Grad-Shafranov Solver using Physics-Informed Neural Networks

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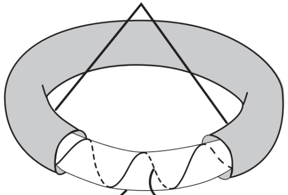
Grad-Shafranov Equation

$$X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\psi}{\partial X}\right)+\frac{\partial^2\psi}{\partial Y^2}=-\mu_0\frac{R_0}{\Psi_0^2}x^2\frac{dp}{d\psi}-\frac{R_0^2}{\Psi_0^2}F\frac{dF}{d\psi}$$

Ideal MHD Equilibrium:

$$J\times B=\nabla p$$

Symmetry Direction:



Divergence Free:

$$\nabla\cdot B=0$$

GS Equation and Hick's Equation

Grad-Shafranov Equation

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2}=-\mu_0 x^2\frac{dp}{d\psi}-F\frac{dF}{d\psi}$$

Hick's Equation

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2}=r^2\frac{dH}{d\psi}-\Gamma\frac{d\Gamma}{d\psi}$$

ψ	$pressure$	F
Flux Function (B)	p	$F = rB_\theta$
Stream Function (v)	H	$\Gamma = rv_\theta$

Grad-Shafranov Equation

$$X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\psi}{\partial X}\right)+\frac{\partial^2\psi}{\partial Y^2}=-\mu_0\frac{R_0}{\Psi_0^2}x^2\frac{dp}{d\psi}-\frac{R_0^2}{\Psi_0^2}F\frac{dF}{d\psi}$$

2D MHD equilibrium

Two free functions:

- p (Pressure)
- F (Current)

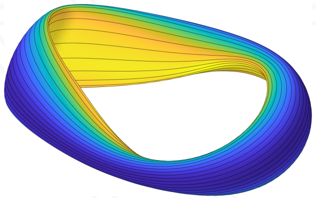
Flux Function:

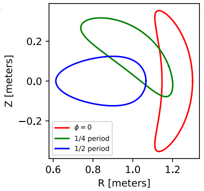
$$\psi = -\mathrm{r}A_\theta$$

$$B = \nabla \times A$$

Applications(Fusion Reactor):

- Active Control
- Reactor Design





Grad-Shafranov Equation

J. P. Freidberg, "Ideal MHD" (2014)
ISBN: 978-1-107-00625-6
J. Wesson, "Tokamaks" (1997)
ISBN: 978-0-19-856293-1

2D MHD equilibrium

Two free functions:

- p (Pressure)
- F (Current)

$$X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\psi}{\partial X}\right)+\frac{\partial^2\psi}{\partial Y^2}=-\mu_0\frac{R_0}{\Psi_0^2}X^2\frac{dp}{d\psi}-\frac{R_0^2}{\Psi_0^2}F\frac{dF}{d\psi}$$

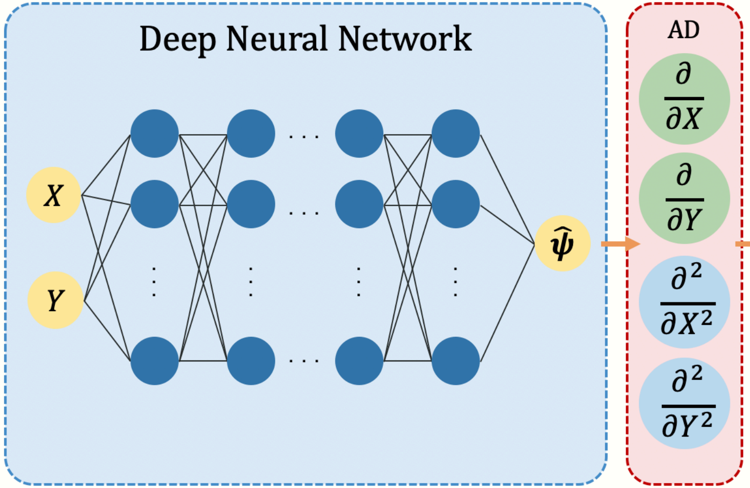
$$\mu_0\frac{R_0}{\Psi_0^2}X^2\frac{dp}{d\psi}=-C \qquad \frac{R_0^2}{\Psi_0^2}F\frac{dF}{d\psi}=-A$$

⬅ Solov'ev Profile

$$\boxed{A+C=1}$$

$$\boxed{X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\psi}{\partial X}\right)+\frac{\partial^2\psi}{\partial Y^2}=CX^2+A} \rightarrow \boxed{X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\psi}{\partial X}\right)+\frac{\partial^2\psi}{\partial Y^2}=(1-A)X^2+A}$$

