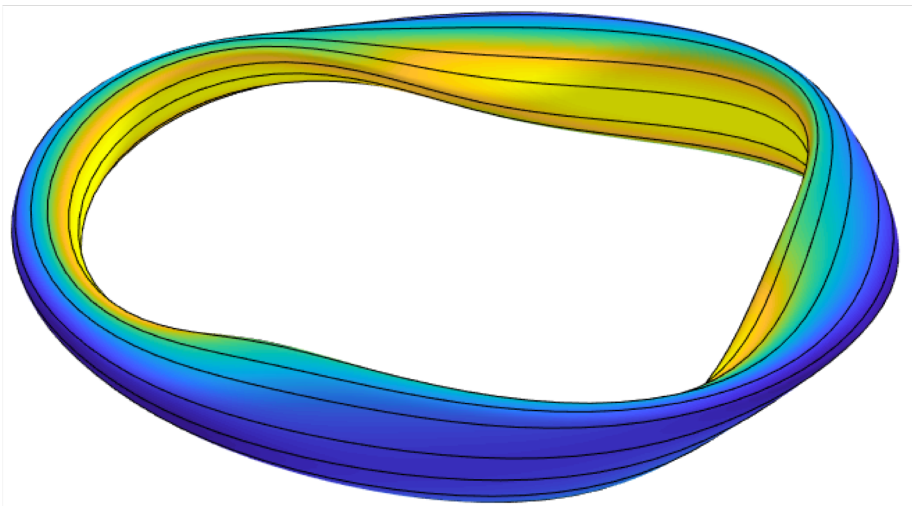
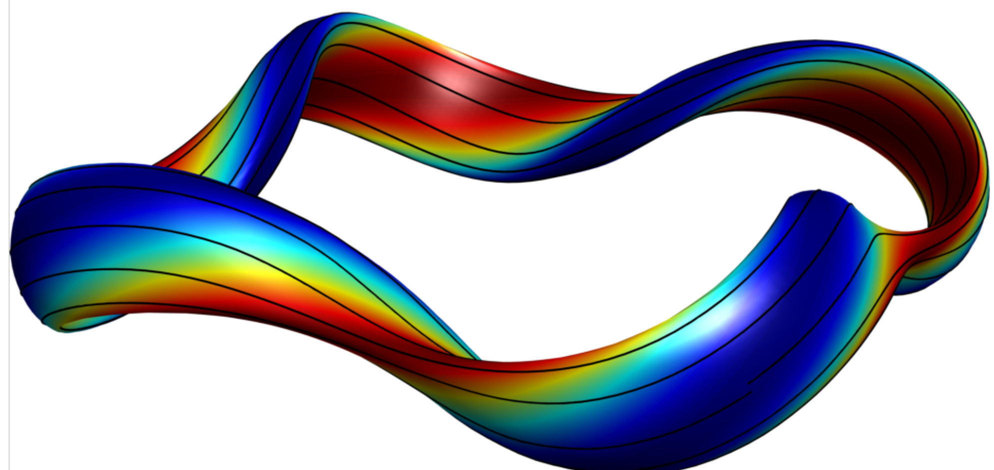
Confining charged particle orbits using hidden symmetry

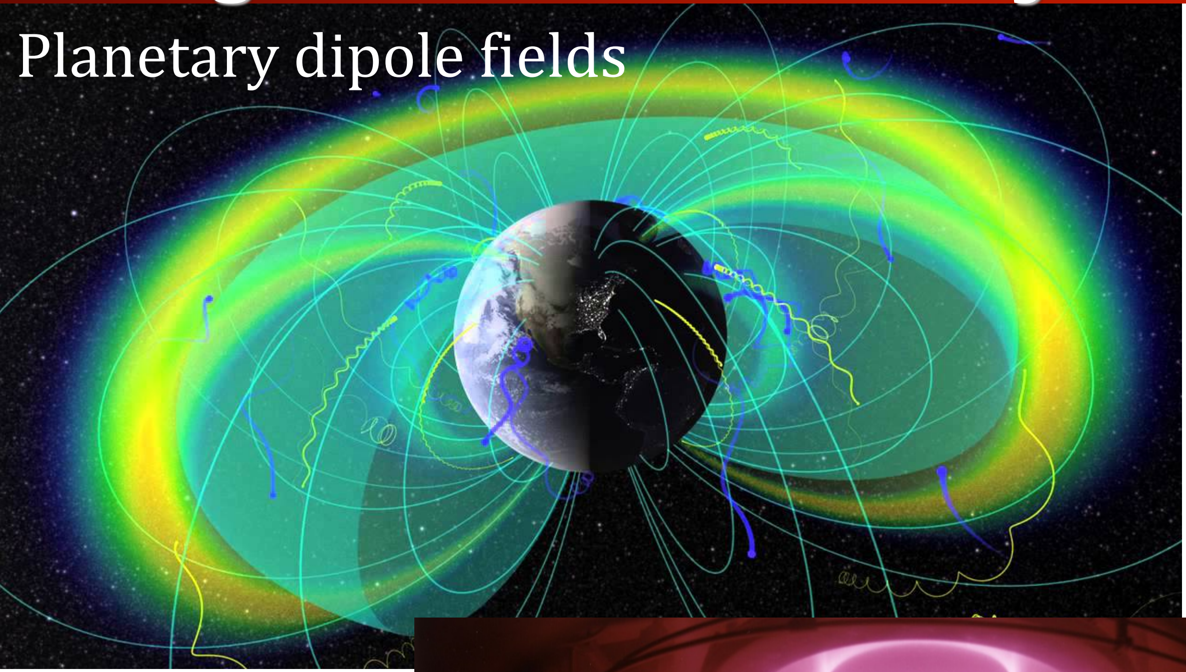
Matt Landreman, UMD IREAP

Outline

- Magnetic confinement, & pros/cons of axisymmetry
- Integrability of non-axisymmetric **B** fields
- Quasi-symmetry
- Finding quasi-symmetric fields

