-1}(FP)$ is exactly k. Then $F^*([X]) = \sum_{P\in\mathcal{P}} [FP] = k[X]$. Thus F^* is multiplication by k on $G(S) = \mathbf{Z}[1/p_1...p_m]$ and thus (since F^* is an isomorphism) k has at most the primes of N.

b) Let F be an expansive endomorphism of the full N-shift S. By Corollary 3.8, F is eventually conjugate to the full k-shift S_k. Thus, by Theorem 4.5d, for large enough n, $G(F) = G(F^n) = G((S_k)^n) = G(S_k) = \mathbf{Z}[1/k]$. Since $G(S) = G(F)$ and $G(S) = \mathbf{Z}[1/N]$ this shows that k has the same primes as N. □

Remark 5.4 Another approach, to showing that the primes dividing k in part(a) also divide N, is to identify k with the multiplier R(F) defined for a block code F by Welch, and then apply [H, Thm. 14.9].

The previous and the following theorem together characterize the actions on $G(S)$ induced by locally injective endomorphisms of S and classify up to eventual conjugacy (as in [BM]) the expansive endomorphisms of S.

Theorem 5.5 a) For any positive integer k which has at most the primes of N, that is $k = (p_1)^{s(1)}(p_2)^{s(2)}...(p_m)^{s(m)}$ with $s(i) \geq 0$ for all i, there is a locally injective endomorphism F_k of S such that $(F_k)^*$ is multiplication by k on $G(S) = \mathbf{Z}[1/p_1...p_m]$.

b) If k has the same primes as N (i.e. $s(i) \geq 1$ for all i) then F_k can be choosen to be conjugate to the full k-shift.

Proof. Let $n = p_1p_2...p_m$. Let S_i denote the full p_i-shift and write the n-shift as $T = S_1 \times S_2 \times ... \times S_m$. For $1 \leq i \leq m$ let V_i denote the endomorphism of T which is the shift map in the i´th coordinate of T and the identity elsewhere. Then $V = (V_1)^{r(1)}\ldots(V_m)^{r(m)}$ is conjugate to the full N-shift S. Let $W = (V_1)^{s(1)}\ldots(V_m)^{s(m)}$, then $VW = WV$. Thus, W is a locally injective endomorphism of V and W is constant-k-to-1, this proves a) by Theorem 5.3a). If $s(i) \geq 1$ for all i, then W is conjugate to the full k-shift. This proves b). □

Example 5.6 Expansive endomorphisms of full shifts need not be full shifts (although they are eventually conjugate to full shifts, [BM] and also Corollary 3.8).

Let $A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$. All columns are distinct, so A is a total column amalgamation. Thus the MSFT F defined by A is not conjugate to a full shift. But $A^2 = \begin{pmatrix} 8 & 8 & 8 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}$ has total amalgamation (16), i.e. is the full 16-shift. Viewing F as an endomorphism of the full 16-shift F^2, gives the example. □

We have characterized the expansive endomorphisms of the full N-shift up to eventual conjugacy. We have seen that expansive endomorphisms of full shifts are eventually conjugate to some full shift, but need not be conjugate to a full shift. So very natural problems arise:

Problem 1 When is a MSFT S_B conjugate to an endomorphism of a full shift on N symbols ? We know B must be primitive with just one nonzero eigenvalue k, where k is divisible by the same primes as N and every column sum of B equals k. Is this sufficient?

Problem 2 More generally: Given two mixing SFTs. When can they be realized as a commuting pair ? (For significant constraints, see Theorem 7.1.)

6. The preimages group. So far we have used the eventual vertex set to see that expansive endomorphisms of full shifts are eventually conjugate to full shifts, and then used their action on the images group to determine their eventual conjugacy classes. For the general case of mixing SFTs we need, besides the images group, to introduce the preimages group to characterize eventual conjugacy of SFT in general, see section 8.

Let (X, S) be a dynamical system such that S is a local homeomorphism. We shall associate to S an ordered group, the preimages group $\widetilde{G}(S)$, and an induced group homomorphism $\widetilde{S}$. Endomorphisms of S which are local homeomorphisms induce actions on this group. In the SFT case the preimages group $\widetilde{G}(S)$ is isomorphic to the free abelian group generated by the columns of the total amalgamation matrix of A^{α}, where A is an $\alpha \times \alpha$ matrix decribing S. The induced homomorphism $\widetilde{S}$ is multiplication by A. From this it follows easily that the direct limit over $(\widetilde{G}(S), \widetilde{S})$ is the dual of the dimension group of A [BMT]. Furthermore we exhibit a natural factoring from the preimages group of an endomorphism onto that of the underlying SFT.

We shall now define the preimages group as a generalization of the idea of the eventual vertex set. We consider a compact, metric zero-dimensional space X and a surjective map $S : X \to X$, which is a local homeomorphism (we could give the definition for just locally injective S). Let $\langle X \rangle$ denote the free group with generating set X, that is the elements of $\langle X \rangle$ are formal sums $\Sigma\, n_x x$, where $n_x \in \mathbf{Z}$ and $n_x \neq 0$ only for finitely many $x \in X$. Let $\widetilde{R}$ denote the equivalence relation on $\langle X \rangle$ given by $\Sigma\, n_x x \approx \Sigma\, m_x x$ iff for any clopen set C there is some non-negative integer N such that for all $n \geq N$ it holds that $\Sigma\, n_x \cdot \mathrm{card}\,\{S^{-n}x \cap C\} = \Sigma\, m_x \cdot \mathrm{card}\,\{S^{-n}x \cap C\}$. To indicate the map we use $\approx_S$ instead of $\approx$.

Definition 6.1 The preimages group $\widetilde{G}(S)$ of S is $\langle X \rangle / \langle \widetilde{R} \rangle$. It is an ordered group where the positive elements are those equivalence classes which contain an element $\Sigma\, n_x x$ with coefficients $n_x \geq 0$ for all $x \in X$. The equivalence class of $\Sigma\, n_x x$ is denoted by $[\Sigma\, n_x x]$.

Let $F \in \mathrm{end}(S)$ be a local homeomorphism. Let C be a clopen set such that F is injective on C. Then $\mathrm{card}\,\{S^{-n}F^{-1}x \cap C\} = \mathrm{card}\,\{S^{-n}x \cap FC\}$ for any point x. Thus, since FC is clopen as well, $[\Sigma\, n_x x] = [\Sigma\, m_x x]$ implies $[\Sigma\, n_x \sum_{y \in F^{-1}x} y] = [\Sigma\, m_x \sum_{y \in F^{-1}x} y]$ and we can define an induced homomorphism $\widetilde{F}$ as follows.

Definition 6.2 Let $F \in \mathrm{end}(S)$ be a local homeomorphism. The induced ordered-group homomorphism $\widetilde{F} : \widetilde{G}(S) \to \widetilde{G}(S)$ is defined by $\widetilde{F}[\Sigma\, n_x x] = [\Sigma\, n_x \sum_{y \in F^{-1}x} y]$.

Lemma 6.3 Let (X,S) and (Y,F) be local homeomorphisms and let $g : X \to Y$ be a conjugacy of S and F. Then g induces a group isomorphism $\widetilde{g} : \widetilde{G}(S) \to \widetilde{G}(F)$ such that $\widetilde{S} = \widetilde{g}^{-1}\, \widetilde{F}\, \widetilde{g}$ and $\widetilde{g}[x]_S = [gx]_F$.

Proof. Define $\widetilde{g} : \widetilde{G}(S) \to \widetilde{G}(F)$ by $\widetilde{g}[\Sigma\, n_x x]_S := [\Sigma\, n_x g(x)]_F$. A similar argument as in Proposition 3.2 shows that $\widetilde{g}$ is a welldefined group isomorphism. It follows that $\widetilde{S} = \widetilde{g}^{-1}\widetilde{F}\widetilde{g}$. □

Lemma 6.4 $\widetilde{F} : \widetilde{G}(S) \to \widetilde{G}(S)$ is injective.

Proof. Consider two classes $[\Sigma\, n_x x] \neq [\Sigma\, m_x x]$. Then $\Sigma\, n_x x$ and $\Sigma\, m_x x$ are not equivalent, i.e. there is a clopen set C such that $\Sigma\, n_x \cdot \text{card}\,\{S^{-n}x \cap C\} \neq \Sigma\, m_x \cdot \text{card}\,\{S^{-n}x \cap C\}$ for infinitely many n. Since F is a local homeomorphism, by Lemma 2.8 there is a clopen set D such that F is injective on D and $FD = C$. Since $\text{card}\,\{S^{-n}F^{-1}x \cap D\} = \text{card}\,\{S^{-n}x \cap FD\} = \text{card}\,\{S^{-n}x \cap C\}$ for every point x this implies $\Sigma\, n_x \cdot \text{card}\,\{S^{-n}F^{-1}x \cap D\} \neq \Sigma\, m_x \cdot \text{card}\,\{S^{-n}F^{-1}x \cap D\}$ for infinitely many n. Thus $\widetilde{F}$ is injective. □

Remark 6.5 $\widetilde{S} : \widetilde{G}(S) \to \widetilde{G}(S)$ is not surjective in general: For the 2-shift the preimages group is isomorphic to the integers $\mathbf{Z}$ and $\widetilde{S}$ is multiplication by 2.

For later use we record that the action of an endomorphism on the images group determines the action on the preimages group.

Lemma 6.6 Let $F, U \in \text{end}(S)$ be local homeomorphisms with $F^* = U^* : G(S) \to G(S)$. Then $\widetilde{F} = \widetilde{U} : \widetilde{G}(S) \to \widetilde{G}(S)$.

