SYMBOLIC EXTENSION ENTROPY: C^r EXAMPLES, PRODUCTS AND FLOWS

MIKE BOYLE AND TOMASZ DOWNAROWICZ

Abstract. Adapting techniques of Misiurewicz, for $1 \le r < \infty$ we give an explicit construction of C^r maps with positive residual entropy. We also establish the behavior of symbolic extension entropy with respect to joinings, fiber products, products, powers and flows.

1. Introduction

Let (X,T) denote a system, i.e. a selfhomeomorphism T of a compact metric space X. We let $K(T)$ denote the space of T-invariant Borel probabilities on X. Now suppose (Y,S) is a symbolic system, i.e. S is the shift map on a closed shift-invariant set Y of doubly infinite sequences on some finite alphabet. If $\varphi : Y \to X$ is a continuous surjection such that $\varphi S = T\varphi$, then we write $\varphi : (Y,S) \to (X,T)$ and we say that φ is a *symbolic extension* of (X,T). The extension entropy function of φ is the function h^{φ}_{ext} on $K(T)$ defined by the rule

$$h^{\varphi}_{\text{ext}} : \mu \mapsto \max\{h_\nu(S) : \nu \in K(S), \varphi\nu = \mu\} \ .$$

The symbolic extension entropy function h^T_{sex} is the infimum of the extension entropy functions h^{φ}_{ext}, taken over all symbolic extensions φ [1]. (If there is no symbolic extension, then we set $h^T_{\text{sex}} \equiv \infty$. Also, we use "sex entropy" as an abbreviation for "symbolic extension entropy".) The function h^T_{sex} is a very fine reflection of the way that complexity emerges in (X,T) along refining scales, and leads to the study of *entropy structure*, a master invariant for entropy theory [5].

A related invariant is $\mathbf{h}_{\text{sex}}(T)$, the topological sex entropy of T. This is the infimum of the topological entropies of (Y,S), taken again over all symbolic extensions $\varphi : (Y,S) \to (X,T)$. If T is C^∞, then $\mathbf{h}_{\text{sex}}(T) = \mathbf{h}_{\text{top}}(T)$ [2, Theorem 7.8]. If T is only C^1, then $\mathbf{h}_{\text{sex}}(T)$ can be infinite (i.e., there is no symbolic extension) [6]. For $1 < r < \infty$, it is not known whether there exists a C^r system (X,T) with $\mathbf{h}_{\text{sex}}(T) = \infty$. In [6], it is shown that within certain families of C^r systems $(1 < r < \infty)$, $\mathbf{h}_{\text{sex}}(T) > \mathbf{h}_{\text{top}}(T)$ is a generic property. Understanding the impact of intermediate smoothness seems to us the most pressing open problem for the theory of sex entropy.

In this paper, we reexamine the earliest construction of C^r systems with no measure of maximal entropy, due to Misiurewicz [10]. In Section 4, we show for these that the sex entropy function equals the upper semicontinuous envelope of the entropy function; consequently, $\mathbf{h}_{\text{top}} = \mathbf{h}_{\text{sex}}$. Then, elaborating the construction of

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[10] with techniques from [4, 9], in Section 5 we construct C^r systems $(1 \leq r < \infty)$ with $\mathbf{h}_{\text{top}} < \mathbf{h}_{\text{sex}}$. Although [6] provides a large collection of C^r systems with $\mathbf{h}_{\text{top}} < \mathbf{h}_{\text{sex}}$, we think it is still of interest to see explicit examples which do not rely on generic existence arguments or invoke infinitely nested geometric structures.

For the analysis of these systems, we require some basic tools of the sex entropy theory. For the Misiurewicz construction, we make use of the characterization of sex entropy by superenvelopes (a functional analytic structure reviewed in Section 2). For the new C^r construction, we establish general inequalities for sex entropy in fiber products, whose proof uses the transfinite sequence characterization of sex entropy (also reviewed in Section 2).

In addition to considering fiber products, in Section 3 we establish the behavior of sex entropy with respect to joinings, products, powers and flows. We thank Joe Auslander and Sheldon Newhouse for turning our attention to these functorial issues.

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2. Preliminaries on sex entropy and superenvelopes

Let (X, T) be a system.

We use h^T, or just h, to denote the entropy map on $K(T)$; so, $h^T(\mu) = h_\mu(T)$, the measure theoretic entropy of T with respect to μ. The *topological residual entropy* of T is $\mathbf{h}_{\text{res}}(T) = \mathbf{h}_{\text{sex}}(T) - \mathbf{h}_{\text{top}}(T)$. The residual entropy function of T is the function h^T_{res} on $K(T)$ defined by $h^T_{\text{res}} = h^T_{\text{sex}} - h^T$. Residual entropy and symbolic extension entropy are essentially alternate notations, each with its advantages. In the sequel, for typographical reasons we will alternatively use the notation $h_{\text{sex}}(T, \mu)$ in place of $h^T_{\text{sex}}(\mu)$, and similarly $h_{\text{ext}}(\mu, \varphi)$ in place of $h^\varphi_{\text{ext}}(\mu)$. We will make use of the Sex Entropy Variational Principle [1, Theorem 8.1]:

$$\mathbf{h}_{\text{sex}}(T) = \sup\{h_{\text{sex}}(T, \mu) : \mu \in K(T)\} \ . \tag{2.1}$$

This supremum is in fact a maximum, because the function h^T_{sex} is upper semicontinuous [1, Theorem 8.1]. The maximum is achieved on the closure of the ergodic measures [1, Theorem 8.3]; however, the supremum over ergodic measures may be strictly smaller [1, Theorem 8.5].

Entropy structures. 2.2. An *entropy structure* for (X, T) is an allowed nondecreasing sequence of functions h_n on $K(T)$ which converges pointwise to h^T. There is a large collection of sequences which are allowed to serve as an entropy structure, and the class of such sequences is a master invariant for entropy. See [5] for the development of this theory and its justification. We give two examples of such structures.

A partition of X is *essential* if it is a finite partition and the boundary of every set in the partition has measure zero for every measure in $K(T)$. A sequence $(\mathcal{P}_n)$ of partitions is *refining* if, for each n, $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, and the largest diameter of an element of $\mathcal{P}_n$ converges to zero as n goes to infinity. Now, suppose $(\mathcal{P}_n)$ is a refining sequence of essential partitions of X. (Such a sequence exists for many but not for all systems [1, Theorem 7.6]). Define $h_n(\mu) = h_\mu(T, \mathcal{P}_n)$. The sequence $\mathcal{H} = (h_n)$ is then an entropy structure for (X, T).