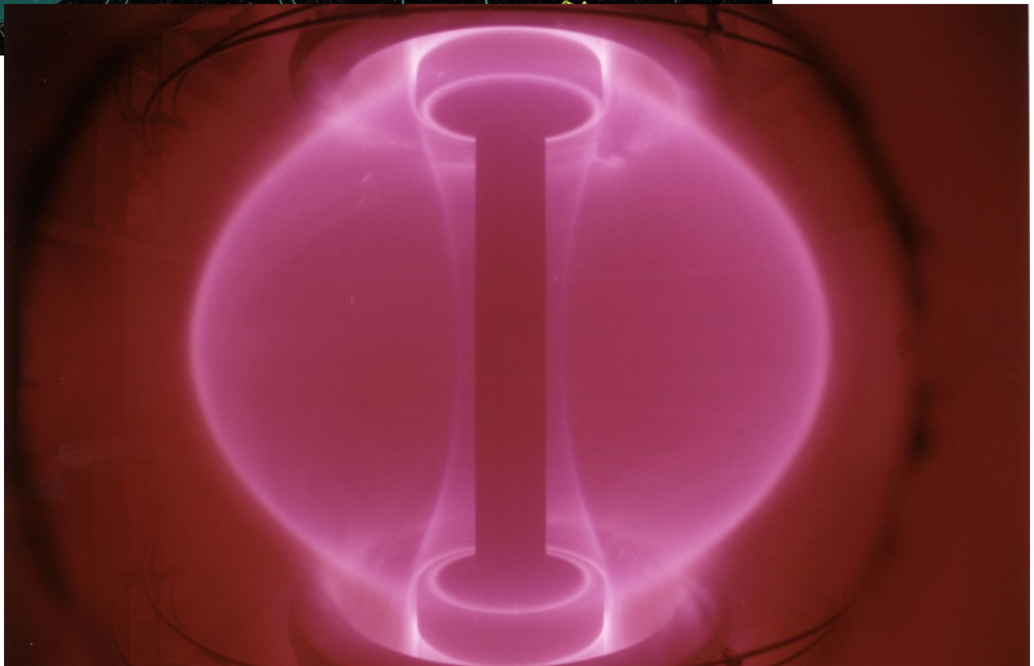
Charged particles can be confined by magnetic fields in many contexts.

Planetary dipole fields



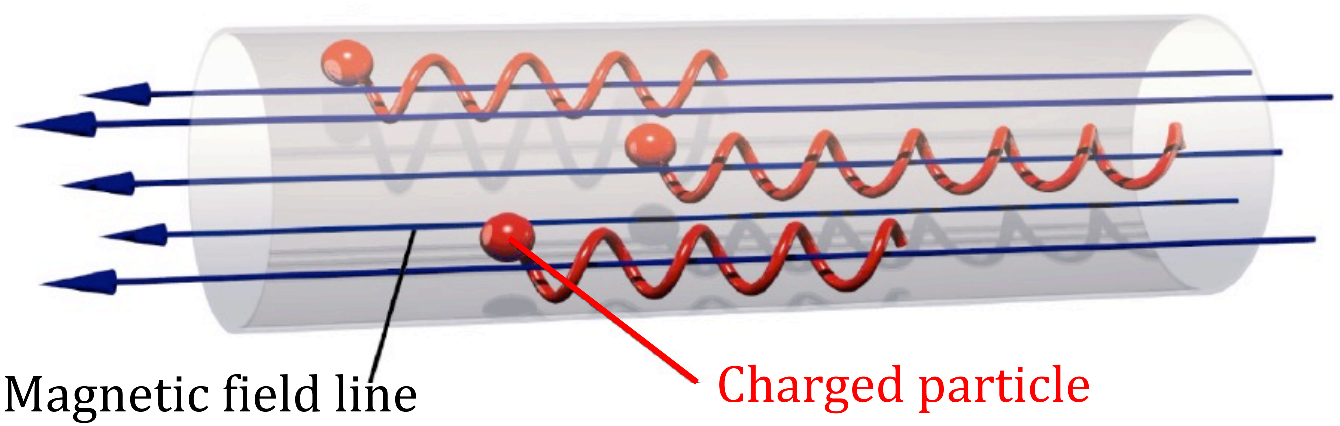
Particle traps for basic physics:

Hot laboratory plasmas, fusion



Confining charged particles with a magnetic field is tricky.

Uniform straight **B**: confinement ⊥ to **B**, but end losses.



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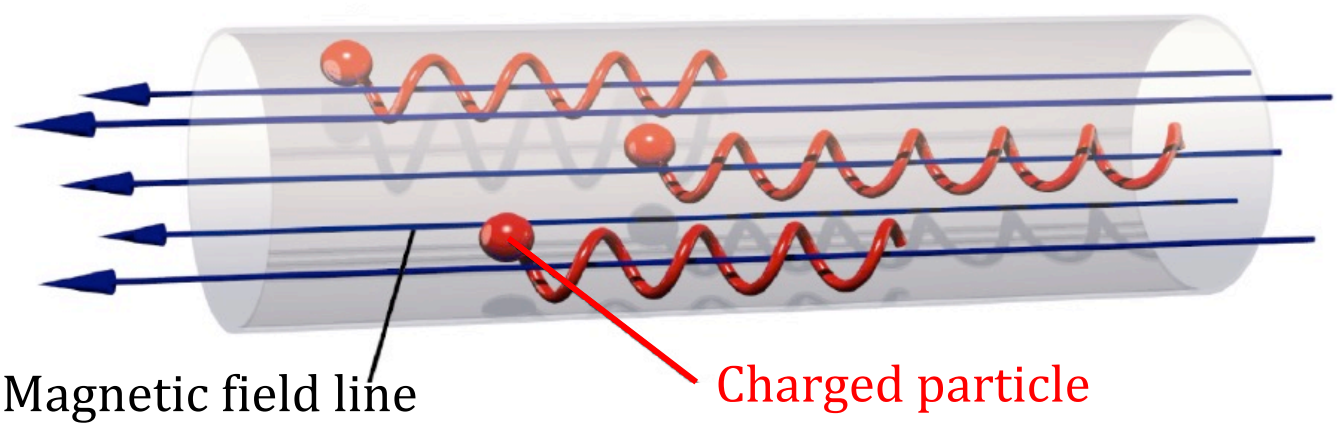

But if field lines are bent, particles drift off them.