Proof. It suffices to show $F^{-1}x \approx U^{-1}x$ for all $x \in X$. First we show that whenever $[\Sigma\, n_i C_i] = [\Sigma\, m_i D_i]$ in $G(S)$, then $\Sigma\, n_i\, \text{card}\,\{S^{-n}x \cap C_i\} = \Sigma\, m_i\, \text{card}\,\{S^{-n}x \cap D_i\}$ for all large enough n. We may assume that $\Sigma\, n_i C_i$ and $\Sigma\, m_i D_i$ differ by an elementary relation. If C is partitioned into clopen sets C_i then $\text{card}\,\{S^{-n}x \cap C\} = \Sigma_i\, \text{card}\,\{S^{-n}x \cap C_i\}$ and if C, D are clopen such that S^m is injective on C and on D and $S^mC = S^mD$ then for all $n \geq m$ it holds that $\text{card}\,\{S^{-n}x \cap C\} = \text{card}\,\{S^{-n+m}x \cap S^mC\} = \text{card}\,\{S^{-n+m}x \cap S^mD\} = \text{card}\,\{S^{-n}x \cap D\}$.

Now let C be a clopen set. Partition it into clopen sets C_i on which F and U are both injective. Then $[\Sigma FC_i] = F^*[C] = U^*[C] = [\Sigma UC_i]$. By the above, $\Sigma_i\, \text{card}\,\{S^{-n}x \cap FC_i\} = \Sigma_i\, \text{card}\,\{S^{-n}x \cap UC_i\}$ for all large enough n. Since $\text{card}\,\{S^{-n}F^{-1}x \cap C_i\} = \text{card}\,\{S^{-n}x \cap FC_i\}$, the first sum is $\text{card}\,\{S^{-n}F^{-1}x \cap C\}$, similarly the second is $\text{card}\,\{S^{-n}U^{-1}x \cap C\}$, thus $F^{-1}x \approx U^{-1}x$. □

We shall see now that for a SFT $S = S_A$ the preimages group $\widetilde{G}(S)$ is isomorphic to the $\mathbf{Z}$-span of the distinct columns of A^α and that $\widetilde{S}$ corresponds to the multiplication by A.

Theorem 6.7 Let S be an irreducible SFT given by an $\alpha \times \alpha$ matrix A with nonnegative integer entries. Let $F \in \text{end}(S)$ be a locally injective endomorphism of S.

a) $(\widetilde{G}(S), \widetilde{S})$ is isomorphic to $(A^\alpha \mathbf{Z}^\alpha, A)$. In particular, the direct limit of $(\widetilde{G}(S), \widetilde{S})$ is isomorphic to $(G(A^T), G(A^T)_+, (A^T)^*)$, which is the dual of the dimensiontriple of A.

b) $\widetilde{R}_F$ refines $\widetilde{R}_S$, and the identity map $X \to X$ induces an onto ordered-group homomorphism $\widetilde{G}(F) \to \widetilde{G}(S)$.

c) If S is mixing and F is expansive then $\widetilde{G}(S) = \widetilde{G}(F)$.

Proof. a) For each atom V of the eventual vertex set V(S) fix a point from this atom, say $x_V \in V$. Then every equivalence class contains an element of the form $\Sigma_{V\in V(S)}\, n_V\, x_V$. Let c_V denote the column of A^α corresponding to V. Then, the map $[\Sigma_{V\in V(S)}\, n_V\, x_V] \to \Sigma_{V\in V(S)}\, n_V\, c_V$ yields a group isomorphism from $\widetilde{G}(S)$ to $A^\alpha\mathbf{Z}^\alpha$ which interwines the actions $\widetilde{S}$ and A and preserves positivity.

b) Since S is a SFT, F is a local homeomorphism by Theorem 2.4 and thus $\widetilde{F}$ is defined. It suffices to show that $\Sigma\, n_x x \approx_F \Sigma\, m_x x$ implies $\Sigma\, n_x x \approx_S \Sigma\, m_x x$. Let $\Sigma\, n_x x \approx_F \Sigma\, m_x x$. Then, there is some $i \in \mathbf{N}$ such that for all sets $V \in V(S)$ it holds that $\Sigma\, n_x \operatorname{card}\{ F^{-i}x \cap V \} = \Sigma\, m_x \operatorname{card}\{ F^{-i}x \cap V \}$. Thus by definition of V(S), $\Sigma\, n_x F^{-i}x \approx_S \Sigma\, m_x F^{-i}x$, i.e. $\widetilde{F}^i[\Sigma\, n_x x]_S = \widetilde{F}^i[\Sigma\, m_x x]_S$. This shows $[\Sigma\, n_x x]_S = [\Sigma\, m_x x]_S$, since $\widetilde{F}$ is injective by Lemma 6.4.

c) This follows by symmetry from b), since an expansive endomorphism of a mixing SFT is a mixing SFT as well, by Lemma 2.9. □

7. Expansive endomorphisms of SFTs. We show that any two commuting mixing SFTs give rise to a commuting semigroup of matrices of equal rank. From this we prove an eventual conjugacy theorem for expansive endomorphisms of a mixing SFT, generalizing the result of the full shift case.

Theorem 7.1 Let S be a mixing SFT and F an expansive endomorphism of S. Let $\{ V_1, \ldots, V_d \} = V(S)$ be the eventual vertex set of S. Let $C(n)_{i,k} = \operatorname{card}\{ S^{-n}x \cap V_i \}$ and $D(n)_{i,k} = \operatorname{card}\{ F^{-n}x \cap V_i \}$ for all $1 \le i, k \le d$ and any $x \in V_k$.

Then there is some N such that $(C(n)D(m))_{i,k} = \operatorname{card}\{ S^{-n}F^{-m}x \cap V_i \}$ for all $1 \le i, k \le d$, $x \in V_k$, $n, m \ge N$. Furthermore, the matrices $C(n), D(n), n \ge N$ generate a commutative semigroup of matrices of equal rank. They all have common row image $\mathbf{Q}^d C(N)$ and common column image $C(N)\mathbf{Q}^d$.

Proof. F is a mixing SFT with $V(S) = V(F)$, by Theorem 3.7. By Lemma 3.6 we can choose N so large that $\operatorname{card}\{ F^{-n}x \cap V_i \}$ and $\operatorname{card}\{ S^{-n}x \cap V_i \}$ do not depend on $x \in V_k$, for all $1 \le i, k \le d$ and $n \ge N$. Let $m, n \ge N$ and write $C = C(n)$, $D = D(m)$. Fix for each $1 \le k \le d$ a point $x(k) \in V_k$. We compute

$$(CD)_{i,k} = \sum_{j=1}^{d} C_{i,j}\, D_{j,k} = \sum_{j=1}^{d} \text{card } \{ S^{-n}x(j) \cap V_i \}\cdot\text{card } \{ F^{-m}x(k) \cap V_j \}$$

$$= \sum_{j=1}^{d} \sum_{\{ y \in F^{-m}x(k) \cap V_j \}} \text{card } \{ S^{-n}y \cap V_i \} = \text{card } \{ S^{-n}F^{-m}x(k) \cap V_i \}.$$

Therefore $CD = DC$, since F and S commute.

Replacing now F by S in the above computation shows that $C(n)C(m) = \text{card } \{ S^{-n}S^{-m}x(k) \cap V_i \} = C(n+m) = C(m)C(n)$. Similarly $D(n)D(m) = D(n+m) = D(m)D(n)$.

Let $m, n \geq N$ and write $C = C(n), D = D(m)$. Then, S_C is conjugate to $(S_A)^n$ and thus, by Theorem 4.5a), rank C = rank G(S) and since G(S) = G(F), we get rank C = rank G(F) = rank D. It remains to show that rank CD = rank C. S^nF^m is an expansive endomorphism of S and thus, for large enough integers a, by Lemma 3.6, we can compute as above that $(C^aD^a)_{i,k} = \text{card } \{ S^{-na}F^{-ma}x(k) \cap V_i \}$. Since $G(S^nF^m) = G(S)$, we get that $\text{rank } (C^aD^a) = \text{rank } G(S^nF^m) = \text{rank } C$. Thus, $\text{rank } C = \text{rank } C^aD^a \leq \text{rank } CD \leq \text{rank } C$. Thus, rank C = rank CD. Similarly, rank CD = rank D. Now $C\mathbf{Q}^d \supset DC\mathbf{Q}^d = CD\mathbf{Q}^d \subset D\mathbf{Q}^d$ and the rank condition forces these containments to be equalities. Thus the rank and row space, and similarly the column space, is the same for C,D and CD. Because m and n can be varied independently, this finishes the proof. □

Let $S = S_A$ be a mixing SFT, where A is an $\alpha \times \alpha$ matrix. Let $V(S)$ denote the eventual vertex set of S and $C(n)$ the standard amalgamation of A^n, for $n \geq \alpha$. We shall see first that $C(n)$ describes how the elements $[V] \in G(S)$ are mapped under $(S^n)^*$, where V is an eventual vertex.

Lemma 7.2 For $n \geq \alpha$ it holds that $(S^n)^*[V] = [\Sigma_{V' \in V(S)} C(n)_{V,V'}\, V']$, $V \in V(S)$.

Proof. $C(n)_{V,V'} = \text{card}\{S^{-n}x \cap V\}$ for any $x \in V'$, thus S^n is constantly $C(n)_{V,V'}$-to-1 on the set $S^{-n}V' \cap V$. Thus applying Lemma 2.8 to S^n and the partition $R := \{S^{-n}V' \cap V , (S^{-n}V' \cap V)^c \}$ shows, $(S^n)^*[S^{-n}V' \cap V] = C(n)_{V,V'}[V' \cap S^nV] = [C(n)_{V,V'}V']$, since $S^nV = X$. The lemma follows from linearity of $(S^n)^*$. □

Lemma 7.3 For each $[\Sigma\, n_c\, C] \in G(S)$ there is some N such that for all $n \geq N$ it holds that $(S^n)^*[\Sigma\, n_c\, C] = [\Sigma_{V \in V(S)}\, m_V\, V]$ for suitable $m_v \in \mathbf{Z}$.