Physics-Informed Neural Network



Loss Terms (Residual Network)

1. Governing Equations

$$\mathcal{R}(X,Y,\hat{\psi}) = X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\hat{\psi}}{\partial X}\right) + \frac{\partial^2\hat{\psi}}{\partial Y^2} - \alpha - (1-\alpha)X^2$$

$$\mathcal{L}_{PDE} = \frac{1}{N_{PDE}}\sum_{j=1}^{N_{PDE}}\left|\mathcal{R}(X_j, Y_j, \hat{\psi})\right|^2 \ on\ \mathcal{D}$$

2. Boundary Condition (Fixed Boundary)

$$\hat{\psi}(X,Y) - \psi_b = 0 \ \ on\ \partial\mathcal{D}$$

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}}\sum_{j=1}^{N_{BC}}\left|\hat{\psi}(X,Y) - \psi_b\right|^2 \ on\ \partial\mathcal{D}$$

$$L_{total} = \lambda_{PDE}\mathcal{L}_{PDE} + \lambda_{BC}\mathcal{L}_{BC} + (\lambda_{Data}\mathcal{L}_{Data})$$

M. Raissi, P. Perdikaris, G. Karniadakis, *J. Comp. Phys.* **2019**, 378:686–707 `(PINN)`
TensorFlow https://www.tensorflow.org/ `(ML/AI platform)`
L. Lu, X. Meng, Z. Mao, G, Karniadakis, *SIAM Rev.* **2021**, 63:208–228 `(PINN Wrapper)`

Data vs. Collocation Points

- Data
    - Ground Truth of the solution from experiments or numerical simulations

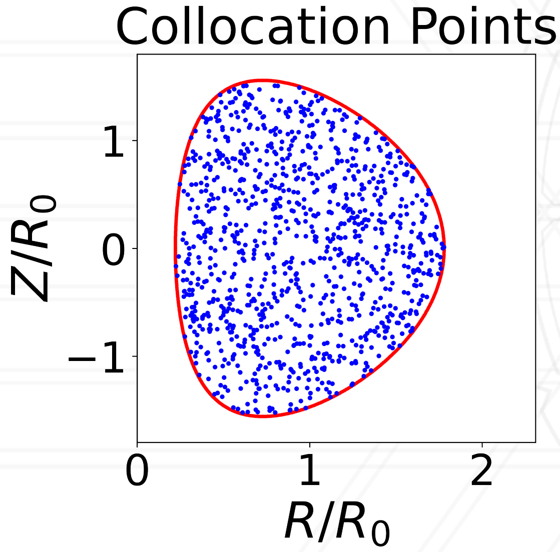
$$\mathcal{L}_{Data} = \frac{1}{N_{Data}} \sum_{j=1}^{N_{Data}} \left|\hat{\psi}(X,Y) - \psi_{true}\right|^2 \; on\ Data\ Points$$

- Collocation Points
    - Points inside a domain where residual of governing equation is calculated

$$\mathcal{L}_{DE} = \frac{1}{N_{DE}} \sum_{j=1}^{N_{DE}} |\mathcal{R}(in, sol_{nn})|^2 \; on\ \mathcal{D}$$

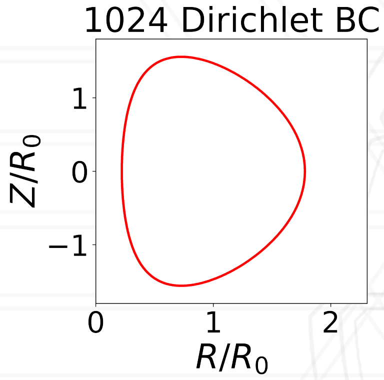
$$\mathcal{R}(X,Y,\hat{\psi}) = X\frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial\hat{\psi}}{\partial X}\right) + \frac{\partial^2\hat{\psi}}{\partial Y^2} - \alpha - (1-\alpha)X^2$$

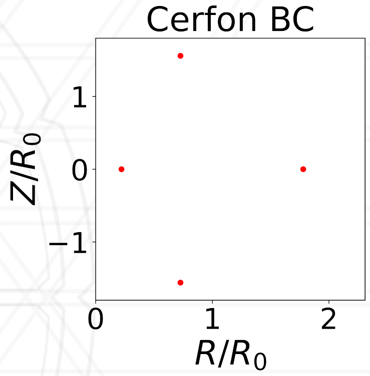
Collocation Points



- Collocation points are where the PDE residuals are evaluated
- Possible to add data from numerical simulation or experiment