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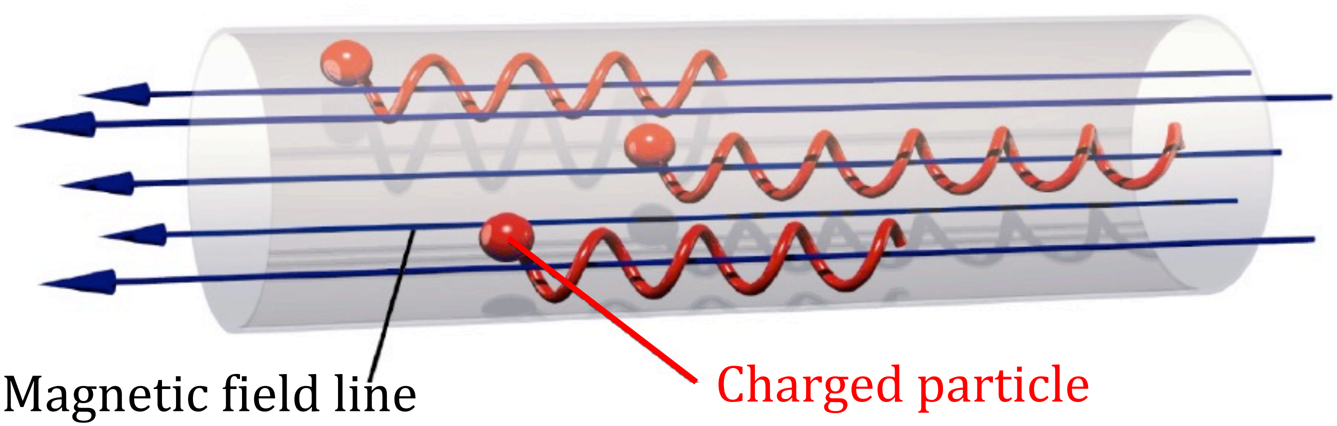

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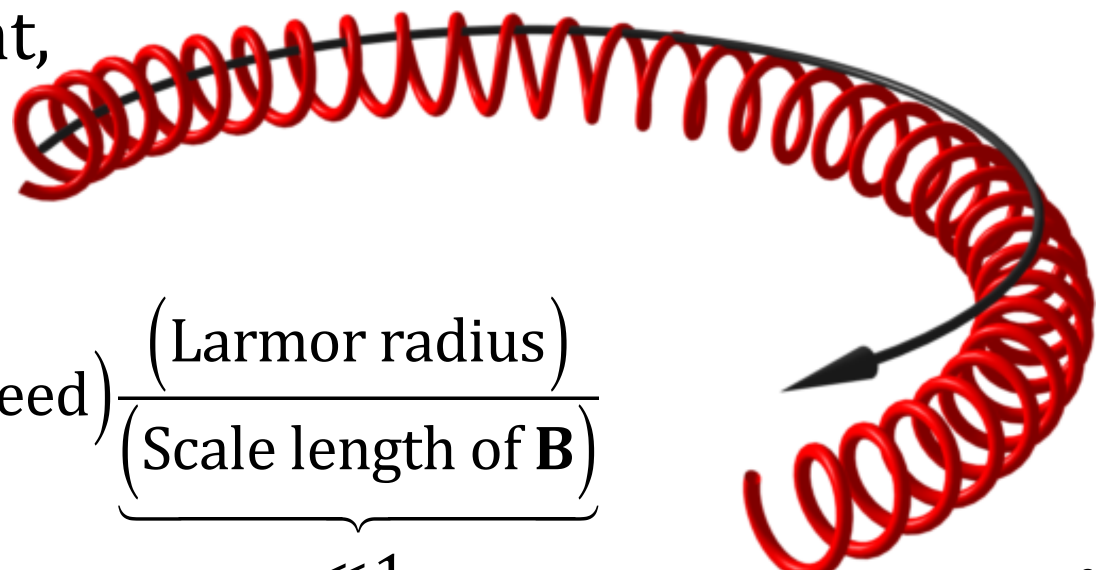


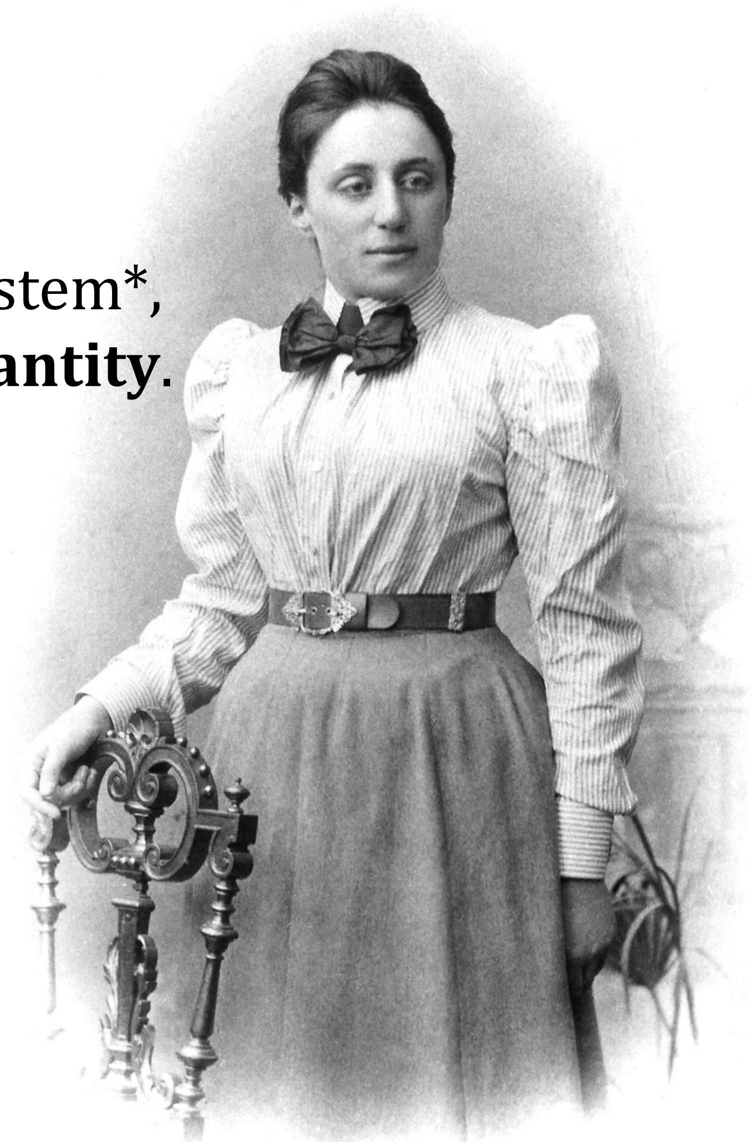
$$\left(\text{Drift velocity}\right) \sim \left(\text{Particle speed}\right) \underbrace{\frac{\left(\text{Larmor radius}\right)}{\left(\text{Scale length of } \mathbf{B}\right)}}_{\ll 1}$$

To confine particles, we can constrain their position with a conservation law.