Proof. It suffices to prove the lemma for a cylinder set C of length, m say, and a given $n \geq m + \alpha$. Let k be the terminal state of C, that is $S^mC = W(k)$, where $W(k)$ is the set of points starting in state k. For each eventual vertex set V, choose a state i with $W(i) \subset V$, and define $m_V = (A^{n-m})_{k,i}$. By Proposition 3.3, because $n - m \geq \alpha$, the number m_V does not depend on the choice of $W(i)$ in V. Then $(S^n)^*[C] = (S^{n-m})^*[W(k)] = [\Sigma_i (A^{n-m})_{k,i} W(i)] = [\Sigma_{V\in V(S)} m_V \Sigma_{\{i \mid W(i) \subset V\}} W(i)] = [\Sigma_{V\in V(S)} m_V V]$. □

In general two mixing SFT F and U which have isomorphic dimension triples need not be eventually conjugate. For an example let F be the 3-shift and U be the SFT given by the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. But, if F and U act as expansive endomorphisms on a mixing SFT S and induce the same action on the dimension group of S, then they will be eventually conjugate as the following theorem shows.

Theorem 7.4 Let S be a mixing SFT and $F, U \in \mathrm{end}(S)$ expansive. Then $F^* = U^*$ iff $\widetilde{F} = \widetilde{U}$ iff the standard amalgamations of F and U eventually coincide. In this case, F and U are eventually conjugate.

Proof. By Lemma 2.9 F and U are mixing SFTs. Let A, B, B' be matrices such that $S \approx S_A$, $F \approx S_B$ and $U \approx S_{B'}$. Permuting the matrices B and B' if necessary we may assume that the presentations S_B and $S_{B'}$ not only share their eventual vertex set with S_A, but also the natural ordering of the eventual vertices (see Lemma 3.6.). We can fix N so large that the standard amalgamations $C(n), D(n), D'(n)$ of $A^n, B^n, (B')^n$, $n \geq N$, satisfy the conditions in Lemma 3.6. Let $\{ V_1, \ldots, V_d \} = V(S)$ be the eventual vertex set of S. For $1 \leq j \leq d$ fix a point $x(j) \in V_j$.

If $F^* = U^*$ on $G(S)$, then $\widetilde{F} = \widetilde{U}$ on $\widetilde{G}(S)$, by Lemma 6.6.

Now suppose $\widetilde{F} = \widetilde{U}$ on $\widetilde{G}(S)$. Let $n \geq N$. Then, $\widetilde{F}^n = \widetilde{U}^n$ and thus for all $1 \leq j \leq d$ $\widetilde{F}^n[x(j)] = \widetilde{U}^n[x(j)]$, which implies card $\{ S^{-m}F^{-n}x(j) \cap V_k \}$ = card $\{ S^{-m}U^{-n}x(j) \cap V_k \}$ for all $1 \leq k \leq d$ and all $m \geq N$. Fix such an m. Then by Theorem 7.1 $C(m)D(n) = C(m)D'(n)$ and $C(m)\mathbf{Q}^d = D(n)\mathbf{Q}^d = D'(n)\mathbf{Q}^d$. Thus the columns of $D(n)$ and of $D'(n)$ are in $C(m)\mathbf{Q}^d$. Since $C(m)$ is injective on $C(m)\mathbf{Q}^d$, this shows $D(n) = D'(n)$, for $n \geq N$. Thus F is eventually conjugate to U, since $D(n), D'(n)$ are total amalgamations of $B^n, (B')^n$.

Now suppose the standard amalgamations of F and U eventually coincide. Fix $n \geq N$ such that $D(n) = D'(n)$ and $D(n+1) = D'(n+1)$. Then, by Lemma 7.2, $(F^n)^*[V] = [\Sigma_{V'\in V(S)} D(n)_{V,V'} V'] = [\Sigma_{V'\in V(S)} D'(n)_{V,V'} V'] = (U^n)^*[V]$ and similarly $(F^{n+1})^*[V] = (U^{n+1})^*[V]$, $V \in V(S)$. Thus, since $(F^n)^* = (F^*)^n$ and F^* is bijective, we get $F^*[V] = U^*[V]$ for all $V \in V(S)$. Now let $[\Sigma n_c C] \in G(S)$, then there is some n such that $(S^n)^*[\Sigma n_c C] = [\Sigma_{V\in V(S)} m_V V]$ for suitable $m_v \in \mathbf{Z}$, by Lemma7.3. Thus, $F^*[\Sigma n_c C] = (S^*)^{-n}F^*(S^n)^*[\Sigma n_c C] = (S^*)^{-n}F^*[\Sigma_{V\in V(S)} m_V V] = (S^*)^{-n}U^*[\Sigma_{V\in V(S)} m_V V] = U^*[\Sigma n_c C]$, which shows $F^* = U^*$ on $G(S)$. □

Example 7.5 Two expansive endomorphisms on a mixing SFT which are conjugate, but do not have the same action on the images group.

Let S be the mixing SFT given by $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then $A^3 = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$. Let $\pi_{i,k}$ be the automorphism of S exchanging state i and state k, $1 \le i < k \le 3$. Let $F = \pi_{1,2} \circ S$ and $U = \pi_{2,3} \circ S$. Then $F, U \in \mathrm{end}(S)$ are both expansive. Then $U = \pi_{1,3} \circ F \circ \pi_{1,3}$ and since $(\pi_{1,3})^2 = \mathrm{id}$, F is conjugate to U. Let $W(k)$, $1 \le k \le 3$, denote the clopen set of points starting in state k. Since S, F and U are injective on $W(k)$, we get $F^*[W(1)] = [FW(1)] = [\pi_{1,2} \circ SW(1)] = [SW(2)] = S^*[W(2)]$ and $U^*[W(1)] = S^*[W(1)]$. Now, $[W(1)] \neq [W(2)]$ in $G(S)$ and thus, since S^* is injective $F^*[W(1)] \neq U^*[W(1)]$, showing $F^* \neq U^*$ on $G(S)$. □

Theorem 7.6 Let S be a mixing SFT and $F \in \mathrm{end}(S)$ expansive. Then the number of actions on $G(S)$ induced by expansive endomorphisms $U \in \mathrm{end}(S)$, which are eventually conjugate to F, is finite.

Proof. Let $D(n)$ be the standard amalgamation of F^n for large enough n. For each permutation P of the eventual vertex set $V(S)$, let $E_P = \{ U \in \mathrm{end}(S) \mid U \text{ is expansive and for all large enough } n \text{ the standard amalgamation of } U \text{ is } D'(n) = PD(n)P^{-1} \}$. By Theorem 8.2 in the next section, the union of the sets E_P is exactly the set of endomorphisms of S which are eventually conjugate to F. By Theorem 7.4 all elements in E_P have the same action on $G(S)$. This proves the theorem. □

Example 7.7 A mixing SFT with two expansive endomorphisms which have conjugate actions on the images group, are twosided conjugate, but not onesided eventually conjugate.

Let $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Then $A^2 = B^2 = \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$.

Subtract from columns 3 and 4 in matrix A the columns 1 and 2, afterwards add to rows 1 and 2 the rows 3 and 4. Then subtract from row 1 and 2 in matrix B the rows 3 and 4, afterwards add to columns 3 and 4 the columns 1 and 2. The resulting matrices are

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ respectively } \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

and are strong shift equivalent to A, respectively B. These matrices are the same up to a permutation. Thus A and B give twosided conjugate shifts, thus $(G(A),A^*)$ is conjugate to $(G(B),B^*)$. The conjugacy does not respect the eventual vertices. A and B are not onesided eventually conjugate, by Theorem 8.2 and the fact that A and B are invertible. Since $A^2 = B^2$, A and B give onesided endomorphism of the onesided MSFT given by matrix A^2 which have conjugate action on $G(A^2)$, but are not eventually onesided conjugate. □

8. Eventual conjugacy of SFTs. Eventual conjugacy has long been an important relation for twosided SFTs, since they are eventually conjugate if and only if their defining matrices are shift equivalent [LM]. In this section we give two characterizations of eventual conjugacy for onesided SFTs. The first (Theorem 8.2) is a straightforward matrix characterization. The second (Theorem 8.3), in the mixing case, abstractly by the paired actions on the images and preimages groups.

Recall that by a standard amalgamation of A^n we mean the total 1-step amalgamation $C(n)$ of the form $A^n = R_nS_n$ and $S_nR_n = C(n)$ where R_n contains the distinct columns of A^n in the same order as in A^n, and S_n is a (uniquely determined) zero-one matrix with every column sum 1. Let $N(A)$ denote the least positive integer n such that rank A^n = rank A^{n+1}. Recall there is a matrix S such that $S_n = S$ whenever $n \geq N(A)$.

Proposition 8.1 With the notation above, suppose $n, k \geq N(A)$ and $r = \text{rank } A^{N(A)}$. Then (1) $C(n)C(k) = C(n+k)$.

(2) rank $C(n) = r$.

(3) $C(n)$ and $C(k)$ have the same row and column spaces.

Proof. These facts can be extracted from the proof of Theorem 7.1, but we give self-contained elementary arguments now.

(1) $C(n)C(k) = SR_nSR_k = SA^nR_k = SR_{n+k} = C(n+k)$.

(2) rank $C(n)$ = rank $SR_n \leq$ rank R_n = rank $A^n = r$ =rank A^{2n} = rank R_nSR_nS = rank $R_nC(n)S \leq$ rank $C(n)$.