Boundary Conditions





$$X_b = 1 + \varepsilon \cos(\tau + \alpha \sin \tau) \qquad Y_b = \varepsilon\kappa \sin(\tau)$$

$$\hat{\psi}(X, Y) - \psi_b = 0 \;\; on\ \partial\mathcal{D}$$

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}} \sum_{j=1}^{N_{BC}} \left|\hat{\psi}(X, Y) - \psi_b\right|^2$$

$$\psi(1+\epsilon) = 0 \qquad \psi_{YY}(1+\epsilon) = -N_1\, \psi_X(1+\epsilon)$$

$$\psi(1-\epsilon) = 0 \qquad \psi_{YY}(1-\epsilon) = -N_2\, \psi_X(1-\epsilon)$$

$$\psi(1-\delta\epsilon) = 0 \qquad \psi_X(1-\delta\epsilon) = 0$$

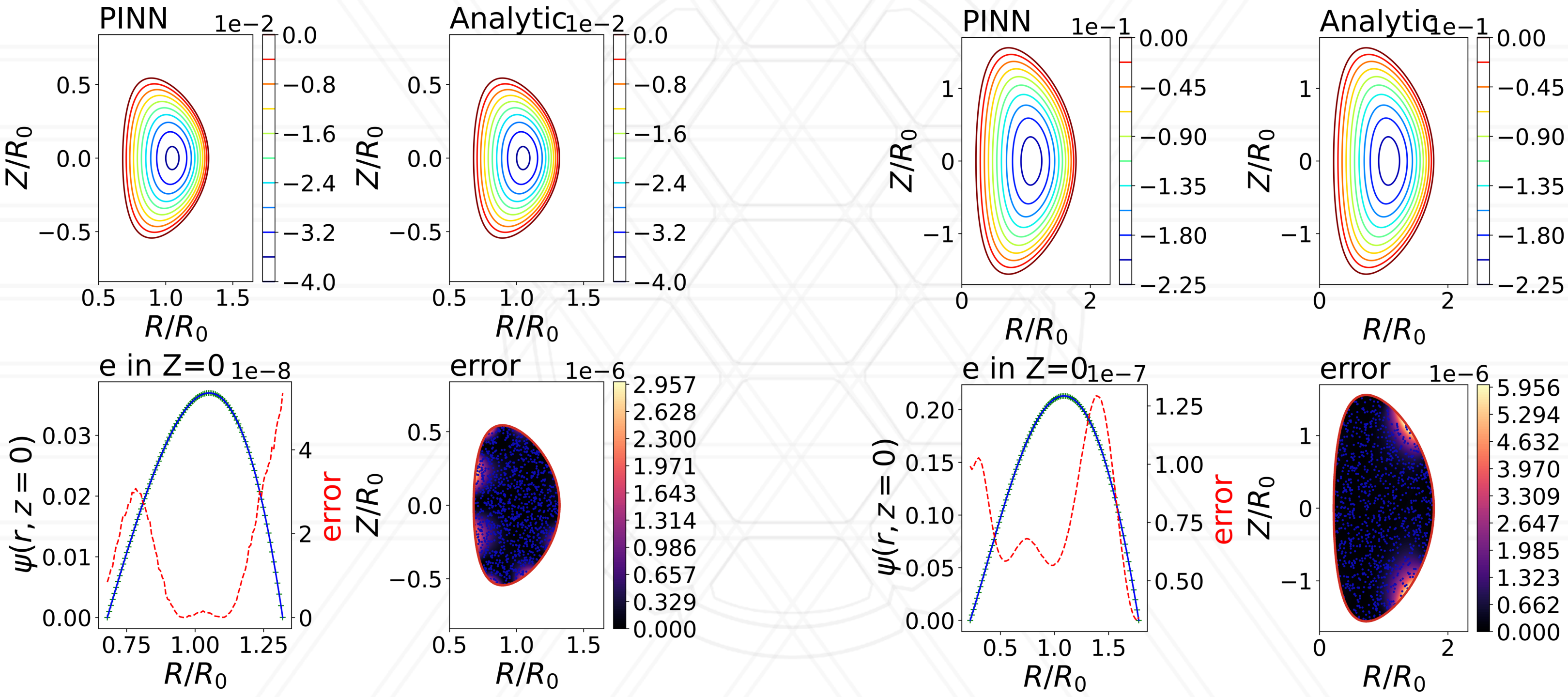
$$\psi_{XX}(1-\delta\epsilon) = -N_3\, \psi_Y(1-\delta\epsilon)$$