Noether's theorem:

For each **continuous symmetry** of a system*, there is a corresponding **conserved quantity**.

* For this talk: Lagrangian is independent of a coordinate.



Emmy Noether (1882-1935)

Axisymmetry + Noether's Theorem is one way to achieve magnetic confinement.

Continuous rotational symmetry $\Rightarrow$ Canonical angular momentum is conserved.

$$L_\phi = m v_\phi R + q A_\phi R = \text{constant}$$

vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$

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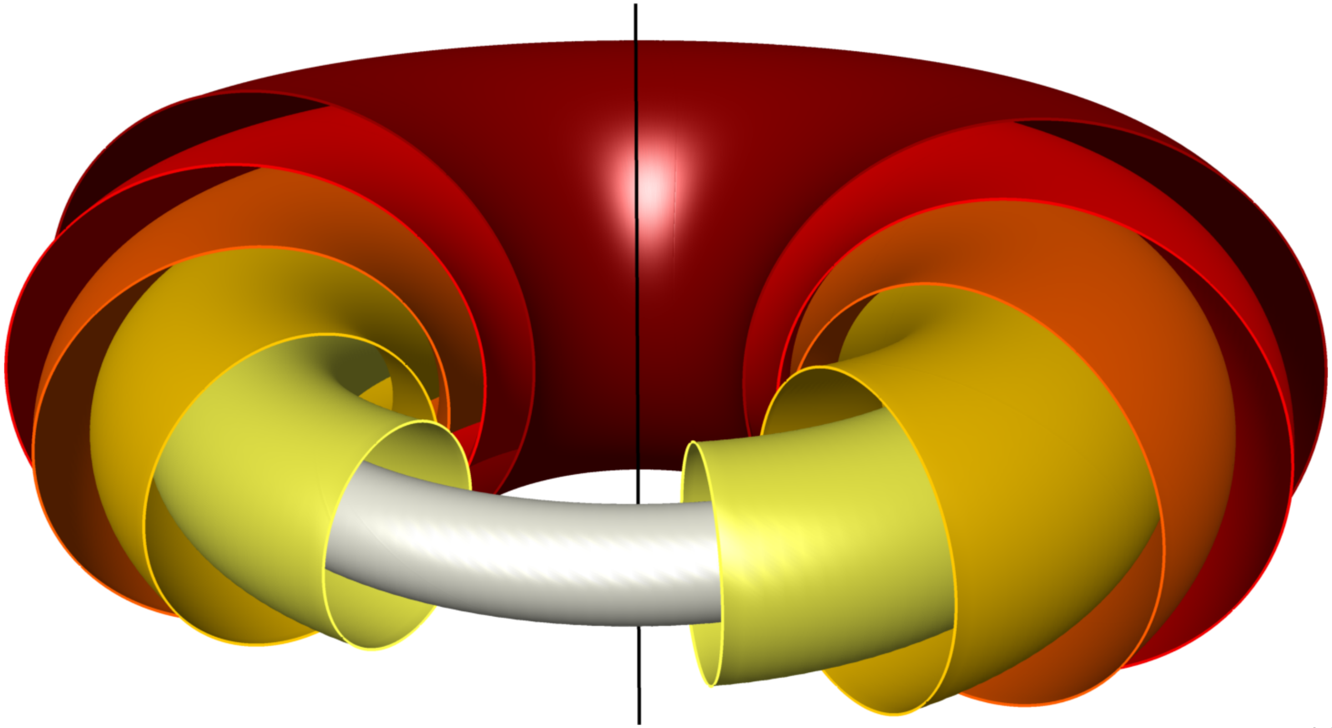
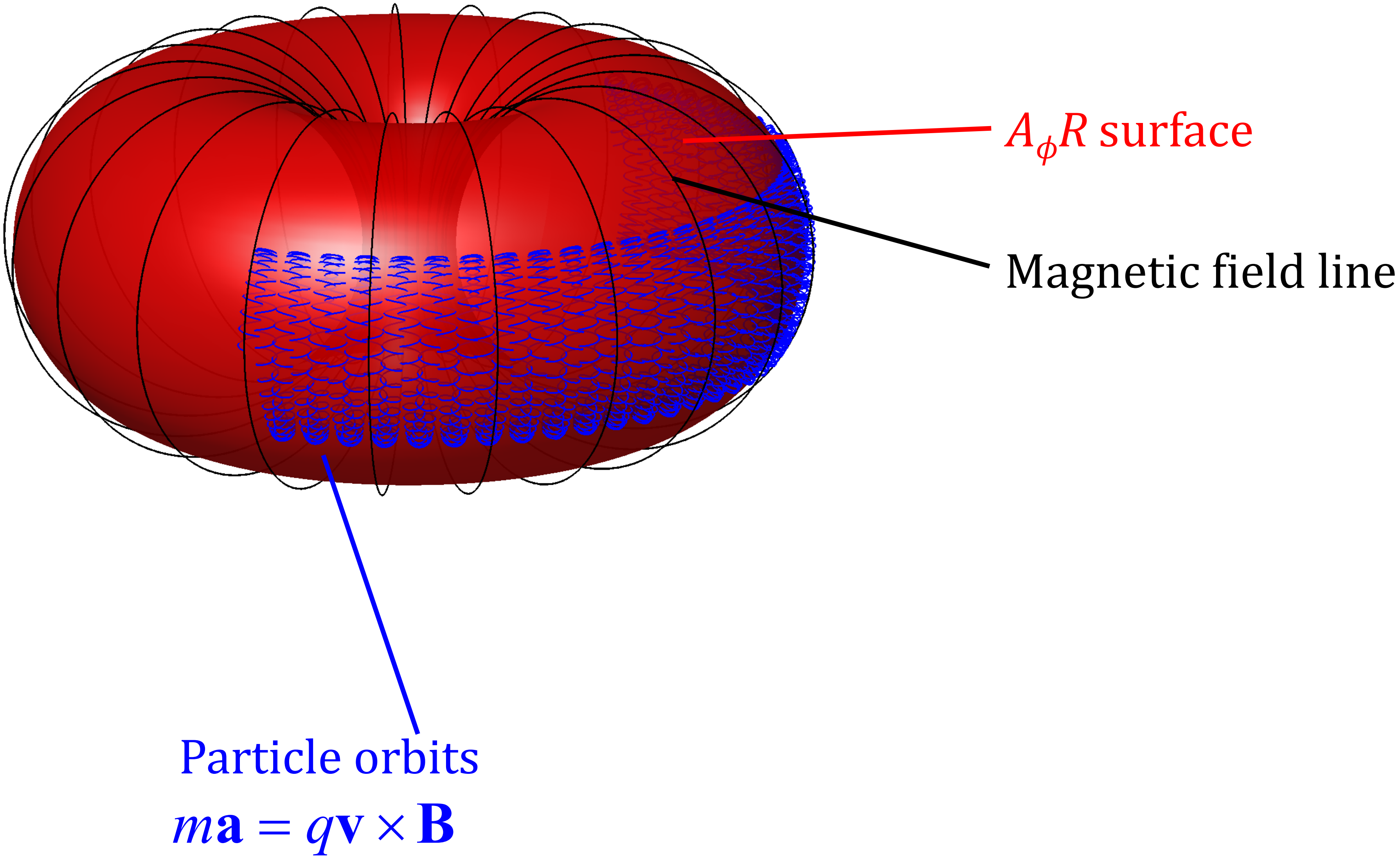
If $A_\phi R$ surfaces are bounded like this, then particles will be confined:



In axisymmetry, particles are confined (close) to $A_\phi R$ surfaces, despite complicated orbits.



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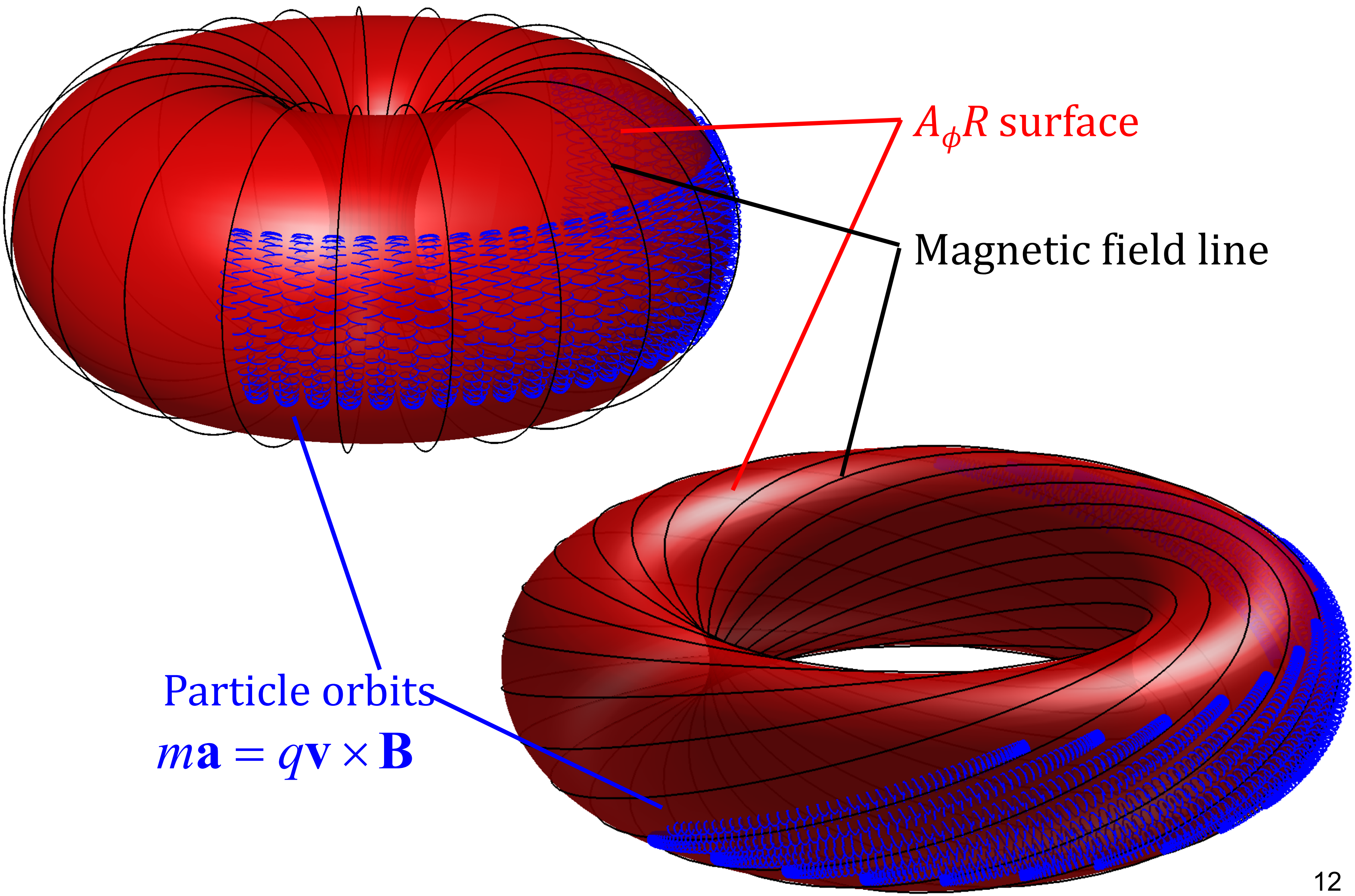
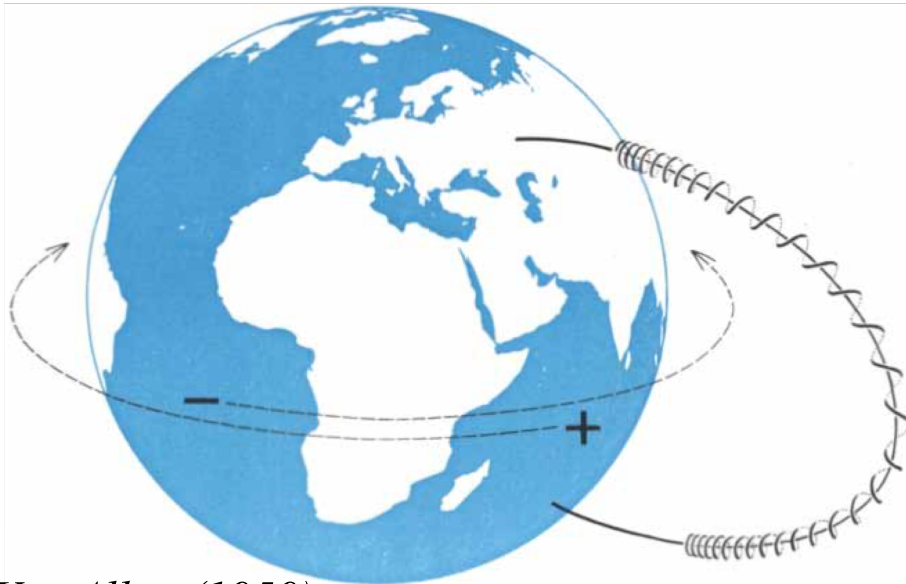
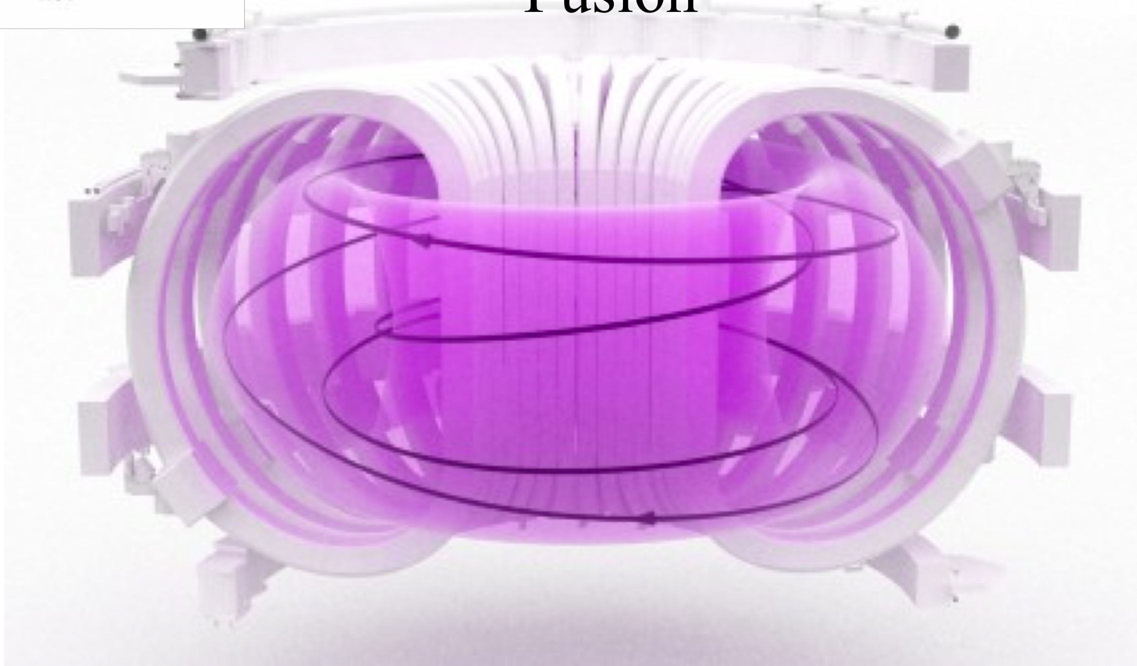