Another entropy structure is provided by functions defined with respect to families of continuous functions, as follows. Let $f : X \to [0,1]$ be a continuous function. The sets $\{\langle x,t\rangle : t \le f(x)\}$ and $\{\langle x,t\rangle : t > f(x)\}$ form a two-element partition $\mathcal{A}_f$ of $X \times [0,1]$. If $\mathcal{F}$ is a finite family of functions f as above, then we let $\mathcal{A}_{\mathcal{F}} = \bigvee_{f\in\mathcal{F}} \mathcal{A}_f$. (Note that $\mathcal{A}_{\mathcal{F}'\cup\mathcal{F}''} = \mathcal{A}_{\mathcal{F}'} \vee \mathcal{A}_{\mathcal{F}''}$.) For $\mu \in K(T)$ we define its entropy *with respect to* $\mathcal{F}$ by $h_\mu(T,\mathcal{F}) = h_{\mu\times\lambda}(T \times \mathrm{Id}, \mathcal{A}_{\mathcal{F}})$ in the product system on $X \times [0,1]$, where Id is the identity map on $[0,1]$ and λ denotes the Lebesgue measure there. If now $(\mathcal{F}_n)$ is a sequence of finite families of continuous functions arranged so that the partitions $\mathcal{A}_{\mathcal{F}_n}$ refine in the product space, and we denote $h_n(\mu) = h_\mu(T,\mathcal{F}_n)$, then $\mathcal{H} = (h_n)$ is an entropy structure for (X,T).

The above examples of entropy structures have additional affinity and semicontinuity properties: for each $n \ge 1$ the function h_n is affine and the difference function $h_n - h_{n-1}$ (we set $h_0 \equiv 0$) is upper semicontinuous. These properties are not strictly required in the general definition of an entropy structure in [5] but their presence allows us to simplify the forthcoming exposition on further properties.

The key results [1, Theorems 5.5 and 8.1] in the entropy theory of symbolic extensions assert that the extension entropy functions of symbolic extensions are exactly the so called *affine superenvelopes* of the entropy structure $\mathcal{H}$, and that the sex entropy function coincides with the *minimal superenvelope* of $\mathcal{H}$:

$$h_{\mathrm{sex}} \equiv \mathrm{E}\mathcal{H}.$$

The definition and basic properties of superenvelopes are provided after the discussion of simplices.

Simplices. 2.3. We recall a few basic facts concerning simplices, harmonic and supharmonic functions and u.s.c. envelopes. The reader may consult [1] and the references therein for a more complete exposition on these subjects.

Consider a compact metrizable set K endowed with a continuous affine structure which makes it convex. For every Borel probability measure μ on K there exists a unique $x \in K$ such that

$$f(x) = \int f(y)\,d\mu(y)$$

for every continuous affine real-valued function f on K. We then say that x is the *barycenter* (or *expected value*) of μ. A function f on a convex K is *harmonic* (*supharmonic*) if

$$\int f d\mu = (\le) f(x),$$

for every μ as above and its barycenter x. Every affine (concave) u.s.c. function is harmonic (supharmonic). (The entropy function h on the simplex of invariant measures is harmonic, without generally being u.s.c.; see [4, Theorem 13.3].) For a real-valued function f on K we let $\widetilde{f}$ denote the u.s.c. envelope of f, i.e. the pointwise infimum of all continuous functions $g \ge f$. The function $\widetilde{f}$ is u.s.c., hence attains its supremum on every compact set. Notice that if f is affine then $\widetilde{f}$ is concave and thus supharmonic.

The set K is called a *Choquet simplex* or just *simplex* if for every point $x \in K$ there is a unique Borel probability measure μ_x supported by the set $\mathrm{ex}K$ of the extreme points of K, with barycenter at x.

A simplex K is *Bauer* if its extreme set $\mathrm{ex}K$ is compact. In this case the restriction map $f \mapsto f' = f|_{\mathrm{ex}K}$ defines a bijection between all u.s.c. affine functions on K and all u.s.c. functions on $\mathrm{ex}K$. The inverse map is provided by the *integral extension* $g \mapsto I(g)$

$$I(g)(x) = \int g(y)\, d\mu_x(y) \ .$$

If K is a general simplex, we will at times refer to the Bauer simplex M of all Borel probability measures supported by the closure $\overline{\mathrm{ex}K}$ of the extreme set of K. The barycenter map provides a continuous and affine surjection $\pi : M \to K$ which is usually not injective except when K is itself a Bauer simplex (then M and K are identical). If now f is a function on M then the *push-down* $f^{[K]}$ is defined on K at each point x as $\sup\{f(z) : z \in \pi^{-1}(x)\}$. It is not hard to see that in such a situation, if f is u.s.c., so is $f^{[K]}$.

We exercise the above notions and facts in proving a simple lemma (which has direct applications to entropy).

Lemma 2.4. *Let h be a harmonic function on a simplex K. Then*

$$\widetilde{h} = (I(\widetilde{h'}))^{[K]},$$

where h' denotes the restriction of h to $\overline{\mathrm{ex}K}$, $I(g)$ denotes the integral extension of a function g defined on $\overline{\mathrm{ex}K}$ onto the Bauer simplex M of all probability measures supported by $\overline{\mathrm{ex}K}$, and $f^{[K]}$ is the push-down onto K via the barycenter map of a function f defined on M.

Proof. Let $x \in K$. One of the measures supported by $\overline{\mathrm{ex}K}$ with barycenter at x is μ_x, so the right hand term evaluated at x is not smaller than $I(\widetilde{h'})(\mu_x)$, which is not smaller than $I(h')(\mu_x)$, which equals $\int h\, d\mu_x$. Because h is harmonic, this is $h(x)$. So the right hand side function is not smaller than h and since it is u.s.c. it is not smaller than $\widetilde{h}$. Conversely, the right hand side evaluated at x equals $I(\widetilde{h'})(\mu)$ for some μ supported by $\overline{\mathrm{ex}K}$ with barycenter at x. The last equals $\int \widetilde{h'}\, d\mu$. Notice that the meaning of "$\widetilde{\ }$" is different on the two sides of the displayed formula. On the right hand side one has to consider majorizing continuous functions defined only on $\overline{\mathrm{ex}K}$. Thus, at each point of $\overline{\mathrm{ex}K}$, $\widetilde{h'}$ is not larger than $\widetilde{h}$ (which is evaluated in the wider context of K). Thus the last integral does not exceed $\int \widetilde{h}\, d\mu$, which, by the supharmonic property of $\widetilde{h}$ does not exceed $\widetilde{h}(x)$. □

U.s.c.d.-sequences. 2.5. Let K be a compact metric set. A nondecreasing sequence $\mathcal{H} = (h_n)_{n\geq 0}$ of nonnegative functions defined on K will be called a *u.s.c.d.-sequence* if $h_0 \equiv 0$ and the differences $h_n - h_{n-1}$ are upper semicontinuous (u.s.c.), for all $n \geq 1$. We will use h to denote the pointwise limit of h_n, allowing ∞ in the range of h. (For example, the entropy structures described above are u.s.c.d.-sequences defined on $K(T)$.)