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}} \sum_{j=1}^{N_{BC}} \left|\hat{\psi}(X, Y) - \psi_b\right|^2$$

Up-Down Symmetric Configurations

Using 1024 Points
(Dirichlet Boundary Condition)

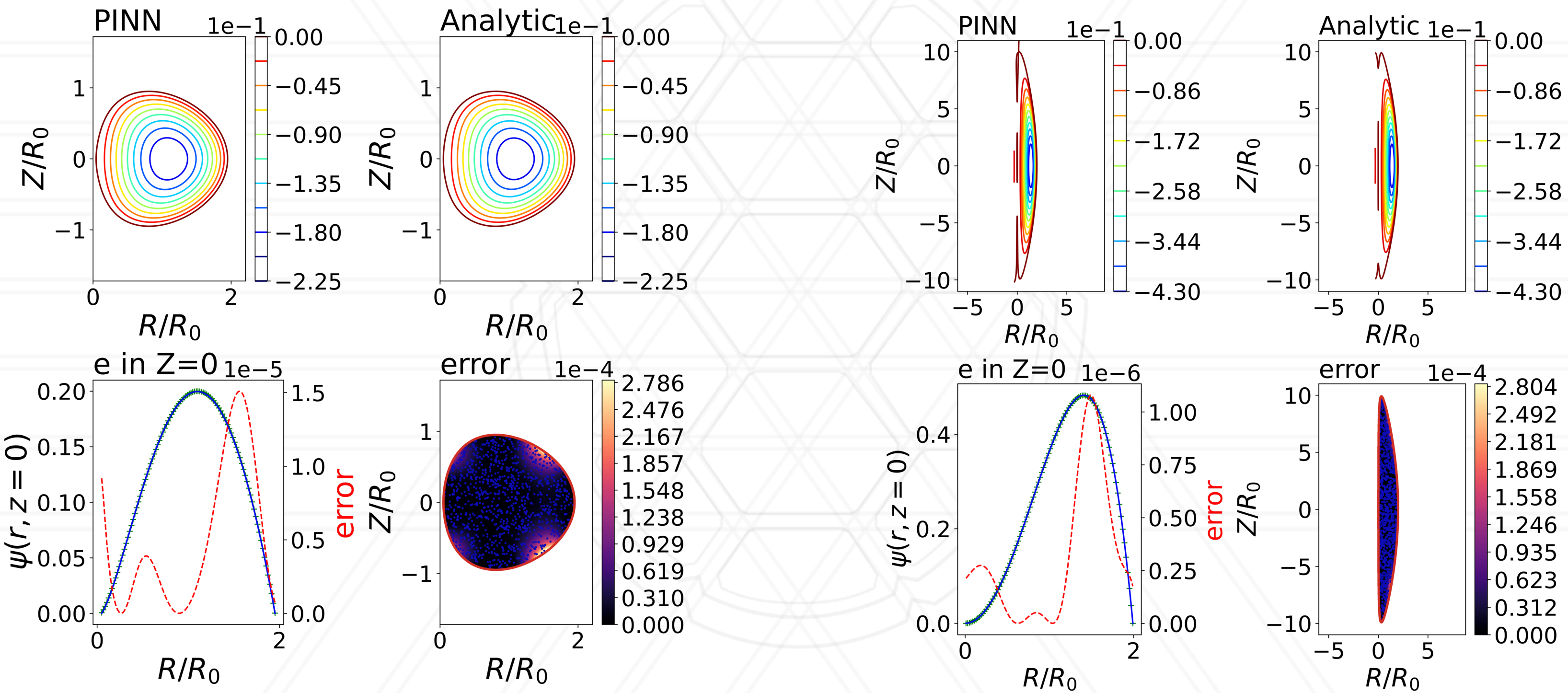
Results – ITER and NSTX



Error

$$error = \left(\psi_{PINN} - \psi_{analytic}\right)^2 / \psi_a^2 \text{ where}$$
$$\psi_a = \psi_{PINN}\ in\ magnetic\ axis$$