Particles are actually confined this way in nature and in the laboratory.



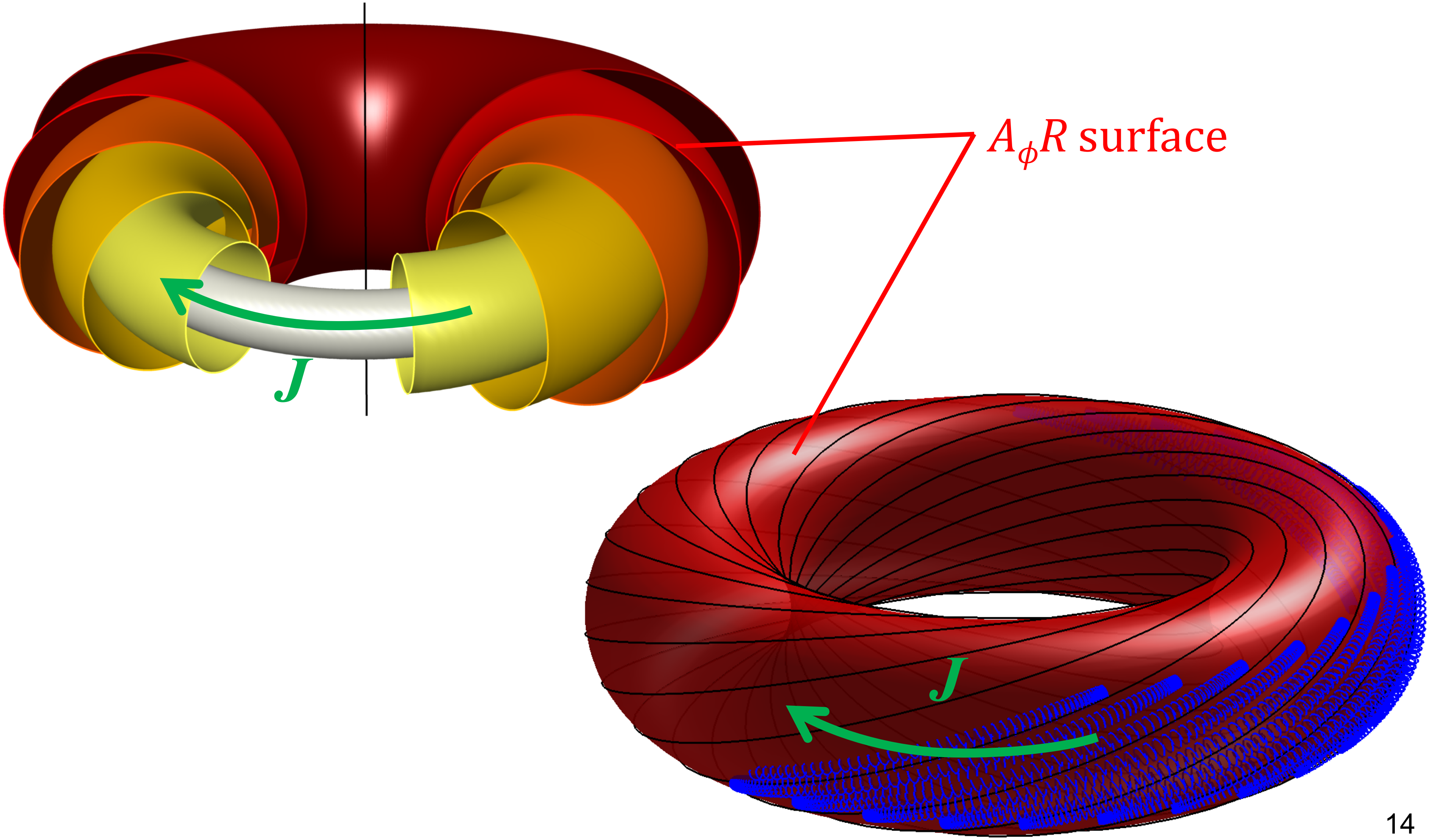
Van Allen (1959)