(3) Suppose $C(N(A))$ is $d \times d$ and consider the row spaces. Since $C(n+k) = C(n)C(k) = C(k)C(n)$, we have $\mathbf{Q}^dC(n) \supset \mathbf{Q}^dC(n+k) \subset \mathbf{Q}^dC(k)$, and equality follows from equality of rank. □

Theorem 8.2 Let A, B be nonnegative square integer matrices and fix any positive integer n such that $n \geq M := \max\{ N(A), N(B) \}$. Let $C(k)$, $D(k)$ denote the standard amalgamations of A^k, B^k. Then the following are equivalent.

(1) A and B define eventually conjugate SFTs.

(2) There is a permutation matrix P such that $C(k) = PD(k)P^{-1}$, for $k = n$ and $k = n+1$.

(3) There is a permutation matrix P such that $C(k) = PD(k)P^{-1}$, for $k \geq M$.

Proof. Clearly (3) implies (1) and (2). Now suppose (1) holds. Then there is an $n_0 \geq n$ such that for all $k \geq n_0$ there is a permutation matrix $P(k)$ such that $C(k) = P(k)D(k)P(k)^{-1}$. Now consider the set W of permutation matrices P such that $C(k) = PD(k)P^{-1}$ for some $k \geq n_0$. For each element $P \in W$ fix a $k_P \geq n_0$ such that $C(k_P) = PD(k_P)P^{-1}$. Then let N be the product of all k_P, $P \in W$. Since $N+1 \geq n_0$, there is a $P \in W$ such that $C(N+1) = PD(N+1)P^{-1}$. Since N is a multiple of k_P, Proposition 8.1 (1) implies $C(N) = PD(N)P^{-1}$. Thus from either (1) or (2), we have some $n \geq M$ satisfying (2), and it remains to show this implies (3). Suppose $k \geq M$. Applying Proposition 8.1(1), we see that $C(j)=PD(j)P^{-1}$ whenever j is a nonnegative integral combination of n and $n+1$, in particular for all large enough j. Choose i positive such that $C(ik) = PD(ik)P^{-1}$ and $C(i(k+1))=PD(i(k+1)P^{-1}$. Now $C(k)^i = PD(k)^iP^{-1}$, so the rank $C(k) = \text{rank } D(k)$, by Proposition 8.1(2). Also, $C(k)^iPD(k)P^{-1} = PD(k)^iP^{-1}PD(k)P^{-1} = PD(k)^{i+1}P^{-1} = PD(k)P^{-1}C(k)^i$, so the row space of $C(k)$ must equal the row space of $PD(k)P^{-1}$. Now $C(k)C(k)^i = C(k)^{i+1} = PD(k)^{i+1}P^{-1} = PD(k)P^{-1}PD(k)^iP^{-1}= PD(k)P^{-1}C(k)^i$, and thus $C(k) = PD(k)^iP^{-1}$, since $C(k)$ is injective on its row space. □

Note that the condition (2) in Theorem 8.2 gives an extremely easy test for determining eventual conjugacy for given matrices A and B. Also, if $N(A) = N(B) = 1$, then eventual conjugacy for A and B implies conjugacy.

Now we turn to the abstract characterization of eventual conjugacy in the mixing case.

Theorem 8.3 Let (X,S) and (Y,F) be mixing SFTs with eventual vertex sets $V(S)$ resp. $V(F)$. Then S is eventually conjugate to F iff there is

a) a bijection $\pi : V(S) \to V(F)$

b) a group isomorphism $\tau : G(S) \to G(F)$ such that $S^* = \tau^{-1} F^* \tau$ and $\tau [V]_S = [\pi V]_F$

c) a group isomorphism $\tilde{\tau} : \tilde{G}(S) \to \tilde{G}(F)$ such that $\tilde{S} = \tilde{\tau}^{-1} \tilde{F} \tilde{\tau}$ and $\tilde{\tau} [x]_S = [y]_F$, where $x \in V$, $y \in \pi(V)$.

Proof. First let S and F be eventually conjugate. Then there is some N, such that S^n and F^n are conjugate, for all $n \geq N$. Fix for each $n \geq N$, a conjugacy $g_n : X \to Y$ with $S^n = g_n^{-1}F^ng_n$. Since $V(S) = V(S^n)$ and $V(F) = V(F^n)$, by Proposition 3.2 g_n induces a bijection $\pi_n : V(S) \to V(F)$, by $\pi_n(V) = g_n(V)$, for $V \in V(S)$. Let $P := \{ \pi : V(S) \to V(F) \mid \text{there is some } n \geq N \text{ such that } \pi_n = \pi \}$. For each $\pi \in P$ fix $k(\pi) \geq N$ such that $\pi_{k(\pi)} = \pi$.

Let $n := \prod_{\pi \in P} k(\pi)$ and let $\pi := \pi_{n+1}$. Since n is a multiple of $k(\pi)$, say $n = mk(\pi)$, we have that $S^n = (S^{k(\pi)})^m = ((g_{k(\pi)})^{-1} F^{k(\pi)}g_{k(\pi)})^m = (g_{k(\pi)})^{-1} F^{k(\pi)m}g_{k(\pi)} = (g_{k(\pi)})^{-1} F^ng_{k(\pi)}$.

Thus, we have shown that there is some $n \geq N$, a bijection $\pi : V(S) \to V(F)$, a conjugacy, say g, of S^n and F^n inducing $\pi : V(S) \to V(F)$ and a conjugacy, say f, of S^{n+1} and F^{n+1} inducing also π. Since $\widetilde{G}(S^n) = \widetilde{G}(S)$ and $\widetilde{G}(F^n) = \widetilde{G}(F)$, by Lemma 6.3 g induces a conjugacy $\widetilde{g}$ of $(\widetilde{G}(S),\widetilde{S}^n)$ and $(\widetilde{G}(F),\widetilde{F}^n)$ and f induces a conjugacy $\tilde{f}$ of $(\widetilde{G}(S),\widetilde{S}^{n+1})$ and $(\widetilde{G}(F),\widetilde{F}^{n+1})$. Now $\widetilde{g}[x]_S = [gx]_F = [fx]_F = \tilde{f}[x]_S$, since $gx, fx \in \pi V$, if $x \in V$. Thus, by linearity $\widetilde{g} = \tilde{f}$. Thus, we have $\widetilde{S}^{n+1} = \tilde{f}^{-1}\,\widetilde{F}^{n+1}\,\tilde{f} = \widetilde{g}^{-1}\,\widetilde{F}^n\,\widetilde{g}\widetilde{g}^{-1}\,\widetilde{F}\,\widetilde{g} = \widetilde{S}^n\widetilde{g}^{-1}\,\widetilde{F}\,\widetilde{g}$, and thus, since $\widetilde{S}^n$ is injective, we get that $\widetilde{S} = \widetilde{g}^{-1}\,\widetilde{F}\,\widetilde{g}$. Let $\tilde{\tau} := \widetilde{g}$, this proves a), c).

By Lemma 4.4 the conjugacy g of S^n and F^n induces a conjugacy g^* of $(G(S),(S^n)^*)$ and $(G(F),(F^n)^*)$ with $g^*[V]_S = [\pi V]_F$ and the conjugacy f of S^{n+1} and F^{n+1} induces a conjugacy f^* of $(G(S),(S^{n+1})^*)$ and $(G(F),(F^{n+1})^*)$ with $f^*[V]_S = [\pi V]_F$ for any eventual vertex V of S. Let $[\Sigma\, n_c\, C] \in G(S)$. By Lemma 7.3 applied to $S^{n(n+1)}$, there is some m such that $(S^{n(n+1)m})^*[\Sigma\, n_c\, C] = [\Sigma_{V\in V(S)}\, m_V\, V]$ for suitable $m_v \in \mathbf{Z}$. Let $p = n(n+1)m$. Then $g^*[\Sigma\, n_c\, C] = ((F^p)^*)^{-1}g^*(S^p)^*[\Sigma\, n_c\, C] = ((F^p)^*)^{-1}g^*[\Sigma_{V\in V(S)}\, m_V\, V] = ((F^p)^*)^{-1}[\Sigma_{V\in V(S)}\, m_V\, \pi V] = ((F^p)^*)^{-1}f^*[\Sigma_{V\in V(S)}\, m_V\, V] = f^*[\Sigma\, n_c\, C]$, and thus $g^* = f^*$. Let $\tau := g^*$. Then, since S^* and F^* are both bijective, τ conjugates S^* and F^*, thus b) is proved.

Thus, we have shown, if S and F are eventually conjugate then a)-c) holds. Now assume that a)-c) holds. Let $C(n)$ denote the standard amalgamation of S^n, i.e. for all n large enough card $\{S^{-n}x \cap V\} = C(n)_{V,W}$ for $V, W \in V(S)$ and $x \in W$. Similarly, let $D(n)$ denote the standard amalgamation of F^n. Then for large n, $(S^n)^*[V] = [\Sigma_{V'\in V(S)}\, C(n)_{V,V'}\, V']$, $V \in V(S)$, by Lemma 7.2. And $\tau^{-1}(F^n)^*\,\tau[V] = \tau^{-1}(F^n)^*[\pi V] = \tau^{-1}[\Sigma_{W\in V(F)}\, D(n)_{\pi V,W}\, W] = [\Sigma_{W\in V(F)}\, D(n)_{\pi V,W}\, \pi^{-1}W] = [\Sigma_{V'\in V(S)}\, (PD(n)P^{-1})_{V,V'}\, V']$, where P is the permutation matrix of size card $V(S)$, which has $P_{V,W} = 1$ iff $\pi V = W$. Thus $S^* = \tau^{-1}F^*\tau$ implies $[\Sigma_{V'\in V(S)}\, C(n)_{V,V'}\, V'] = [\Sigma_{V'\in V(S)}\, (PD(n)P^{-1})_{V,V'}\, V']$, and since S is a SFT this implies $C(n)C(m) = PD(n)P^{-1}C(m)$ for all large enough m.