Definition 2.6. By a *superenvelope* of a u.s.c.d.-sequence $\mathcal{H}$ we shall mean any real-valued function E on K such that $E \geq h$ and $E - h_n$ is u.s.c. for every $n \geq 0$. We also allow the constant infinity function to be a superenvelope.

Notice that a real-valued superenvelope is itself u.s.c., hence bounded. Such superenvelopes may but need not exist (for example they do not exist if h is unbounded). Let $\mathrm{E}\mathcal{H}$ (or sometimes $\mathrm{E}(\mathcal{H})$) denote the pointwise infimum of all superenvelopes. It is immediate to see that $\mathrm{E}\mathcal{H}$ is itself a superenvelope and we call it the *minimal superenvelope* of $\mathcal{H}$.

We will be especially interested in *u.s.c.d.a.-sequences*: the u.s.c.d.-sequences which consist exclusively of affine functions defined on a simplex. (Again, the two examples of entropy structures mentioned above are sequences of this kind.) In [1, Theorem 4.3] it is proved that for a u.s.c.d.a.-sequence $\mathcal{H}$, $\mathrm{E}\mathcal{H}$ is equal to the pointwise infimum of all affine superenvelopes of $\mathcal{H}$ (by convention, the constant infinity function is affine).

Now assume for a moment that K is a *Bauer simplex.* If $\mathcal{H} = (h_n)$ is a u.s.c.d.a.-sequence on K then $\mathcal{H}' = (h_n')$ is clearly a u.s.c.d.-sequence defined on the compact set $\mathrm{ex}K$. By invertibility of the restriction map it is now almost immediate to see that the superenvelope $\mathrm{E}(\mathcal{H}')$ on $\mathrm{ex}K$ is equal to the restriction $(\mathrm{E}\mathcal{H})'$ of $\mathrm{E}\mathcal{H}$, in other words,

$$\mathrm{E}\mathcal{H} = I(\mathrm{E}(\mathcal{H}')),$$

and hence $\mathrm{E}\mathcal{H}$ is necessarily an affine function.

Since the theory of superenvelopes does not work on a noncompact domain, the above formula cannot be applied if the extreme set of K is not compact. However, we can modify the above approach to understand how $\mathrm{E}\mathcal{H}$ is determined by the restriction of $\mathcal{H}$ to the closure of $\mathrm{ex}K$. As in lemma 2.4, let M be the Bauer simplex consisting of all probability measures supported by $\overline{\mathrm{ex}K}$. Every u.s.c.d.a.-sequence $\mathcal{H}$ on K lifts (by composition with the barycenter map π) to a u.s.c.d.a.-sequence $\mathcal{H}''$ on M. By what was said about Bauer simplices, the minimal superenvelope $\mathrm{E}(\mathcal{H}'')$ equals $I(\mathrm{E}(\mathcal{H}'''))$, where I denotes the integral extension to M and $\mathcal{H}'''$ is the u.s.c.d.-sequence obtained as the restriction of $\mathcal{H}''$ to the extreme points of M. But this $\mathcal{H}'''$ coincides with the restriction $\mathcal{H}'$ of $\mathcal{H}$ to $\overline{\mathrm{ex}K}$, so we can replace $\mathcal{H}'''$ by $\mathcal{H}'$ and write

$$\mathrm{E}(\mathcal{H}'') = I(\mathrm{E}(\mathcal{H}'))\ .$$

Because the functions (h_n'') are constant on the preimages of points by π, it is immediate to see that the function $(\mathrm{E}(\mathcal{H}''))^{[K]}$ is a superenvelope of $\mathcal{H}$ on K. Also, it must be the minimal one since if there were a smaller one, its lift to M would become a superenvelope of $\mathcal{H}''$ at some point strictly smaller than $\mathrm{E}(\mathcal{H}'')$, contradicting the definition of $\mathrm{E}(\mathcal{H}'')$ as the pointwise infimum of all supervenvelopes. We have proved the following:

Proposition 2.7. *If $\mathcal{H}$ is a u.s.c.d.a.-sequence on a simplex K, then*

$$E\mathcal{H} = (I(E(\mathcal{H}')))^{[K]},$$

where $\mathcal{H}'$ denotes the restriction of $\mathcal{H}$ to $\overline{exK}$, $I(g)$ denotes the integral extension of a function g defined on $\overline{exK}$ onto the Bauer simplex M of all probability measures supported by $\overline{exK}$, and $f^{[K]}$ is the push-down onto K via the barycenter map of a function f defined on M.

Note, if $\mathcal{H}$ is u.s.c.d.a. on a simplex which is not Bauer, then $\mathrm{E}\mathcal{H}$ is a concave function, without necessarily being affine.

The transfinite sequence. 2.8. We recall the (transfinite) inductive characterization of $\mathrm{E}\mathcal{H}$. Given a u.s.c.d.-sequence $\mathcal{H} = (h_n)$ converging to h on a compact set K, define $\tau_n = h - h_n$. Set $u_0 \equiv 0$. Given an ordinal $\alpha \succ 0$, define

$$u_{\alpha+1} = \lim_{n\to\infty} \widetilde{u_\alpha + \tau_n}$$

and for a limit ordinal α define

$$u_\alpha = \widetilde{\sup_{\gamma \prec \alpha} u_\gamma}\ , \tag{2.9}$$

Then $u_\alpha \le u_\beta$ if $\alpha \prec \beta$. In fact, $u_\alpha = u_{\alpha+1} \iff u_\alpha = \mathrm{E}\mathcal{H} - h$; and there will be a countable ordinal α such that $u_\alpha = u_{\alpha+1}$ [1, Theorem 3.3].

Throughout the rest of this note $\mathcal{H}$ will represent an entropy structure of a system (X, T). The collection of all superenvelopes, $\mathrm{E}\mathcal{H}$ and the transfinite sequence u_α do not depend on the choice of a particular entropy structure [5, Theorem 2.3.2]. At times we will write u_α^T to denote the elements of the transfinite sequence for a system (X, T).