Fusion



But, axisymmetric confinement has a big problem: requires an internal current.

$\nabla\times(\nabla\times\mathbf{A}) = \nabla\times\mathbf{B} = \mu_0\mathbf{J}$ so nested $A_\phi R$ surfaces require a J_ϕ.



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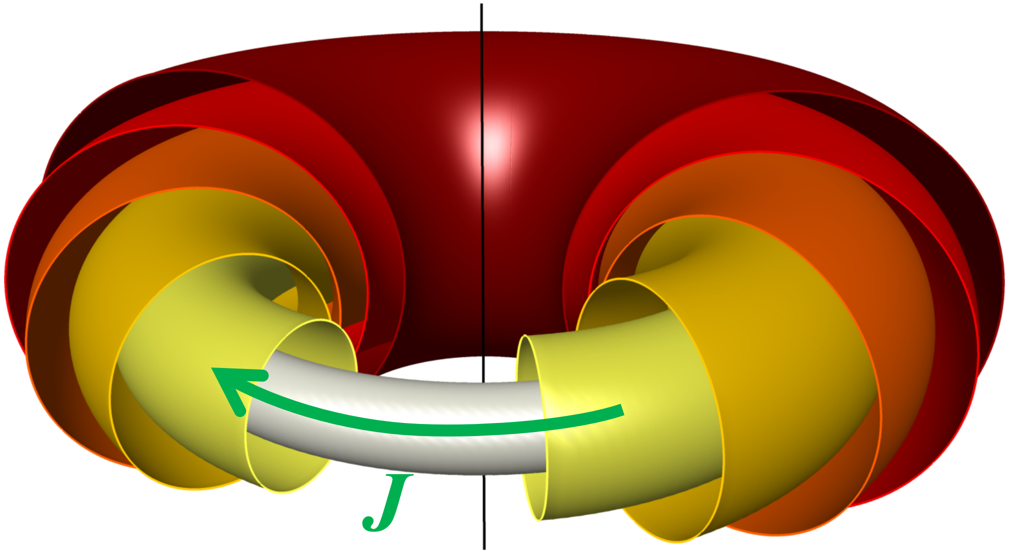

- Sustaining this current in steady-state is hard.
- This current drives instabilities.
- Not possible for low plasma density.

Can we achieve similar confinement without axisymmetry to avoid these problems?

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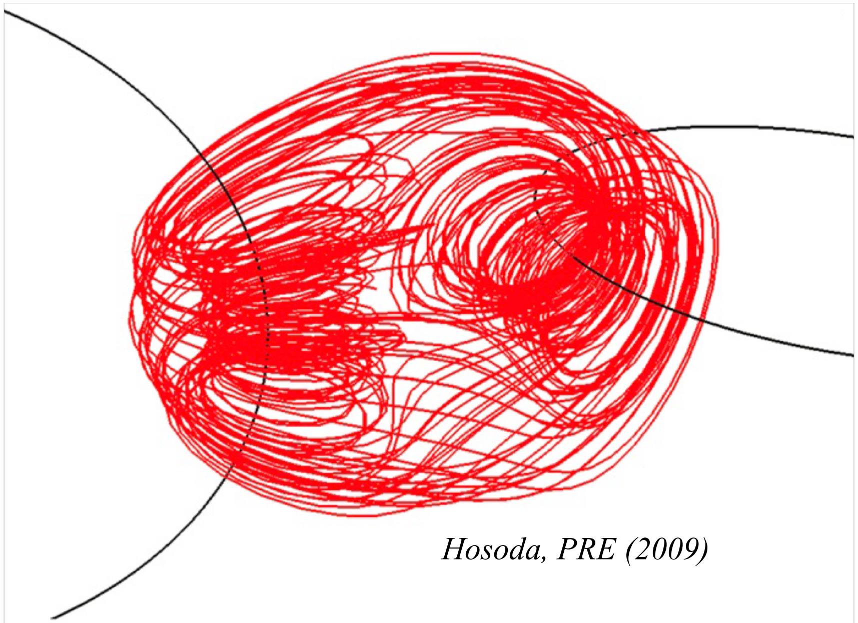
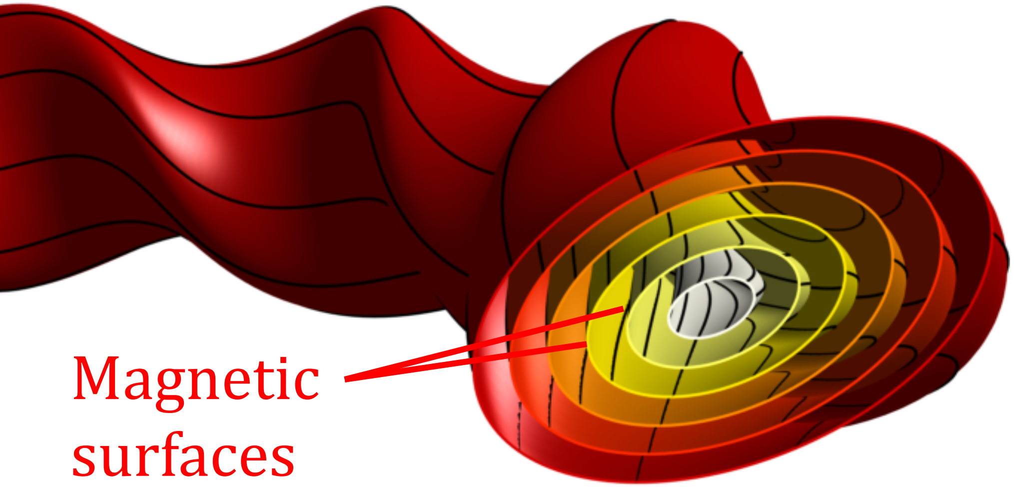
When axisymmetry is broken, we want field lines to still lie on surfaces.