Now fix for each eventual vertex set $V \in V(S)$ a point $x_V \in V$ and similarly a point $y_W \in W$, for each $W \in V(F)$. Now we have $[S^{-n}x_V] = \widetilde{S}^n[x_V] = \tilde{\tau}^{-1}\,\widetilde{F}^n\,\tilde{\tau}[x_V] = \tilde{\tau}^{-1}\,\widetilde{F}^n[y_{\pi V}] = \tilde{\tau}^{-1}[F^{-n}y_{\pi V}] = [\Sigma_{V'\in V(S)}\, x_{V'}\,\text{card}\,\{F^{-n}y_{\pi V} \cap \pi V'\}]$. This implies that for large enough m and all sets $W \in V(S)$ we have card $\{S^{-m-n}x_V \cap W\} = \Sigma_{V'\in V(S)}\text{card}\,\{S^{-m}x_{V'} \cap W\}\text{card}\,\{F^{-n}y_{\pi V} \cap \pi V'\} = \Sigma_{V'\in V(S)}C(m)_{W,V'}D(n)_{\pi V',\pi V} = \Sigma_{V'\in V(S)}C(m)_{W,V'}(PD(n)P^{-1})_{V',V}$ and thus, $C(m)C(n) = C(m)PD(n)P^{-1}$. Since $C(m)C(n) = C(n)C(m)$, by Theorem 7.1, we get thus, $PD(n)P^{-1}C(m) = C(m)PD(n)P^{-1}$. Now the rank of $D(n)$ is that of the group $G(F)$, and thus that of $G(S)$ and thus $\text{rank}D(n) = \text{rank}C(m)$. Therefore $\text{rank}PD(n)P^{-1} = \text{rank}C(m)$. And also $\text{rank}PD(n)P^{-1}C(m) = \text{rank}\; C(n)C(m) = \text{rank}C(n+m) = \text{rank}C(m)$. Thus, row image of $PD(n)P^{-1}$ = row image of $C(m)$. The matrix $C(m)$ is injective on its row image, and thus $C(n)C(m) = PD(n)P^{-1}C(m)$ implies $C(n) = PD(n)P^{-1}$. This holds for all large enough n, and thus S is eventually conjugate to F. □

Example 8.4 In Theorem 8.3 it is essential to have the same bijection π in b) and c). It actually fails if we allow different bijections: Let $A = \begin{pmatrix} 16 & 8 \\ 8 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 10 & 5 \\ 20 & 10 \end{pmatrix}$. A and B are not eventually conjugate, since $A^n = 20^{n-1}A$ is not conjugate to $B^n = 20^{n-1}B$ [W]. But, one checks that the multiplication by 5/4 is an isomorphism from $G(A) = (2,1)\mathbf{Z}[1/20]$ to $G(B) = (2,1)\mathbf{Z}[1/20]$. It sends (16,8) to (20,10), thus $[V_1(A)]_A$ is sent to $[V_2(B)]_B$ and also $[V_2(A)]_A$ to $[V_1(B)]_B$. Mapping $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ to $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ induces an isomorphisms of the preimages groups $\begin{pmatrix} 8 \\ 4 \end{pmatrix}\mathbf{Z}$ and $\begin{pmatrix} 5 \\ 10 \end{pmatrix}\mathbf{Z}$, which sents $[x]_A$ to $[y]_B$, where $x \in V_1(A)$ and $y \in V_1(B)$.

Question 8.5 If F and S are both endomorphisms of the same mixing SFT, is it enough for eventual conjugacy to have a) and b) of Theorem 8.3 ? We know, it is enough in case that the permutation π is the identity, Theorem 7.4.

9. Maximal measures of endomorphisms of SFTs.

Theorem 9.1 Let S be a mixing SFT and $F \in \mathrm{end}(S)$ be expansive. Then S and F have the same measure of maximal entropy.

Remarks 9.2 (1) Theorem 9.1 is false in the twosided setting. Nasu [N1,N2] has constructed an example of a twosided mixing SFT S and a homeomorphism U commuting with S, such that U is also a mixing SFT, and the measures of maximal entropy for U and S are different.

(2) Blanchard and Maass [BM] proved Theorem 9.1 with combinatorial arguments in the special case that S is a full shift. We will give two abstract proofs below. The first relies on the dimension group ideas. The second appeals to Ruelle's Perron-Frobenius Theorem.

1.Proof. Let $h_m(S)$ denote the entropy of S with respect to its measure of maximal entropy m and let Fm be the measure transported by F, that is $Fm(B) = m(F^{-1}B)$ for all Borel sets B. Since S and F commute, the measure Fm is S-invariant. Since F is an onto endomorphism of S, and thus bounded-to-one, we have $h_m(S) = h_{Fm}(S)$, which shows $Fm = m$, since S has a unique measure of maximal entropy. So m is F-invariant as well.

Consider the images group G(S) of S. Up to multiplication with a constant, there is a unique homomorphism $\tau = \tau_S : G(S) \to \mathbf{R}$, which sends a strictly positive element of G(S) to a strictly positive real. Let A be a 0-1 matrix describing S, with Perron value λ and right Perron vector r and left Perron vector l, say. For a cylinder set $C = C(x_0,\dots , x_k)$ we have

$\tau[C] = \lambda^{-k}r(x_k)$, where $r(i)$ denotes the i´th entry of the vector r. Thus, since $m(C) = \lambda^{-k}l(x_0)r(x_k)$ and the Perron vectors are strictly positive, there is some $K > 0$ such that $K^{-1} \leq \frac{\tau[C]}{m(C)} \leq K$ for all cylinder sets C. Since F is an expansive endomorphism of S, it is also a mixing SFT by Lemma 2.9 and $G(S) = G(F)$ by Theorem 4.5. Thus we have $\tau_F = \tau_S$ and thus for the maximal measure of F, say μ, the ratios $\frac{\tau[C]}{\mu(C)}$ (for all clopen sets) are also uniformly bounded between positive constants. It follows that the F-invariant measure m is absolutely continuous with respect to the ergodic F-invariant measure μ. Therefore $\mu = m$.

2.Proof. First we summarize from [B] some consequences of Ruelle's Perron-Frobenius Theorem. For a mixing SFT S, one can define a Ruelle operator L_S on $C(X,R)$ by $(L_Sf)(x) = \Sigma_{\{y,\ Sy=x\}}\ f(y)$. The dual operator L_S^* is defined on the space of finite signed Borel measures on X. There is a unique Borel probability measure μ_S which is an eigenmeasure for L_S^*, say $L_S^*(\mu_S) = (\lambda_S)\mu_S$; here $\lambda_S > 0$; and there is a positive continuous function f_S such that the measure of maximal entropy for S, $\max_S$, is given by $(f_S)\mu_S$. Similarly, one has such objects for F, since F is mixing SFT. Now compute

$$L_S\,(L_Ff)\,(x) = \Sigma_{\{y,\ Sy=x\}}(L_Ff)(y) = \Sigma_{\{y,\ Sy=x\}}\Sigma_{\{z,\ Fz=y\}}f(z) = \Sigma_{\{z,\ SFz=x\}}\ f(z) = L_{SF}f\,(x)\ .$$

Since SF =FS, it follows that L_S and L_F commute, and therefore L_S^* and L_F^* commute. Thus their unique probability eigenmeasures must be the same measure μ.

We now know there is a positive continuous function g such that $\max_S = g \max_F$. Therefore the F-invariant probability $\max_S$ is absolutely continuous with respect to the F-invariant probability $\max_F$, which is ergodic for F. It follows that $\max_S = \max_F$. □

Example 9.3 Theorem 9.1 cannot be extended to locally injective endomorphisms of a onesided mixing SFT. Let (X,S) denote the 5-shift and let μ be its unique measure of maximal entropy. We will define an automorphism U (in particular, a locally injective endomorphism) of S such that $h_\mu(U) = 0$ and $h_{top}(U) = \log 2$. (Background for the argument below is in [DGS].)

First, let T be the twosided 2-shift on symbols {0,1} and let F denote the endomorphism of T given by the block code $(Fx)_0 = x_{-1}+x_1$ (mod 2). Let F' denote the onesided 4-shift on symbols { (0,0), (0,1), (1,0), (1,1) }. It is easy to see that the map $\ldots x_{-1}x_0x_1\ldots \rightarrow y_0y_1\ldots$ given by $y_i = (\,(F^ix)_0, (F^ix)_1)$ defines a conjugacy of F to F'. We use this map to present the 2-shift T as an automorphism T' of F'. It is routine to check that T' is given by the 2-block code $(a,b)(c,d) \rightarrow (b,a+d)$ (mod 2) .

Next extend the 4-shift F' to the 5-shift S by extending the alphabet to include a fifth symbol *. Extend T' to a 2-block code U from S to S by adding the additional coding rules $* \rightarrow *$ and $(a,b)* \rightarrow (b,a)$. Then U is an automorphism of S. Clearly $h_{top}(U) \geq h_{top}(T') = \log 2$.

Let E denote the set of sequences in S in which * occurs infinitely often. Suppose ν is an invariant ergodic Borel probability measure for U such that $\nu(E) > 0$. Because the symbol * is invariant under U, by ergodicity $\nu(E) = 1$, also there is an infinite set Z(*) of positive integers such that $Z(*) = \{i \geq 0 \mid y_i = *\}$ for ν-almost all y. For an integer n in Z(*), consider the partition P_n of X by the words occurring in coordinates 0 through n. Modulo a ν-null set, U permutes the atoms of P_n, so $h_\nu(U,P_n) = 0$. Because these P_n generate, $h_\nu(U)=0$.

The restriction of U to the complement of E is the union of countably many systems which are finite extensions of the 2-shift. It now follows easily from the variational principle that $h_{top}(U) = \log 2$.

