MATH 411 **** Spring 2012 **** Extra Homework Problems

Problem X. From the origin in the plane, travel distance 1 to the east,then 1/2 to the north, then 1/4 to the east, then 1/8 to the north, then 1/16 to the east, etc., forever. What is the point (a,b) you are approaching in the plane?

Problem A. Define a function f from $\mathbb{R}^2$ to $\mathbb{R}$ by the rule

$$\begin{aligned} f(x,y) &= \frac{x^2y^2}{x^2+y^2} \qquad \text{if } (x,y) \neq (0,0) \\ f(0,0) &= 0 \ . \end{aligned}$$

Determine whether f is continuous at the origin, $(0,0)$, and give a proof justifying your answer.
(Hint: for simplicity, polar coordinates may help.)

Problem C. Define functions $f : \mathbb{R} \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ by the rules

$$f(t) = \begin{pmatrix} 1+t \\ 2-3t \end{pmatrix} \qquad g(x,y) = x^2 + y^3x \qquad h(t) = g(f(t)) \ .$$

1. Compute the derivative matrix of f at 0.
2. Compute the derivative matrix of g at $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
3. Use the Chain Rule to compute the derivative of h at 0.
4. Define $u = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Compute the directional derivative

$$\frac{\partial g}{\partial u}(p)$$

where p is the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem B. Suppose $\mathcal{U}$ is an open subset in $\mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}$, and the derivative of f exists and vanishes at every point in $\mathcal{U}$. Let E_c be the set of points in $\mathcal{U}$ at which f assumes the value c.

1. Show that E_c is open as a subset of $\mathcal{U}^{\star\star\star}$.
2. Show that E_c is a closed subset of $\mathcal{U}^{\star\star\star}$.
3. Prove that if $\mathcal{U}$ is connected, then f is constant on $\mathcal{U}$.
4. Suppose now that g and h are differentiable on $\mathcal{U}$ and $Dg(p) = Dh(p)$ at every p in $\mathcal{U}$. Suppose that $\mathcal{U}$ is connected. Prove that there is a number c such that $g(p) = h(p) + c$, for all p in $\mathcal{U}$.

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*** COMMENT:

Here we are considering $\mathcal{U}$ as a metric space. Let V be a subset of $\mathcal{U}$.

V is open in $\mathcal{U}$ if for every v in V, there exists $\epsilon > 0$ such that $\{x \in \mathcal{U} : \text{distance}(x, v) < \epsilon\}$ is contained in V. Because $\mathcal{U}$ is open in $\mathbb{R}^n$, if the ϵ chosen for v is small enough, then

$$\{x \in \mathcal{U} : \text{distance}(x, v) < \epsilon\} = \{x \in \mathcal{R}^n : \text{distance}(x, v) < \epsilon\} \ .$$

So, V is an open subset of $\mathcal{U}$ if and only if V is an open subset of $\mathbb{R}^N$.
(To show E_c is open above, show that if p is in E_c then there is an open ball around p also contained in E_c.)

For V to be closed in $\mathcal{U}$, recall there are two equivalent conditions:

1. $\{x \in \mathcal{U} : x \notin V\}$ is an open subset of $\mathcal{U}$.
2. If a sequence of points in $\mathcal{V}$ converges to a point x in $\mathcal{U}$, then x is in V.

So, for some $\mathcal{U}$, a set V can be closed in $\mathcal{U}$ but not in $\mathbb{R}^n$. For example, let $\mathcal{U}$ be the union of two open balls in $\mathbb{R}^n$, B and B', which have empty intersection. Then each is both open and closed in $\mathcal{U}$.
In Problem B, to show E_c is closed in $\mathcal{U}$, it's easiest to use condition 2.

Problem D. In this problem you can prove a fundamental fact of information theory using Lagrange multipliers.

- Define the function $f : [0, 1] \to \mathbb{R}$ by the rule

$$f(x) = -x\ln(x) \qquad \text{if } 0 < x \leq 1$$
$$f(0) = 0 \ .$$

- Let Δ_n be the set of points $(x_1, ..., x_n)$ in $\mathbb{R}^n$ such that $x_i \geq 0$ for all i and $x_1 + x_2 + \cdots + x_n = 1$. Let g be the function defined on Δ_n by the rule $g(x) = \sum_{i=1}^{n} f(x_i)$.

1. Prove that f is continuous at 0. (Then, obviously, the function g is continuous on Δ_n.)
2. For $n \geq 1$, prove the following by induction on n: the maximum value of g on Δ_n is $\ln(n)$, and it is attained only at the input x for which each $x_i = 1/n$.

Problem E. In this problem, let $B(n, R)$ denote $\{x \in \mathbb{R}^n : ||x|| \leq R\}$. For a positive integer n, let c_n denote the constant such that the n-dimensional volume of $B(n, R)$ equals $c_n R^n$.

1. Use polar coordinates and the Fubini theorem to prove the following claim: if $n \geq 3$, then $c_n = (2\pi/n)c_{n-2}$.
2. For positive integers k, give a formula for c_{2k} and a formula for c_{2k+1}.
3. When $n = 10$, approximately what fraction of the volume of the box $\{x \in \mathbb{R}^{10} : |x_i| = 1,\ 1 \leq i \leq n\}$ is occupied by the unit ball $B(10, 1)$?