Asymptotic h-expansiveness. 2.10. An extension $\varphi : (Y, S) \to (X, T)$ is a *principal extension* if $h^S(\nu) = h^T(\varphi\nu)$ for every ν in $K(S)$. In the theory of symbolic extension entropy, there is a naturally distinguished class of systems, characterized by various equivalent conditions, such as the following ([1, Theorem 8.6], [5, Theorem 9.0.2]):

- (X, T) is asymptotically h-expansive [4, 10, 11]
- (X, T) has a symbolic extension which is a principal extension
- $h_{\text{sex}}^T = h^T$
- An/every entropy structure $\mathcal{H}$ on (X, T) converges uniformly to h^T
- $u_\alpha^T \equiv 0$ for every ordinal α.

Buzzi, following work of Yomdin, showed that every C^∞ system is asymptotically h-expansive [3].

3. Joinings, products, powers and flows

In this section, we will show that symbolic extension entropy satisfies the relations one would hope for with respect to joinings, powers, products (to some extent also fiber products) and flows.

Consider two systems (X', T') and (X'', T'') having a common factor (Z, U). Let $\varphi' : X' \to Z$ and $\varphi'' : X'' \to Z$ be the corresponding factor maps. The *fiber product* (X, T) of (X', T') and (X'', T'') over (Z, U) is defined as the set

$$X = \{\langle x', x''\rangle \in X' \times X'' : \varphi'(x') = \varphi''(x'')\}$$

with the product action $T(\langle x', x''\rangle) = \langle T'(x'), T''(x'')\rangle$. The direct product of two systems can be regarded as their fiber product over the trivial (one-point) factor.

Theorem 3.1 (Fiber Products)**.** *Let (X, T) be the fiber product of (X', T') and (X'', T'') over their common factor (Z, U). Then*

(1) $h_{\text{sex}}^T(\mu) \le h_{\text{sex}}^{T'}(\mu') + h_{\text{sex}}^{T''}(\mu'') - h^U(\xi)$,

where μ is an invariant measure supported on the fiber product X, while μ', μ'' and ξ are its projections onto X', X'' and Z, respectively.

Now assume also that (X'', T'') is asymptotically h-expansive and that μ is the relatively independent joining of μ' and μ'' over ξ. Then

(2) $h_{\text{sex}}^T(\mu) \ge h_{\text{sex}}^{T'}(\mu') + h_{\text{sex}}^{T''}(\mu'') - h_{\text{sex}}^U(\xi)$.

In 3.1(2), $h_{\text{sex}}^{T''}(\mu'')$ can be replaced simply by $h^{T''}(\mu'')$, because T'' is asymptotically h-expansive. The inequality 3.1(2) can fail if T'' is not asymptotically h-expansive. An appropriate example (3.6), in which additionally $h^U \equiv 0$ (hence also $h_{\text{sex}}^U \equiv 0$), is provided after the proof below of Theorem 3.1.

Theorem 3.2 (Joinings and Products)**.** *Suppose (X,T) is the product of finitely or countably many systems (X_k,T_k) such that $\sum_k \mathbf{h}_{\rm sex}(T_k) < \infty$, and $\mu \in K(T)$. Let μ_k in $K(T_k)$ be the coordinate projection of μ. Then the following hold.*

(1) $h_{\rm sex}(T,\mu) \le \sum_k h_{\rm sex}(T_k,\mu_k)$.
(2) *If μ is the product measure $\Pi_k\mu_k$, then* $h_{\rm sex}(T,\mu) = \sum_k h_{\rm sex}(T_k,\mu_k)$.
(3) $\mathbf{h}_{\rm sex}(T) = \sum_k \mathbf{h}_{\rm sex}(T_k)$.

Theorem 3.3 (Powers)**.** *Suppose $0 \neq n \in \mathbb{Z}$. Then for any system (X,T) the following hold.*

(1) *The restriction of $h_{\rm sex}(T^n,\cdot)$ to $K(T)$ equals $|n|h_{\rm sex}(T,\cdot)$.*
(2) $\mathbf{h}_{\rm sex}(T^n) = |n|\mathbf{h}_{\rm sex}(T)$.

By a *flow* we mean a continuous $\mathbb{R}$ action on a compact metric space. Given a flow T, we let T^t denote the homeomorphism which is the time t map of the flow.

Theorem 3.4 (Flows)**.** *Suppose T is a flow on X, and let a and b be distinct nonzero real numbers. Then the following hold.*

(1) *If $\mu \in K(T^a)\cap K(T^b)$, then* $h_{\rm sex}(T^a,\mu) = |a/b|h_{\rm sex}(T^b,\mu)$.
(2) $\mathbf{h}_{\rm sex}(T^a) = |a/b|\mathbf{h}_{\rm sex}(T^b)$.

Remark 3.5. It is easy to see that all the conclusions of Theorems 3.3 and 3.4, and also (2) and (3) of Theorem3.2, remain valid if $h^T_{\rm sex}$ is replaced with $h^T_{\rm res}$ and $\mathbf{h}_{\rm sex}(T)$ is replaced with $\mathbf{h}_{\rm res}(T)$.

Proof of Theorem 3.1. If (Y',S') and (Y'',S'') are symbolic extensions of (X',T') and (X'',T''), respectively, then the fiber product (Y,S) of (Y',S') and (Y'',S'') over (Z,U) provides an obvious symbolic extension of (X,T). The entropy of an invariant measure ν on Y equals at most $h(\nu')+h(\nu'')-h(\xi)$, where ν',ν'' and ξ are the projections of ν respectively onto Y', Y'' and Z. This easily implies (1).