The measure μ is invariant for U but not ergodic for U. With respect to U, μ has an ergodic decomposition $\{\mu_t\}$ such that $\mu_t(E) = 1$ for all t. For each μ_t, as argued above $h_{\mu_t}(U) = 0$, and therefore $h_\mu(U) = 0$. □

Remark 9.4 Marek Rychlik has pointed out to us that the images group G(S) for a mixing SFT S can be interpreted in terms of the Ruelle operator L_S (defined in the second proof of Theorem 9.1), as follows. Let L denote the restriction of L_S to $C(X,\mathbf{Z})$ and let $K = \cup_{\{n>0\}} \mathrm{Ker}(L^n)$. Then one can check that the map sending a clopen set to its indicator function induces an isomorphism from $G(S) = CO(X)/\langle R\rangle$ to $C(X,Z)/K$.

It is natural to speculate about generalizations. In particular, L_S is the operator associated to the zero function, and one could consider Ruelle operators for nonconstant functions.

Remark 9.5 We'll sketch now an "expansive subdynamics" proof for Theorem 9.1, which provides some perspective on the difference between the onesided and twosided cases. Theorem 9.1 fails in general for commuting twosided SFTs, but Theorem 9.1 is true for twosided mixing SFTs in the same "expansive component" of directions for a $\mathbf{Z}^2$ action [BL, Cor. 8.5].

For S and F in Theorem 9.1, let S' denote the twosided SFT which is the natural extension of S and let F' denote the endomorphism of S' induced by F. Then for the $\mathbf{Z}^2$ action generated by S' and F', it is not difficult to check that the direction corresponding to the action of F' is either inside or on the boundary of the expansive component of directions containing the direction corresponding to S'. (The corresponding result in Nasu's setting appears in the final remark of [N2, Sec. 8].) In any case, S'F' and S' have the same maximal measure, and it follows that SF and S have the same maximal measure. By symmetry, SF and F have the same maximal measure, and therefore so do F and S.

10. The images group is a dimension group. A *dimension group* is an ordered group (G, G_+) where G is a direct limit group $\lim_{\rightarrow}(\mathbf{Z}^{n(k)}, i_k)$ where i_k is a positive homomorphism from the ordered group $\mathbf{Z}^{n(k)}$ into $\mathbf{Z}^{n(k+1)}$,([E]). We have seen that the images group of a SFT is a dimension group. In this section we show that for any local homeomorphism S its images group is actually a dimension group. Thus, in particular to any endomorphism of a mixing SFT, which is a local homeomorphism (not necessarily expansive!) we can associate a dimension group. As an example, we shall compute the dimension group for the multiplication by 2 in the full 6-shift in section 11.

The theorem is based on the following Lemma, where we use $\mathrm{diam}(\mathcal{P}) := \sup\{\mathrm{diam}(P) \mid P \in \mathcal{P}\}$ for a partition $\mathcal{P}$.

Lemma 10.1 Let S be a local homeomorphism. Then there is a sequence of clopen partitions $\mathcal{P}_k$ of X with $\mathrm{diam}(\mathcal{P}_k) \rightarrow 0$ and

(1) S is injective on the atoms of $\mathcal{P}_1$

(2) $\mathcal{P}_k$ refines $S^{-1}\mathcal{P}_{k-1} \vee \mathcal{P}_{k-1}$ for all $k > 1$

(3) For any pair of atoms P, Q in $\mathcal{P}_k$ and all $0 \le i \le k$ it holds that S^iP and S^iQ are either disjoint or equal, for all k.

Note that (1) and (2) imply that S^k is injective on each of the atoms of $\mathcal{P}_k$.

Proof Since S is locally injective, there is a clopen partition $\mathcal{R}$ such that S is injective on the atoms of $\mathcal{R}$. For $\mathcal{P}_1$ we choose a partition refining $\mathcal{R}$ according to Lemma 2.8 .

Now assume $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are clopen partitions satisfying (1) - (3). We show how to construct $\mathcal{P}_{k+1}$. Let $\mathcal{P}_{k+1}$ be a partition refining $S^{-1}\mathcal{P}_k \vee \mathcal{P}_k$ according to Lemma 2.8 with S replaced by S^{k+1}. Then (2) are satisfied for k+1. We may in addition choose $\mathcal{P}_{k+1}$ with diameter less than 1/k (say), to guarantee $\mathrm{diam}(\mathcal{P}_k) \rightarrow 0$. It remains to check (3). Let P, Q be any pair of atoms in $\mathcal{P}_{k+1}$. By the definition of $\mathcal{P}_{k+1}$ we have that $S^{k+1}P$ is either disjoint from $S^{k+1}Q$ or $S^{k+1}P = S^{k+1}Q$. Assume there is some $1 \le i \le k$ and a point $x \in S^iP \cap S^iQ$. Then $S^{k+1}P = S^{k+1}Q$ and S^iP and S^iQ are contained in the same $\mathcal{P}_{k+1-i}$-atom, by (2). But the hypothesis implies that S^{k+1-i} is injective when restricted to an $\mathcal{P}_{k+1-i}$-atom. Since $S^{k+1}P = S^{k+1}Q$, we get that $S^iP = S^iQ$. □

An *order unit* in an ordered group (G, G_+) is an element u such that for any g in G, there is a positive integer n such that $nu > g$.

A *unital ordered group* is an ordered group with a distinguished order unit, formally it is a triple (G, G_+, u). For completeness, from here we regard the images group of a local homeomorphism $X \rightarrow X$ as a unital ordered group, where the distinguished order unit is [X].

Let S be a local homeomorphism and let $\mathcal{P}_k$ be any sequence of partitions as in Lemma 10.1. For fixed k call two atoms of $\mathcal{P}_k$ equivalent if they have the same image under S^k. Choose one atom from each equivalence class. This gives a collection E_k of atoms of $\mathcal{P}_k$ such that for any atom P of $\mathcal{P}_k$ there is a unique element Q of E_k with $S^kP = S^kQ$. Define a projection $f_k : \mathcal{P}_k \to E_k$ by $f_k(P) = Q$ if Q is equivalent to P.

Let G_k be the free abelian group with generators E_k. Thus G_k is isomorphic to $\mathbf{Z}^{n(k)}$, where $n(k)$ = number of atoms in E_k, and we may think of the elements in G_k as integer combinations $\sum_{P \in E_k} n_P \cdot P$, with $n_P \in \mathbf{Z}$. Note that any generator $P \in E_k$ splits into $\mathcal{P}_{k+1}$-atoms by (2), thus we may define $i_k(P) := \sum_{Q \in \mathcal{P}_{k+1},\, Q \subset P} f_{k+1}(Q)$, which extends to a group homomorphism $i_k : G_k \to G_{k+1}$ by linearity.

Theorem 10.2 Let S be a local homeomorphism. Then $G(S)$ is a unital dimension group. Specifically, with the notation as above, there is an order preserving group isomorphism from $G(S)$ onto the direct limit group $\lim_{\to}(G_k, i_k)$.

Proof Let $G := \lim_{\to}(G_k, i_k)$. Recall that the elements in G are of the form $[(g, k)]$, where $k \geq 1$ is an integer and $g \in G_k$, where $[(g, k)] = [(h, m)]$ if for some $n \geq \max(k,m)$ it holds that $i_n \ldots i_{k+1} i_k(g) = i_n \ldots i_{m+1} i_m(h)$.

Recall that $CO(X)$ denotes the set of clopen subsets of X. We define a map $\Psi : CO(X) \to G$. Let $C \in CO(X)$. Since C and its complement are both clopen, and thus compact, they have a positive distance and thus for large enough $k \geq 1$ an atom in $\mathcal{P}_k$ is contained either in C or in the complement of C. For such k, $C = \bigcup_{P \in \mathcal{P}_k,\, P \subset C} P$, and we define $\Psi(C) := [(\sum_{P \in \mathcal{P}_k,\, P \subset C} f_k(P), k)] \in G$. We check that Ψ is well defined, independent of the choice of k.

If $P \in \mathcal{P}_k$ then $S^kP = S^k f_k P$, and since S^k is injective on the atoms P and f_kP and (3) holds, there is a bijection $g : \{ Q \in \mathcal{P}_{k+1} \mid Q \subset P \} \to \{ Q' \in \mathcal{P}_{k+1} \mid Q' \subset f_kP \}$ such that $S^kQ = S^kg(Q)$ all $Q \in \mathcal{P}_{k+1}$ with $Q \subset P$. Thus also $S^{k+1}Q = S^{k+1}g(Q)$ which implies $f_{k+1}Q = f_{k+1}g(Q)$ for all $Q \in \mathcal{P}_{k+1}$ with $Q \subset P$. This shows $i_k(f_kP) = \sum_{Q \in \mathcal{P}_{k+1},\, Q \subset P} f_{k+1}(Q)$, thus Ψ is well defined.

Extend Ψ to a group homomorphism $\Psi : \langle CO(X) \rangle \to G$. Since $\Psi(P) = [(P, k)]$ for any atom $P \in E_k$ we have that Ψ is onto. We shall show that the kernel of Ψ is the group generated by the relations $\mathcal{R}$.

If C is a disjoint union of sets C_i, then $\Psi(C) = \Sigma\, \Psi(C_i)$, because for some k the C and C_i are unions of atoms of $\mathcal{P}_k$.