Results – Spheromak and FRC

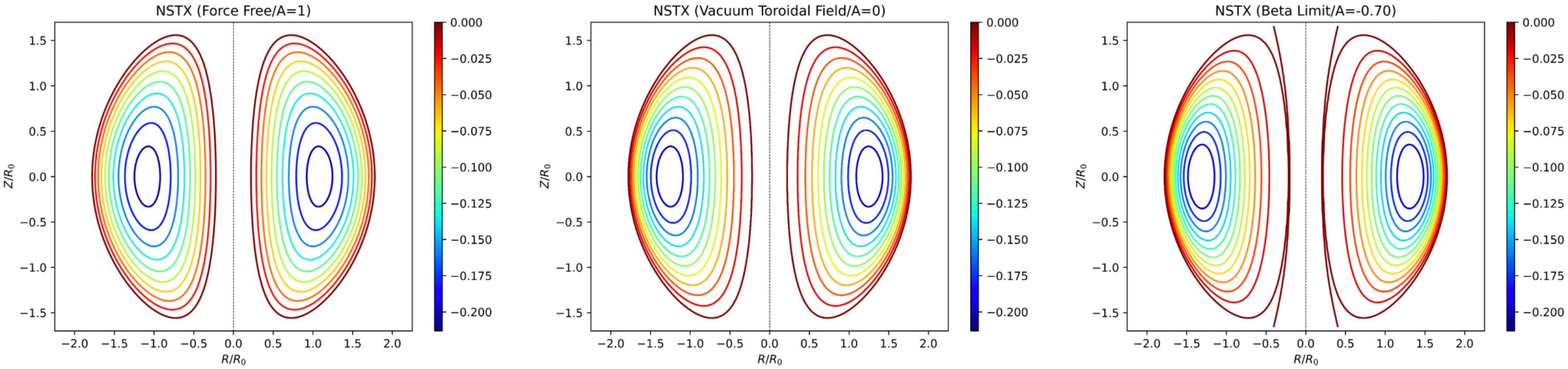


Shafranov Shift

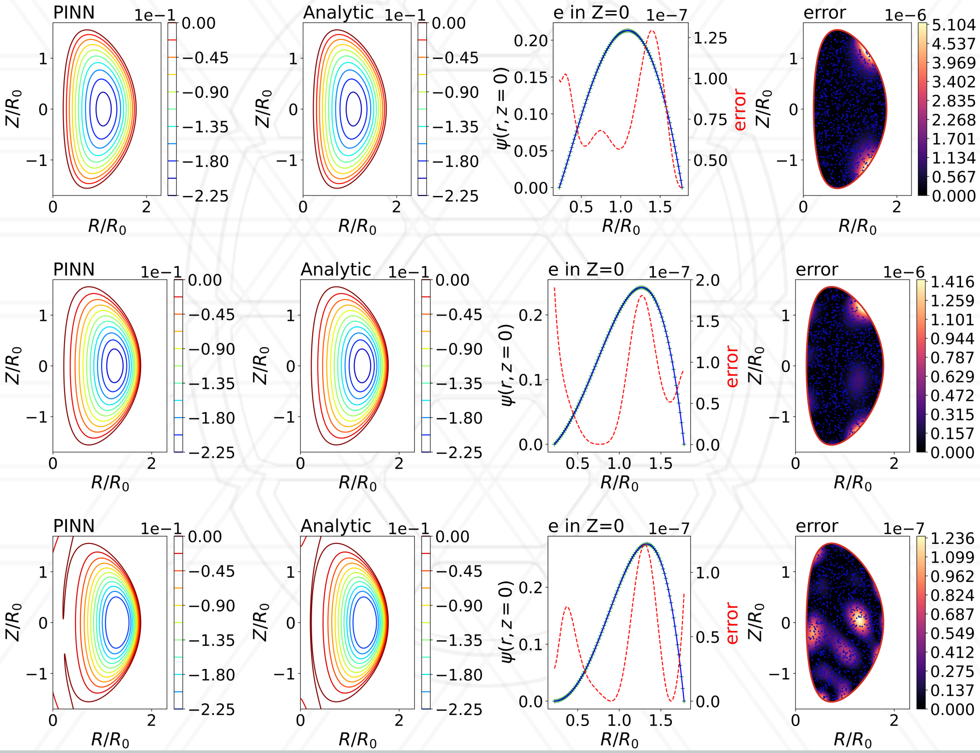
Using 1024 Points

(Dirichlet Boundary Condition)