The proof of (2) will be based on the transfinite characterization of $h_{\rm sex}$, hence we need to study relations among the entropy structures in X', X'', X and Z. Let $(\mathcal{F}_n)$, $(\mathcal{F}'_n)$ and $(\mathcal{F}''_n)$ be sequences of finite families of continuous functions on Z, X' and X'', respectively, whose associated sequences of partitions are refining in the corresponding product spaces with $[0,1]$. The lifts of these families to extending spaces will be denoted by the same letters. With this convention, the sequences of families $(\mathcal{F}'_n\cup\mathcal{F}_n)$, $(\mathcal{F}''_n\cup\mathcal{F}_n)$ and $(\mathcal{F}'_n\cup\mathcal{F}''_n\cup\mathcal{F}_n)$ also refine in X', X'' and X, respectively. So, the sequences of entropy functions with respect to these families are entropy structures on the corresponding systems. We have

$$\begin{aligned}
h^T_n(\mu) &= h_\mu(\mathcal{F}'_n\cup\mathcal{F}''_n\cup\mathcal{F}_n) = h_\mu(\mathcal{F}'_n\cup\mathcal{F}''_n|\mathcal{F}_n) + h_\xi(\mathcal{F}_n)\\
&\le h_{\mu'}(\mathcal{F}'_n|\mathcal{F}_n) + h_{\mu''}(\mathcal{F}''_n|\mathcal{F}_n) + h_\xi(\mathcal{F}_n) + h_\xi(\mathcal{F}_n) - h_\xi(\mathcal{F}_n)\\
&= h_{\mu'}(\mathcal{F}'_n\cup\mathcal{F}_n) + h_{\mu''}(\mathcal{F}''_n\cup\mathcal{F}_n) - h_\xi(\mathcal{F}_n)\\
&= h^{T'}_n(\mu') + h^{T''}_n(\mu'') - h^U_n(\xi)\ .
\end{aligned}$$

(This holds whenever μ projects to μ' and μ'', not only for their relative independent product; asymptotic h-expansiveness has not been used yet, either.) If now μ is the relatively independent joining, then $h^T(\mu) = h^{T'}(\mu')+h^{T''}(\mu'')-h^U(\xi)$, hence $\tau^T_n(\mu) \ge \tau^{T'}_n(\mu')+\tau^{T''}_n(\mu'')-\tau^U_n(\xi)$.

We will now apply transfinite induction to prove $u^T_\alpha(\mu) \ge u^{T'}_\alpha(\mu') - u^U_\alpha(\xi)$ for all ordinals α. Clearly it is true for $\alpha = 0$. Suppose it holds for some ordinal $\alpha \ge 0$.

Then

$$\begin{aligned}\left(u_\alpha^T(\mu)+\tau_n^T(\mu)\right)^\sim &\geq \left(u_\alpha^{T'}(\mu')+\tau_n^{T'}(\mu')+\tau_n^{T''}(\mu'')-u_\alpha^U(\xi)-\tau_n^U(\xi)\right)^\sim\\ &\geq \left(u_\alpha^{T'}(\mu')+\tau_n^{T'}(\mu')+\tau_n^{T''}(\mu'')\right)^\sim-\left(u_\alpha^U(\xi)+\tau_n^U(\xi)\right)^\sim .\end{aligned}$$

It is now that we invoke asymptotic h-expansiveness of (X'', S''). It implies that the $\tau_n^{T''}$ converge uniformly to zero, and therefore can be omitted in the limit. Taking limits as n goes to infinity we get $u_{\alpha+1}^T(\mu) \geq u_{\alpha+1}^{T'}(\mu') - u_{\alpha+1}^U(\xi)$. The inductive step for limit ordinals is even easier and we skip it. This concludes the induction.

Now, choosing α sufficiently large, we have

$$\begin{aligned}h_{\text{sex}}^T(\mu) &= u_\alpha^T(\mu)+h^T(\mu) \geq u_\alpha^{T'}(\mu')-u_\alpha^U(\xi)+h^{T'}(\mu')+h^{T''}(\mu'')-h^U(\xi)\\ &= h_{\text{sex}}^{T'}(\mu')+h^{T''}(\mu'')-h_{\text{sex}}^U(\xi) .\end{aligned}$$

□

Example 3.6. Let Z consist of two sequences (a_n) and (b_n) converging to a common limit c. Let U be the identity map on Z. Let (X', T') be an extension of (Z, U) obtained as the union of $\{b_1, b_2, \dots, c\}$ and a sequence of invariant sets A_n each carrying a unique measure μ'_n, of entropy 1. We arrange that the A_n shrink in diameter to zero and converge to c as n grows. The factor map from X' onto Z is defined to be the identity on $\{b_1, b_2, \dots, c\}$ and it sends each A_n to a_n. The system (X'', T'') and its factor map onto Z are defined analogously, with the roles of a_n's and b_n's exchanged. The fiber product has one measure $\mu'_n \times \delta_{a_n}$ above each a_n and one measure $\delta_{b_n} \times \mu''_n$ above each b_n. These converge to $\delta_c \times \delta_c = \delta_{\langle c,c\rangle}$. As in [5, Example 8.3.1], the sex entropy function at δ_c equals 1 in both X', X'' and also the sex entropy of $\delta_{\langle c,c\rangle}$ is 1 in X. Clearly, $h_{\text{sex}}^U(\delta_c) = 0$, hence the inequality (2) of Theorem 3.1 fails for $\mu = \delta_{\langle c,c\rangle}$.

Proof of Theorem 3.2(1). We may suppose $(X, T) = \prod_{k=1}^\infty (X_k, T_k)$. Take $\epsilon > 0$. For each k, choose a symbolic extension $\varphi_k : (Y_k, S_k) \to (X_k, T_k)$ such that

$$h_{\text{ext}}(\mu_k, \varphi_k) \ < \ h_{\text{sex}}(T_k, \mu_k) + \epsilon 2^{-k} .$$

Choose M such that $\sum_{k=M+1}^\infty \mathbf{h}_{\text{sex}}(T_k) < \epsilon$, and choose symbolic extensions $\varphi'_k : (Y'_k, S'_k) \to (X_k, T_k)$ such that $\mathbf{h}_{\text{top}}(S'_k) < \mathbf{h}_{\text{sex}}(T_k) + \epsilon 2^{-k}$. Consider the extension $\psi_M = \left(\prod_{k=1}^M \varphi_k\right) \times \left(\prod_{k=M+1}^\infty \varphi'_k\right)$. Let (W_M, R_M) denote the domain of ψ_M. Given a measure ν on the product system (W_M, R_M), let ν_k denote its coordinate projection. If $\psi_M \nu = \mu$, then

$$\begin{aligned}h(\nu, R_M) \ &\leq \ \sum_{k=1}^M h(\nu_k, S_k) + \sum_{k=M+1}^\infty h(\nu_k, S'_k)\\ &< \ \epsilon + \sum_{k=1}^M h_{\text{sex}}(T_k, \mu_k) + \sum_{M+1}^\infty \mathbf{h}_{\text{top}}(S'_k)\\ &< \ 3\epsilon + \sum_{k=1}^M h_{\text{sex}}(T_k, \mu_k) \ \leq \ 3\epsilon + \sum_{k=1}^\infty h_{\text{sex}}(T_k, \mu_k) .\end{aligned}$$