Let C, D be clopen sets such that S^n is injective on C and D and $S^nC = S^nD$. Choose some $k \geq n$ such that C and D both split into $\mathcal{P}_k$-atoms. By injectivity, for any pair P, Q of distinct $\mathcal{P}_k$-atoms in C we have that S^nP is disjoint from S^nQ. The same applies for D. And thus from $S^nC = S^nD$ and (3) it follows that there is a bijective map $g : \{ Q \in \mathcal{P}_k \mid Q \subset C \} \to \{ P \in \mathcal{P}_k \mid P \subset D \}$ with $S^nQ = S^ngQ$, which implies $f_kQ = f_kgQ$. Thus $\Psi(C) = [(\sum_{Q \in \mathcal{P}_k, Q \subset C} f_k(Q), k)] = [(\sum_{P \in \mathcal{P}_k, P \subset D} f_k(P), k)] = \Psi(D)$. Thus we have shown that $<\mathcal{R}>$ is contained in the kernel of Ψ.

Now let $\sum_{C \in CO(X)} i_C \cdot C \in \ker \Psi$. Write $\sum_{C \in CO(X)} i_C \cdot C = \sum_{C \in CO(X)} n_C \cdot C - \sum_{C \in CO(X)} m_C \cdot C$ with $n_C, m_C \geq 0$ for all $C \in CO(X)$. Then $\Psi(\sum_{C \in CO(X)} n_C \cdot C) = \Psi(\sum_{C \in CO(X)} m_C \cdot C)$. Let $\mathcal{C}'$ be the finite collection of clopen sets with $n_C \neq 0$ or $m_C \neq 0$. For some large k any $C \in \mathcal{C}'$ can be partitioned into $\mathcal{P}_k$-atoms.

Thus in $G(S)$ we have $\sum_{C \in CO(X)} n_C \cdot C = \sum_{P \in P_k} r_P \cdot P$ and $\sum_{C \in CO(X)} m_C \cdot C = \sum_{P \in P_k} s_P \cdot P$ for certain $r_P, s_P \geq 0$. Thus $\Psi(\sum_{C \in CO(X)} n_C \cdot C) = [(\sum_{Q \in E_k} t_Q \cdot Q, k)]$ and $\Psi(\sum_{C \in CO(X)} m_C \cdot C) = [(\sum_{Q \in E_k} u_Q \cdot Q, k)]$ with all $u_Q, t_Q \geq 0$ and $[(\sum_{Q \in E_k} t_Q \cdot Q, k)] = [(\sum_{Q \in E_k} u_Q \cdot Q, k)]$. Thus we may assume that k is so large that also $\sum_{Q \in E_k} t_Q \cdot Q = \sum_{Q \in E_k} u_Q \cdot Q$, i.e. $u_Q = t_Q$ for all Q. Now consider $W \in E_k$ with $t_W > 0$. Then there are atoms $P', P'' \in \mathcal{P}_k$ with $f_kP' = W = f_kP''$ and $r_{P'}, s_{P''} > 0$. Thus $S^kP' = S^kP''$ and S^k is injective on P' and P'', thus $P' - P'' \in \mathcal{R}$. Furthermore $\sum_{P \in P_k} r_P \cdot P - P'$ and $\sum_{P \in P_k} s_P \cdot P - P''$ are again formal sums with non negative coefficients and the same Ψ-image $(\sum_{Q \in E_k} t_Q \cdot Q - W, k)$, with non-negative coefficients. Repeating the argument finitely many times shows $\sum_{P \in P_k} r_P \cdot P - \sum_{P \in P_k} s_P \cdot P \in <\mathcal{R}>$ and thus $\sum_{C \in CO(X)} n_C \cdot C - \sum_{C \in CO(X)} m_C \cdot C \in <\mathcal{R}>$. This finishes the proof that $\ker\Psi = <\mathcal{R}>$ and shows that $G(S) = <CO(X)>/<\mathcal{R}>$ is isomorphic to G.

An element in $G(S)$ is positive if it can be presented by a formal sum with non-negative coefficients. By definition this sum is mapped under Ψ to some element $[(\sum_{Q \in E_k} t_Q \cdot Q, k)] \in G$, where all $t_Q \geq 0$. On the other hand for $[(\sum_{Q \in E_k} t_Q \cdot Q, k)] \in G$, where all $t_Q \geq 0$ we have that $\Psi(\sum_{Q\in E_k} t_Q\cdot Q) = [(\sum_{Q\in E_k} t_Q\cdot Q,k)]$ and thus positive elements in G have positive preimages. Thus, this proves that Ψ induces an order preserving group isomorphism from $G(S) = \langle CO(X)\rangle/\langle R\rangle$ to G and thus proves the theorem. □

11. Example: The multiplication CA In this section we shall consider the endomorphism of the 6-shift which describes multiplication by 2. We will see that the images group of the multiplication by 2 in the 6-shift is $\mathbf{Z}[1/2] \oplus \mathbf{Z}^\infty$, where $\mathbf{Z}^\infty$ denotes the direct sum of countably many copies of $\mathbf{Z}$. In particular, the images group is not of finite rank as in the SFT case. Without proof we note that the preimages group of this multiplication can be computed as the free abelian group with uncountably many generators, (we only have a rather tedious argument.)

Let (X, S) be the full onesided 6-shift with $X = \{0,1,\ldots, 5\}^{\mathbf{N}}$. Then S describes multiplication by 6, in that multiplication by 6 (modulo 1) is the quotient of S by the map π : $x \to \sum_{i=0}^{\infty} \frac{x_i}{6^{i+1}}$ (mod 1). For $a \in \{0,1,\ldots, 5\}$ write $2\cdot a = c(a)\cdot 6 + r(a)$, where $c(a) \in \{0, 1\}$, $r(a) \in \{0, 2, 4\}$ with $r(a) = 2\cdot a \bmod 6$. Let $f : \{0,1,\ldots, 5\}^2 \to \{0,1,\ldots, 5\}$ be defined by $f(a,b) = r(a) + c(b)$. Define a continuous, shift commuting map $F : X \to X$ by $F(x)_i = f(x_i,x_{i+1})$ for $i \geq 0$. F is onto and locally injective, and multiplication by 2 is the quotient of F by the map π [BM].

An ordered group G is called *simple* if every nonzero positive element u is an order unit. By a *trace* on a unital ordered group (G, G_+, u) we mean a homomorphism $\tau : G \to \mathbf{R}$ such that $\tau(G_+) \subset \mathbf{R}_+$ and $\tau(u) = 1$. An element g is an infinitesimal if it is in the kernel of every trace; equivalently, for every order unit v and integer n, the element $v + ng$ is an order unit. If there is a unique trace, then its kernel is the subgroup I of infinitesimals.

Below, for a positive integer i, we let H_i denote the free abelian group whose generators are the formal elements $\langle m,i\rangle$ with $m \in (\mathbf{Z}/3^i)^*$, the units of $\mathbf{Z}/3^i\mathbf{Z}$.

In (iii) below, we identify an element $m \in (\mathbf{Z}/3^i)^*$ with its unique representative in $[0, 3^i) \cap \mathbf{Z}$. Let $H = \bigoplus_{i=1}^{\infty} H_i$. Let $\mathbf{Z}^\infty$ denote the direct sum of countably many copies of $\mathbf{Z}$.

Theorem 11.1 Let notations be as above, let G denote the unital ordered group G(F).

(1) As a nonordered group, G is isomorphic to $\mathbf{Z}[1/2] \oplus \mathbf{Z}^\infty$.

(2) G is a simple dimension group with a unique trace τ, which maps G onto $\mathbf{Z}[1/6]$.

(3) The kernel I of τ is isomorphic to $\mathbf{Z}^\infty$ and does not split.

(4) G is isomorphic to $\mathbf{Z}[1/2] \oplus H$. In this presentation,

(i) the distinguished order unit is $1 \in \mathbf{Z}[1/2]$;

(ii) $\tau(x + \Sigma\, c_{<m,i>}<m,i>) = x + \Sigma\, c_{<m,i>} m6^{-i}$, where $x \in \mathbf{Z}[1/2]$, $<m,i> \in H$; and

(iii) F^* is determined by linearity from

$F^*(x) = 2x$ for $x \in \mathbf{Z}[1/2]$

$F^*(<m,i>) = <2m,i>$ for $<m,i> \in H$ with $2m < 3^i$ and

$F^*(<m,i>) = 2^{-i} + <2m-3^i,i>$ for $<m,i> \in H$ with $2m \geq 3^i$.

Proof. We will use Theorem 10.2. We shall define a sequence of clopen partitions P_k of X as in Lemma 10.1 and collections E_k of atoms of P_k.

For P_k we choose the partition of X into cylinder sets of length k, i.e. the atoms are the sets $C(x_0,\ldots,x_{k-1}) := \{ y \in X \mid y[0,k-1] = (x_0,\ldots,x_{k-1}) \}$, $(x_0,\ldots,x_{k-1}) \in \{0,\ldots,5\}^k$. Let $\pi_k : \{0,\ldots,5\}^k \to [0,1]$, $\pi_k(x_0,\ldots,x_{k-1}) = \sum_{i=0}^{k-1} \frac{x_i}{6^{i+1}}$.