NSTX with Different β



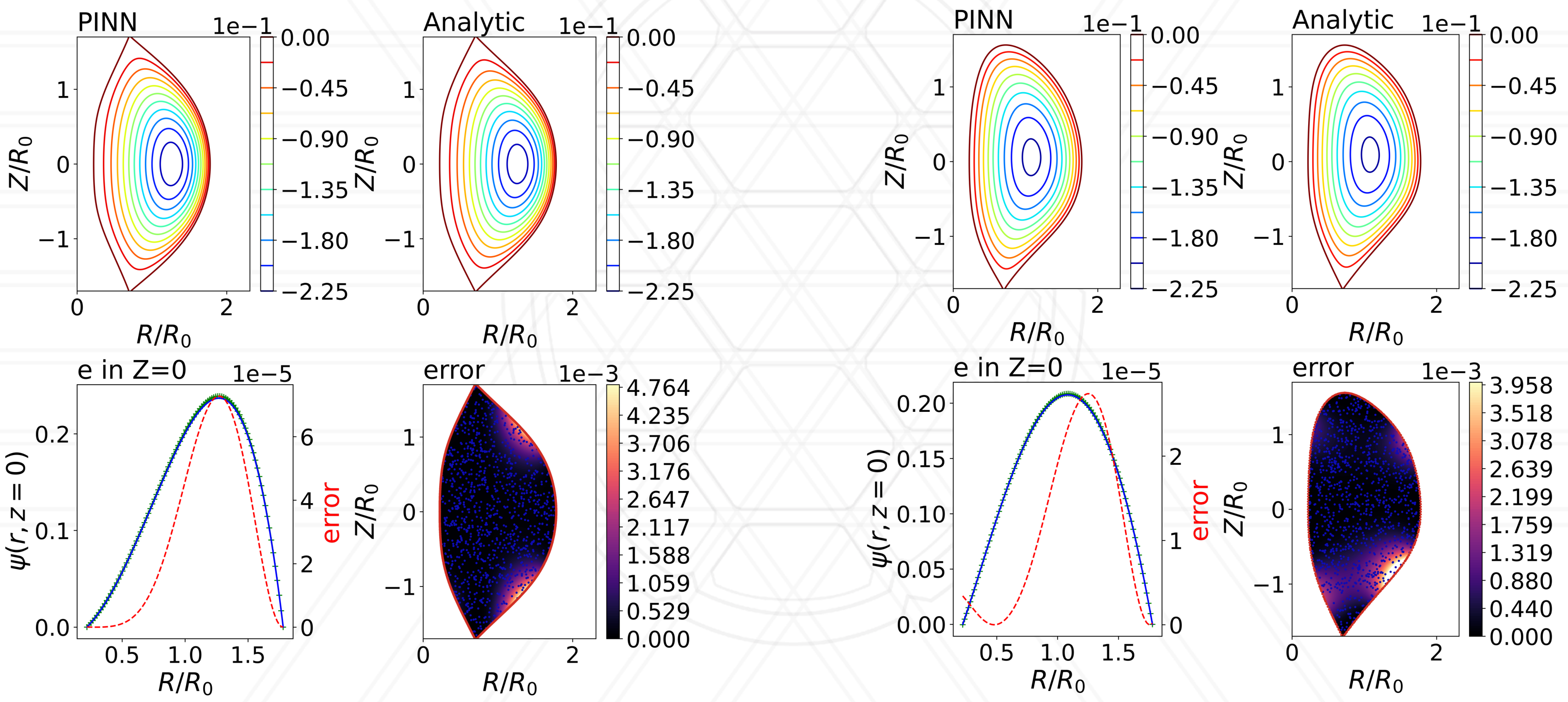
NSTX with Different β



Divertor Configuration – X-points

Using Cerfon's Boundary Conditions

Results – NSTX (Double and Single Null)