Taking the supremum over ν, we could conclude that

$$h_{\text{sex}}(T, \mu) \leq \sum_{k=1}^{\infty} h_{\text{sex}}(T_k, \mu_k)$$

except that the extension (W_M, R_M) is generally not symbolic. However, as a finite entropy product of subshifts, it is asymptotically h-expansive and thus [1, Theorem 8.6] has a principal symbolic extension $\pi : (Y', S') \to (Y, S)$, with $h(\nu', S') = h(\nu, S)$ for every ν' in $K(S')$ such that $\pi\nu' = \nu$. The symbolic extension $\psi_M \pi : (Y', S') \to (X, T)$ defines the same extension entropy function as does ψ_M. This completes the proof of 3.2(1). □

Proof of Theorem 3.2(2). We must show $h_{\text{sex}}(T, \Pi_k \mu_k) \geq \sum_k h_{\text{sex}}(T_k, \mu_k)$. Without loss of generality, it suffices to show this when (X, T) is the product of just two systems. Let $(X, T) = (X', T') \times (X'', T'')$ and $\mu = \mu' \times \mu''$.

As in the proof of the preceding theorem, let $(\mathcal{F}'_n)$ and $(\mathcal{F}''_n)$ be sequences of families of continuous functions giving rise to entropy structures on (X', T') and (X'', T''), respectively. Then $(\mathcal{F}'_n \cup \mathcal{F}''_n)$ (union of lifted families) gives rise to an entropy structure in the product system (X, T).

We have simply $h^T(\mu) = h^{T'}(\mu') + h^{T''}(\mu'')$ and $h^T_n(\mu) = h^{T'}_n(\mu') + h^{T''}_n(\mu'')$, hence also $\tau^T_n(\mu) = \tau^{T'}_n(\mu') + \tau^{T''}_n(\mu'')$. We will complete the proof by showing, using transfinite induction, that for every α

$$u^T_\alpha(\mu) \geq u^{T'}_\alpha(\mu') + u^{T''}_\alpha(\mu'') \ .$$

This inequality is trivial for $\alpha = 0$. Next we suppose it holds for some ordinal α. Consider a given n in $\mathbb{N}$. Pick sequences $\mu'_k \to \mu'$ and $\mu''_k \to \mu''$ such that

$$\lim_k (u^{T'}_\alpha + \tau^{T'}_n)(\mu'_k) = (\widetilde{u^{T'}_\alpha + \tau^{T'}_n})(\mu') \ ,$$
$$\lim_k (u^{T''}_\alpha + \tau^{T''}_n)(\mu''_k) = (\widetilde{u^{T''}_\alpha + \tau^{T''}_n})(\mu'') \ .$$

Then

$$\begin{aligned}
(\widetilde{u^T_\alpha + \tau^T_n})(\mu) &\geq \limsup_k (u^T_\alpha + \tau^T_n)(\mu'_k \times \mu''_k) \\
&\geq \limsup_k \left(u^{T'}_\alpha(\mu'_k) + u^{T''}_\alpha(\mu''_k) + \tau^{T'}_n(\mu'_k) + \tau^{T''}_n(\mu''_k) \right) \\
&= (\widetilde{u^{T'}_\alpha + \tau^{T'}_n})(\mu') + (\widetilde{u^{T''}_\alpha + \tau^{T''}_n})(\mu'') \ .
\end{aligned}$$

Because $u_{\alpha+1} = \lim_n (\widetilde{u_\alpha + \tau_n})$ by definition, we conclude that

$$u^T_{\alpha+1}(\mu) \geq u^{T'}_{\alpha+1}(\mu') + u^{T''}_{\alpha+1}(\mu'') \ .$$

If α is a limit ordinal such that $u^T_\gamma(\mu) \geq u^{T'}_\gamma(\mu') + u^{T''}_\gamma(\mu'')$ when $\gamma \prec \alpha$, then the desired inequality follows directly from the definition (2.9) of u^T_α. This completes the transfinite induction, and the proof. □

Proof of Theorem 3.2(3). We have

$$\begin{aligned}\mathbf{h}_{\mathrm{sex}}(T) &= \sup\{h_{\mathrm{sex}}(T,\mu) : \mu \in K(T)\} \\ &= \sum_k \sup\{h_{\mathrm{sex}}(T_k,\mu_k) : \mu_k \in K(T_k)\} \\ &= \sum_k \mathbf{h}_{\mathrm{sex}}(T_k)\ .\end{aligned}$$

The first and last equalities hold by the Sex Entropy Variational Principle 2.1. The middle equality follows from 3.2(1) and 3.2(2). □

Proof of Theorem 3.3(1). Suppose $\mu \in K(T)$. Because T and T^{-1} define the same sex entropy function, we may assume $n > 0$.

Now suppose $\varphi : (Y,S) \to (X,T)$ is a symbolic extension. Let $\varphi^{(n)}$ denote the symbolic extension $(Y,S^n) \to (X,T^n)$ defined by the same map $X \to Y$. Suppose $\nu \in K(S)$ and $\varphi\nu = \mu$. Then $\nu \in K(S^n)$ and $h_\nu(S^n) = nh_\nu(S)$. Therefore $nh_{\mathrm{ext}}(\mu,\varphi) \le h_{\mathrm{ext}}(\mu,\varphi^{(n)})$. It follows that $nh_{\mathrm{sex}}(T,\mu) \le h_{\mathrm{sex}}(T^n,\mu)$.

Conversely, suppose $\psi : (Y,S) \to (X,T^n)$ is a symbolic extension. Define $Y' = Y \times \{0,1,\dots,n-1\}$ and define $S' : Y' \to Y'$ by the following rules: $(y,i) \mapsto (y,i+1)$ if $0 \le i < n-1$, and $(y,i) \mapsto (S^n y, 0)$ otherwise. Define $\psi' : Y' \to X$ by $(y,i) \mapsto T^i\varphi y$. Then $\psi' : (Y',S') \to (X,T)$ is a symbolic extension. Let ν' be the product of ν with equidistributed probability measure on $\{0,1,\dots,n-1\}$. Then $\nu' \in K(S')$, $\psi'\nu' = \mu$ and $nh_{\nu'}(S') = h_\nu(S)$. Therefore $h_{\mathrm{ext}}(\mu,\psi) \le nh_{\mathrm{ext}}(\mu,\psi')$. It follows that $h_{\mathrm{sex}}(T^n,\mu) \le nh_{\mathrm{sex}}(T,\mu)$. This completes the proof of 3.3(1). □

Proof of Theorem 3.3(2). This follows from 3.3(1) and the Sex Entropy Variational Principle [1, Theorem 8.1], or by reconsidering the constructions in the proof 3.3(1). □

Proof of Theorem 3.4(1). Without loss of generality, we assume $0 < a < b = 1$. By Theorem 3.3, we may also assume that a is irrational.