The property (1) of Lemma 10.1 is obvious. Because F is a 2-block map any $F^{-1}C(x_0,\ldots,x_{k-2})$ is a union of cylinders of length k, thus (2) of Lemma 10.1 holds. We check properties (3). To visualize the action of F on the atoms of P_k assign to each $C(x_0,\ldots,x_{k-1})$ a subinterval $I(x_0,\ldots,x_{k-1}) = [\pi_k(x_0,\ldots,x_{k-1}),\pi_k(x_0,\ldots,x_{k-1}) + 6^{-k})$ of $[0,1)$. Thus we have a bijection of the atoms of the partition P_k to the atoms of the partition I_k of the interval $[0,1)$ into 6^k consecutive intervals of length 6^{-k}. Consider the map $g(x) = 2x \bmod 1$ on $[0,1)$. By definition $F(C(x_0,\ldots,x_{k-1})) = C(f(x_0,x_1),\ldots,f(x_{k-2},x_{k-1}),f(x_{k-1},0)) \cup C(f(x_0,x_1),\ldots,f(x_{k-2},x_{k-1}),f(x_{k-1},3))$. But also $g(I(x_0,\ldots,x_{k-1})) = I(f(x_0,x_1),\ldots,f(x_{k-2},x_{k-1}),f(x_{k-1},0)) \cup I(f(x_0,x_1),\ldots,f(x_{k-2},x_{k-1}),f(x_{k-1},3))$, which shows that F maps any atom of P_k onto the union of two other P_k - atoms just in the same way as g maps the assigned subinterval. Thus, to understand a cover F^iP_k , whose elements are the sets F^iP, $P \in P_k$, it sufficies to understand the g^i-image of the partition I_k. Since g maps both intervals $[0,1/2)$ and $[1/2,1)$ onto $[0,1)$ and since the number of atoms of I_k is even, the g-image is a cover where any two sets are either the same or disjoint. Inductively this follows for all g^i-images $0 \leq i \leq k$. Thus the same holds for the covers F^iP_k $0 \leq i \leq k$, which means that (3) of Lemma 10.1 holds.

Let $E_k = \{ C(x_0,\ldots,x_{k-1}) \mid \pi_k(x_0,\ldots,x_{k-1}) < 2^{-k} \}$. Since $g^k[0,2^{-k}) = [0,1)$ injectively, this also implies that for any atom P of P_k there is a unique element Q of E_k such that $S^kP = S^kQ$.

This also shows that for $P = C(x_0,\ldots,x_{k-1})$ this unique element Q of E_k is the unique cylinder $C(y_0,\ldots,y_{k-1})$ with $\pi_k(y_0,\ldots,y_{k-1}) < 2^{-k}$ and $\pi_k(x_0,\ldots,x_{k-1}) = \pi_k(y_0,\ldots,y_{k-1}) + m2^{-k}$ for some integer $m \geq 0$. Denote this map $C(x_0,\ldots,x_{k-1}) \to C(y_0,\ldots,y_{k-1})$ by ϕ_k.

Let G_k be the free abelian group with generators in E_k, that is $G_k = \mathbf{Z}^{E(k)}$. Define a homomorphism $i_k : G_k \to G_{k+1}$ by $i_k(C(x_0,\dots,x_{k-1})) := \sum_{i=0}^{5} \phi_{k+1}(C(x_0,\dots,x_{k-1}, i))$. By Theorem 10.2, G(F) is isomorphic to the direct limit group of the (G_k, i_k).

Let $n(k) = 3^k$ for $k \geq 1$. Enumerate the elements of E_k by $\{0,...,n(k)-1\}$ in increasing order (according to the π_k-values of these atoms). This induces an isomorphism from G_k to $\mathbf{Z}^{n(k)}$ under which the homomorphism i_k is isomorphic to the map which maps the i´th standard basis vector $e_i, 0 \leq i \leq n(k) - 1$, of $\mathbf{Z}^{n(k)}$ to $\Sigma_{0\leq j\leq 5}\, e_{(6i+j) \bmod n(k+1)}$ in $\mathbf{Z}^{n(k+1)}$, which we denote again by i_k.

Let $B_k \subset \mathbf{Z}^{n(k)}$ be the finite set of vectors $B_k = \{v_k(m) = \Sigma_{0\leq i<m}\, e_i$, where $0 < m < 3^k$ and m is not divisible by $3\}$, let $v_k := \Sigma_{0\leq i<n(k)}\, e_i$. Let $C_1 := B_1$ and, for $k \geq 1$, $C_{k+1} = i_k(C_k) \cup B_{k+1}$. By induction one proves that for all $k \geq 1$, $C_k \cup \{v_k\}$ is an integral basis of $\mathbf{Z}^{n(k)}$. Note that under i_k all elements of C_k are mapped onto elements of C_{k+1} and the vector v_k is mapped to $2v_{k+1}$. This shows that i_k is injective. Now we use this basis to establish a group isomorphism $G \to \mathbf{Z}[1/2] \oplus_k H_k$. Recall that $v_k(m) = \Sigma_{0\leq i<m}\, e_i \in \mathbf{Z}^{n(k)}$.. Then with certain integer coefficients $a_i(m)$, a_k any $v \in \mathbf{Z}^{n(k)} = G_k$, $k \geq 1$, can be uniquely written in the following form:

$$v = a_k v_k + \sum_{i=1}^{k} \sum_{m\in (\mathbf{Z}/3^i\mathbf{Z})^*} a_i(m) v_i(m) \; .$$

Recall that H_i is the free abelian group with free generating elements $<m,i>$, where $m \in (\mathbf{Z}/3^i\mathbf{Z})^*$ and define a map $\theta : G \to \mathbf{Z}[1/2] \oplus_k H_k$ as follows. For v in the above form let

$$\theta[(v,k)] = \frac{a_k}{2^k} + \sum_{i=1}^{k} \sum_{m\in (\mathbf{Z}/3^i\mathbf{Z})^*} a_i(m) <m,i> .$$

Since the maps i_k are injective an element $(v´,k)$ is equivalent to (v,k) (in G) iff $v = v´$ and an element $(v´,k+1)$ is equivalent to (v,k) iff $i_k(v) = v´$. This together with the defintion of the basis shows that θ is well defined. One checks that θ is a bijective group homomorphism.

We now describe the induced action F* on G in this representation. Avoiding additional notation, we denote this map again by F*. Let $a \in \mathbf{Z}$, $a{\cdot}2^{-k} \in \mathbf{Z}[1/2]$. Then $F^*(a{\cdot}2^{-k}) = 2a{\cdot}2^{-k}$. Let $k \geq 1$ and $<m,k> \in H_k$. Then $F^*(<m,k>) = <2m,k>$ if $2m < 3^k$ and $F^*(<m,k>) = 2^{-k} + <2m \bmod 3^k,k> \in \mathbf{Z}[1/2] \oplus H_k$, if $2m \geq 3^k$.

We describe the positive elements in G. First we show that in the presentation of G as a direct limit of the $\mathbf{Z}^{n(k)}$, an element $[(v,k)] \in G$, with $v \in \mathbf{Z}^{n(k)}$, is positive iff the sum of its coefficients with respect to the standard basis vectors of $\mathbf{Z}^{n(k)}$ is positive. To see this observe that each standard basis vector in $\mathbf{Z}^{n(k)}$ is mapped via i_k to a sum of 6 cyclically consecutive standard basis vectors. Mapping the i´th standard basis vector in $\mathbf{Z}^{n(k)}$ m steps forward, yields $N\cdot\Sigma_{0\leq i<n(k+m)}\, e_i + w_i \in \mathbf{Z}^{n(k+m)}$, where w_i is a vector which has coefficients of absolute value at most 1 with respect to the standard basis in $\mathbf{Z}^{n(k+m)}$ and N is the greatest integer less than $6^m/3^{k+m}$. Thus, if the sum of coordinates of v is positive, then v gets mapped eventually to a positive vector, which means $[(v,k)] \in G$ is an order unit (and $[-v,k] \notin G_+$). A vector with coefficent sum zero remains a vector with coefficient sum zero. If the coordinate sum C(v) for v is zero, then [(v,k)] must be an infinitesimal. So, every nonzero positive element is an order unit, and G is simple.

The map $\tau : [(v,k)] \rightarrow 6^{-k}C(v)$ defines a trace, and has image $\mathbf{Z}[1/6]$. Using the maps i_k shows that τ is the unique trace. To see that the kernel I of τ is isomorphic to $\mathbf{Z}^\infty$, note that the map $(x,y) \rightarrow y$ from I into H is injective, and recall an infinitely generated subgroup of $\mathbf{Z}^\infty$ is isomorphic to $\mathbf{Z}^\infty$. The kernel I does not split, because a complementary summand would be isomorphic to $\mathbf{Z}[1/6]$, in which every element is divisible by 6. This finishes the proof. □

REFERENCES

[B] R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Springer Lec. Notes in Math 470 (1975).

[BFK] M.Boyle, J.Franks and B.Kitchens. Automorphisms of one-sided subshifts of finite type. Ergod. Th. & Dynam. Sys. 10 (1990), 421-449.

[BL] M. Boyle and D. Lind. Expansive subdynamics. Trans. AMS, to appear.

[BM] F.Blanchard, A.Maass. Dynamical properties of expansive one-sided cellular automata. Israel J. Math, to appear.

[BMT] M. Boyle, B. Marcus, P.Trow. Resolving maps and the dimension group for shifts of finite type. Memoirs of the AMS 377 (1987)

[DGS] M. Denker, C. Grillenberger and K. Sigmund. Ergodic theory on compact spaces. Springer Lec. Notes in Math. 527 (1976), Springer-Verlag.

[E] E.G.Effros. Dimensions and C*-Algebras. CMBS 46 (1981), AMS

[H] G.A.Hedlund. Endomorphisms and automorphisms of the shift dynamical system. Math. Systems Th. 3 (1969), 320-375.

[K] P.Kurka. Equicontinuity and attractors in linear cellular automata. Ergod. Th. & Dynam. Sys., to appear.

[LM] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding (1995), Cambridge University Press

[N1] M. Nasu. Textile systems for endomorphisms and automorphisms of the shift, Memoirs AMS 546 (1995)

[N2] M. Nasu. Maps in symbolic dynamics,in Lecture Notes of the Tenth KAIST Mathematics Workshop, ed. Geon Ho Choe, Korea Advanced Institute of Science and Technology Mathematics Research Center, Taejon, Korea.

[P] W.Parry. Symbolic dynamics and transformations of the unit interval. Trans. Amer. Soc. **122** (1966) 368-378

[W] R.F.Williams. Classification of subshifts of finite type. Ann. of Math. **98** (1973), 120-153; erratum **99** (1974), 380-381.

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