Hyperparameter Scan

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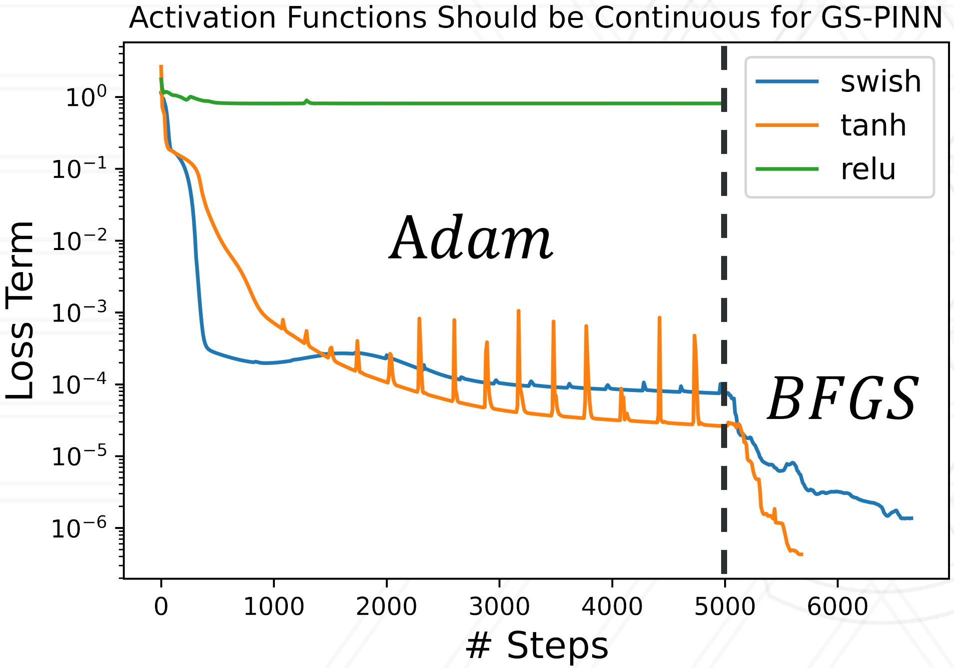
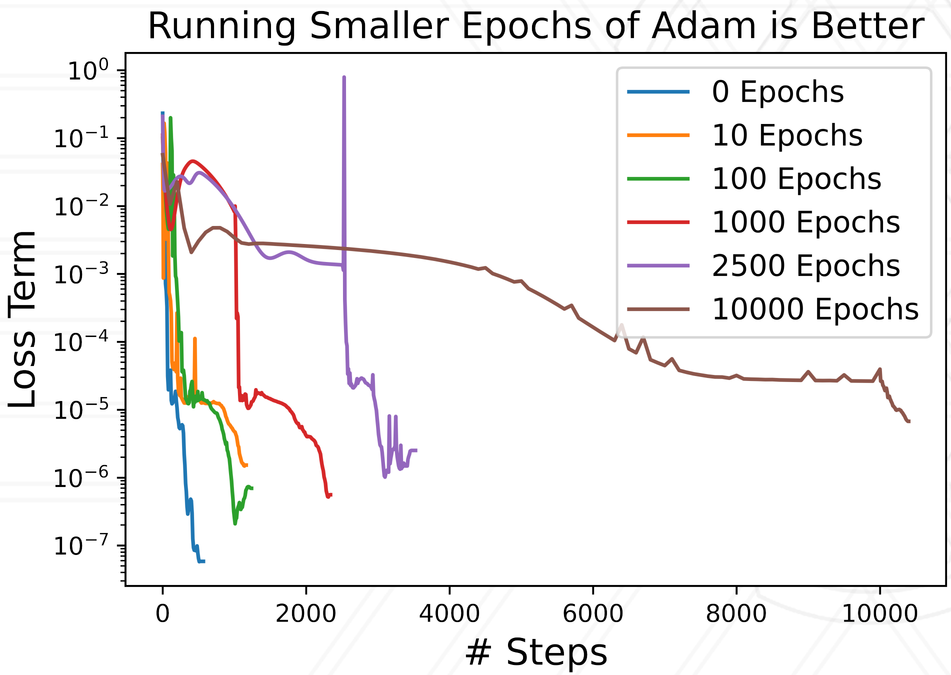
Activation Functions



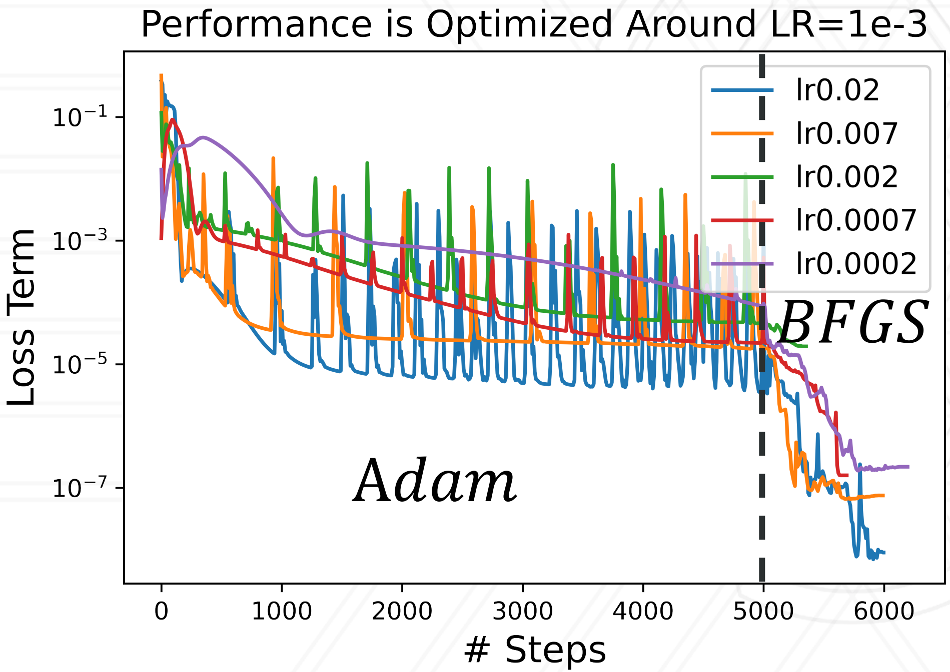
- **Activation Function**
- Activation functions should be differentiable to solve GS equation with PINN