Let $([0,1), R_a)$ denote the rotation of the circle $x \mapsto x + a \bmod 1$. (Here, $[0,1)$ is our additive notation for the circle.) For a definite notation, we spell out a standard construction of a symbolic extension $\varphi_a : (W_a, S_a) \to ([0,1), R_a)$. Choose any x in $[0,1)$ whose R_a-orbit avoids 0. Define a doubly infinite sequence $\tilde{x}$ by setting $\tilde{x}[k] = 1$ if $1 - a < (R_a)^k x < 1$, and setting $\tilde{x}[k] = 0$ otherwise. Then W_a is the orbit closure under the shift of $\tilde{x}$, and the extension φ_a is uniquely determined by requiring $\varphi(\tilde{x}) = x$.

Now suppose $\varphi : (Y,S) \to (X,T^1)$ is a symbolic extension. Define a skew product $S' : W_a \times Y \to W_a \times Y$ by the rule $(x,y) \mapsto (S_a x, S^{x[0]}y)$. Define the map $\varphi' : W_a \times Y \to X$ by $(x,y) \mapsto T^t(\varphi y)$, where $t = \varphi_a(x)$. The map φ' intertwines S' and T^a, giving a symbolic extension $\varphi' : (W_a \times Y, S') \to (X, T^a)$. If $\nu' \in K(S')$ and ν is its image under the coordinate projection $(y,x) \mapsto y$, then $\nu \in K(S)$ and $h_{\nu'}(S') = ah_\nu(S)$. Considering the case $\varphi\nu = \mu$, we conclude that $h_{\mathrm{sex}}(T^a,\mu) \le ah_{\mathrm{sex}}(T^1,\mu)$.

Now choose n in $\mathbb{N}$ such that $na > 1$. The previous argument shows that

$$h_{\mathrm{sex}}(T^1,\mu) \le \frac{1}{na} h_{\mathrm{sex}}(T^{na},\mu)\ .$$

By Theorem 3.3(3) we have $h_{\mathrm{sex}}(T^{na},\mu) = nh_{\mathrm{sex}}(T^a,\mu)$. Therefore $h_{\mathrm{sex}}(T^1,\mu) \le \frac{1}{a} h_{\mathrm{sex}}(T^a,\mu)$. Thus $h_{\mathrm{sex}}(T^a,\mu) = ah_{\mathrm{sex}}(T^1,\mu)$. □

Proof of Theorem 3.4(2). Given a symbolic extension $\varphi : (Y, S) \to (X, T^1)$ and $0 < a < 1$, construct the symbolic extension $\varphi' : (Y \times W_a, S') \to (X, T^a)$ as for part (1). Then $\mathbf{h}_{\text{top}}(S') = a\mathbf{h}_{\text{top}}(S)$, so $\mathbf{h}_{\text{sex}}(T^a) \leq a\mathbf{h}_{\text{sex}}(T^1)$. The reverse inequality is established as in the proof for part (1). $\square$

4. The Example of Misiurewicz

M. Misiurewicz gave the first construction of C^r $(1 \leq r < \infty)$ systems with no measure of maximal entropy (hence not asymptotically h-expansive). In this section, we will show for his construction that the sex entropy function is simply the upper semicontinuous envelope of the entropy function. In particular, the topological residual entropy is still zero.

Fix the positive integer r. The Misiurewicz example (which gives a family of examples due to some freedom in the construction) is constructed in [10, Theorem 1] on the product $V \times S^1$, where V is a certain C^∞ manifold, with D the time-one map of a certain vector field. ($V \times S^1$ is our notation for the space denoted $\widetilde{M} \times S^1 = M$ in [10, p.907], and D is our notation for $\varphi^{(1)}$ there.) The sets $V_y = V \times \{y\}$ are D-invariant and we will call them *fibers*. There is a distinguished element of S^1, which we denote as 0, because it is 0 in the local chart in [10]. For any closed interval J not containing 0, the restriction of D to $V \times J$ is C^∞, and therefore asymptotically h-expansive. Let D_y denote the restriction of D to V_y. Every D_y is C^∞; $\mathbf{h}_{\text{top}}(D_0) = 0$; and $\limsup_{y \to 0} \mathbf{h}_{\text{top}}(D_y) \geq 1/(r+1)$. The maximum of $\widetilde{h^D}$ is achieved on the fiber V_0.

Example 4.1. Given a positive integer r, let D be the C^r Misiurewicz map described above. Then $h^D_{\text{sex}} = \widetilde{h^D}$. In particular, the topological residual entropy of D is zero.

Proof. Let $\mathcal{H} = (h_n)$ be a u.s.c.d.a.-sequence which is an entropy structure for D, with limit $h = h^D$. In order to prove that $h^D_{\text{sex}} = \widetilde{h}$ it suffices to prove $E(\mathcal{H}') = \widetilde{h}'$ for the restrictions $\mathcal{H}' = (h'_n)$ and h' of $\mathcal{H}$ and h, respectively, to $\overline{\text{ex}K(D)}$. (This reduction follows from Lemma 2.4 and Proposition 2.7, because h is harmonic and $h^D_{\text{sex}} = \text{E}\mathcal{H}$.) It suffices to show that each function $\widetilde{h}' - h'_n$ is u.s.c. on $\overline{\text{ex}K(D)}$.

Clearly, every ergodic measure is supported by one fiber. Moreover (and this is a key observation) the same holds for measures in the closure of ergodic measures – every such measure is supported by a single fiber. Thus we have a natural map $P : \overline{\text{ex}K(D)} \to S^1$ which is almost obviously a continuous surjection. Every measure $\mu \in \overline{\text{ex}K(D)}$ projecting by P to a nonzero $y \in S^1$ has a closed neighborhood (preimage by P of a closed arc J not containing zero) on which $\mathcal{H}'$ converges uniformly; hence, in a neighborhood of such a μ, $E(\mathcal{H}') = h = h'$. Because $E(\mathcal{H}')$ is u.s.c., it follows that in a neighborhood of such a μ we have $\widetilde{h}' = h'$, so $\widetilde{h}' - h_n = E(\mathcal{H}') - h'_n$, which is u.s.c.

Finally, suppose $P(\mu) = 0$. Then $(\widetilde{h}' - h'_n)(\mu)$ equals $\widetilde{h}'(\mu)$, and the restriction of $\widetilde{h}'$ to V_0 is u.s.c. Thus it remains to verify u.s.c. behavior of $\widetilde{h}' - h'_n$ along a sequence (μ_k) converging to μ when for every k, $P(\mu_k) \neq 0$. With such a choice of μ_k, we have $(\widetilde{h}' - h_n)(\mu_k) = (h' - h_n)(\mu_k) \leq h'(\mu_k)$, so the upper limit over k of these values is not larger than $\widetilde{h}'(\mu)$ which equals $(\widetilde{h}' - h'_n)(\mu)$, and upper semi-continuity holds. $\square$

5. C^r systems with positive residual entropy

We will modify the Misiurewicz example of the last section to construct C^r ($1 \le r < \infty$) systems with positive topological residual entropy. We will also be using ideas from [9] and [4, pp. 170-172].

As in the last section, 0 will denote the distinguished element of S^1. We begin with a lemma in which we produce a system with entropy function "complementary" to that of the previous example.

Lemma 5.1. *There is a compact manifold W and a C^∞ diffeomorphism $R : W \times S^1 \to W \times S^1$ with the following properties.*

- *Each set $W_y := W \times \{y\}$ is R-invariant.*
- *Let $R_y : W \to W$ be defined by $R_y(w) = R(w, y)$.*
  Then R_y has zero entropy if $y \neq 0$.
- $\mathbf{h}_{\text{top}}(R_0) > 0$.

Proof. We construct R by a slight elaboration of another Misiurewicz example [9]. Let $\overline{R}$ be a C^∞ diffeomorphism of positive entropy on a compact manifold $\overline{X}$. Let W_1 be the mapping torus of $\overline{R}$ (W_1 is the quotient of $\overline{X} \times \mathbb{R}$ obtained by identifying (x, s) and $(\overline{R}^n x, s - n)$ when $x \in \overline{X}$, $n \in \mathbb{Z}$ and $s \in \mathbb{R}$). Let $\alpha^t : W_1 \to W_1$ denote the time t map of the suspension flow on W_1 (induced by $(x, s) \mapsto (x, s + t)$ on $\overline{X} \times \mathbb{R}$). Let β be a C^∞ orientation-preserving flow on S^1, with time t map β^t, such that the flow (and therefore each β^t with $t \neq 0$) has a unique fixed point, p. Let $f : S^1 \to [0, 1]$ be a surjection of class C^∞ such that $f = 0$ in a neighborhood of p. Let $g : S^1 \to [0, 1]$ be a surjection of class C^∞ such that $g(0) = 0$ and otherwise $g(z) > 0$. Define $W = (W_1 \times S^1)$ and define $R : W \times S^1 \to W \times S^1$ by the rule

$$R : (w_1, z, y) \mapsto (\alpha^{f(z)}(w_1), \beta^{g(y)}(z), y) \ .$$

If $y \neq 0$, then the nonwandering set of R_y is contained in $W_1 \times \{p\}$, on which R_y acts by the identity, so for $y \neq 0$ we have $\mathbf{h}_{\text{top}}(R_y) = 0$. On the other hand, $R_0 : (w_1, z) \mapsto (\alpha^{f(z)}, z)$. Because $f(z) = 1$ for some z, $\mathbf{h}_{\text{top}}(R_0) \ge \mathbf{h}_{\text{top}}(\overline{R}) > 0$. □

We are now in a position to explicitly produce a smooth system with positive residual entropy.

Example 5.2. Let r be a positive integer. Let V, D, V_y, D_y and 0 be as in Section 4. Let W, R, W_y and R_y be as in Lemma 5.1. Let $X = V \times W \times S^1$, and set

$$T : \big(v, w, y\big) \mapsto \big(D_y(v), R_y(w), y\big) \ .$$

Then (X, T) is a C^r system with positive topological residual entropy.

Proof. Notice that (X, T) is in fact the fiber product of $(V \times S^1, D)$ and $(W \times S^1, R)$ over (S^1, Id) via the projections $(v, y) \mapsto y$ and $(w, y) \mapsto y$. For each y, either D_y or V_y has zero entropy; so, if μ in $K(T)$ projects to μ' in $K(D)$ and μ'' in $K(R)$, then

$$h^T(\mu) \le h^R(\mu') + h^D(\mu'') \le \max\{\mathbf{h}_{\text{top}}(R), \mathbf{h}_{\text{top}}(D)\}$$

and therefore $\mathbf{h}_{\text{top}}(T) = \max\{\mathbf{h}_{\text{top}}(R), \mathbf{h}_{\text{top}}(D)\}$. From Section 4, we know that $h^D_{\text{sex}} = \widetilde{h^D}$. Also, R is asymptotically h-expansive (because it is C^∞), and $h^R_{\text{sex}} = h^R$. Therefore, because the identity map on S^1 has entropy zero, we may deduce from the two inequalities of Theorem 3.1 that

$$h^T_{\text{sex}}(\mu) = \widetilde{h^D}(\mu') + h^R(\mu'')$$

whenever μ is the relatively independent joining of μ' and μ''. The supremum of $\widetilde{h^D}$ over V_0 is $\mathbf{h}_{\rm top}(D)$, and the supremum of h^R over W_0 is $\mathbf{h}_{\rm top}(R)$. Thus, choosing measures μ' and μ'' on those fibers realizing these suprema, and letting μ be their relatively independent joining (i.e. $\mu' \times \mu''$ extended trivially to X), we obtain

$$\mathbf{h}_{\rm sex}(T) \geq h^T_{\rm sex}(\mu) = \mathbf{h}_{\rm top}(D) + \mathbf{h}_{\rm top}(R) > \mathbf{h}_{\rm top}(T) \ .$$

$\square$

Remark 5.3. The approach above of joining D and R is evident in a simple and instructive example of Denker-Grillenberger-Sigmund [4, pp.170-172]. The DGS example is of a zero-dimensional system for which there is a positive constant c such that $h \equiv c$, but the system still is not asymptotically h-expansive. An analysis of the DGS example as above shows that it has topological residual entropy equal to c. (In fact, for the DGS example, the analysis can be simplified. In that example, the set of invariant measures is a Bauer simplex, and to compute $h_{\rm sex}$ one can restrict the analysis to ergodic measures [1, Theorem 8.3].)

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Department of Mathematics, University of Maryland, College Park, MD 20742-4015, U.S.A.

E-mail address: mmb@math.umd.edu

URL: www.math.umd.edu/∼mmb

Institute of Mathematics, Technical University, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: downar@im.pwr.wroc.pl

URL: www.im.pwr.wroc.pl/∼downar