BFGS vs. Adam



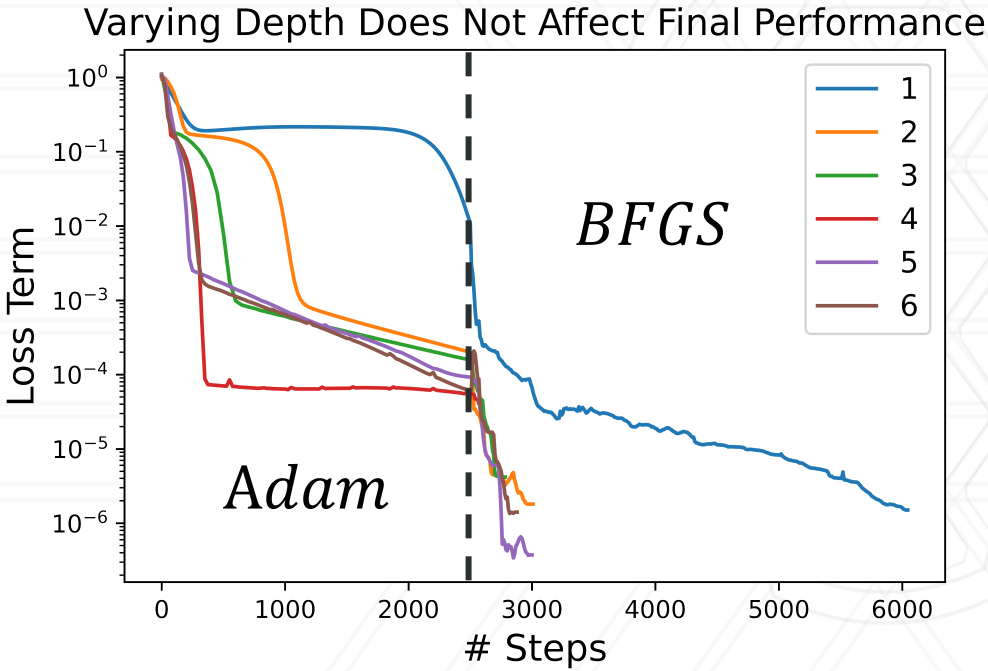
- **BFGS vs. Adam**
- Number of Adam optimization steps does effect BFGS convergence

Learning Rate



- **Learning Rate**
- BFGS performs great regardless of previous learning rates of Adam

Network Depth



- **Network Architecture**
- For Adam, there is no significant improvement after depth=3
- For BFGS, it all converged nicely

Future Works

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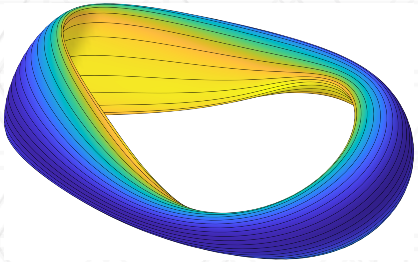
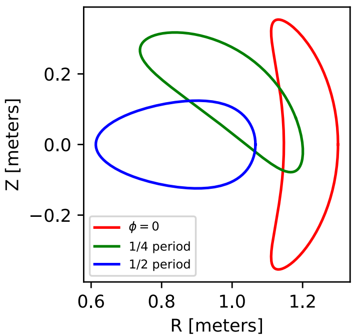
Future Works

- Expand to 3D MHD to reconstruct equilibria for stellarator optimization





- Parametric-PINN to expand our input to included *A*(current/pressure profile) and shape parameters (i.e. eps, kappa, and delta)

Thank You!

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