BAD: Particle motion along **B** allows inside & outside to mix even without cross-**B** drift.



Hosoda, PRE (2009)

GOOD: **B** is "integrable"



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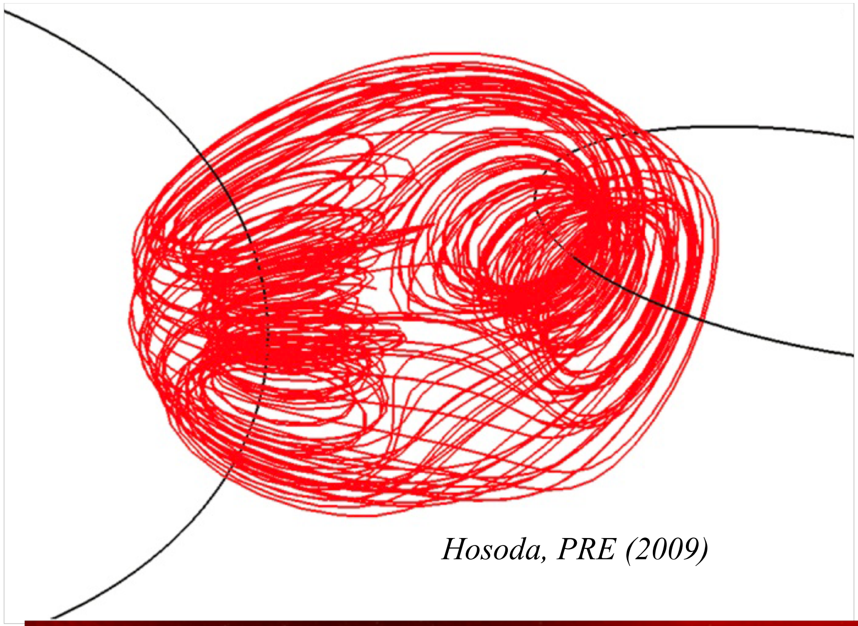

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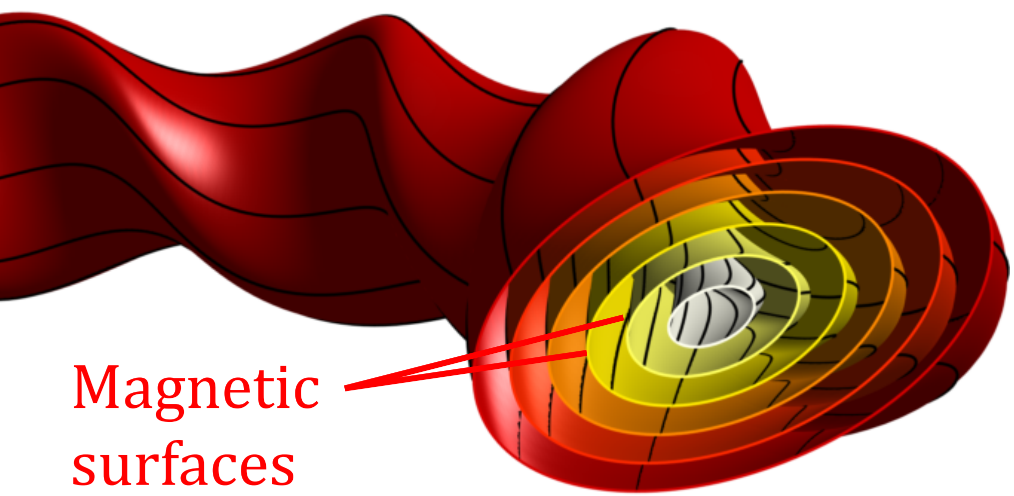
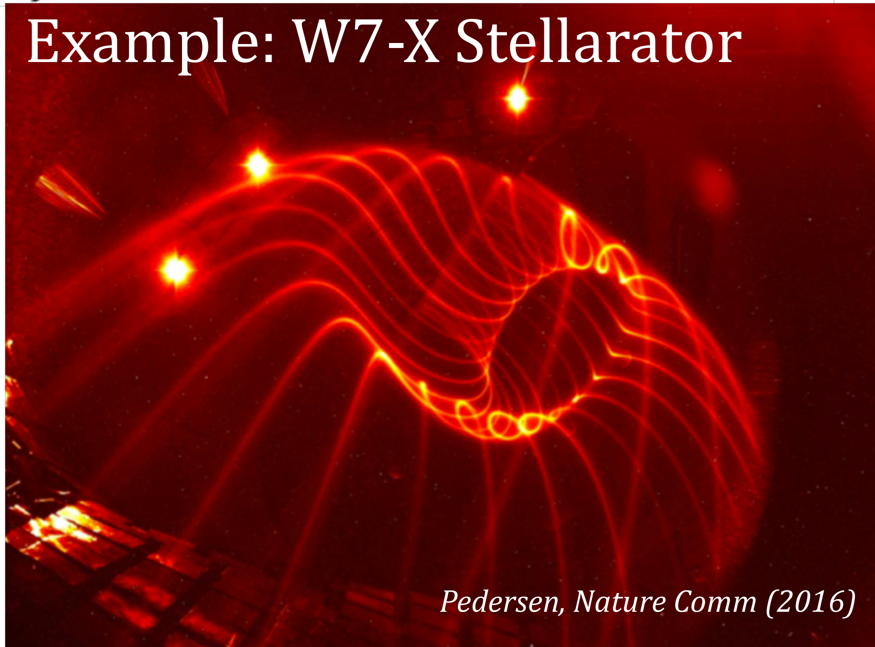
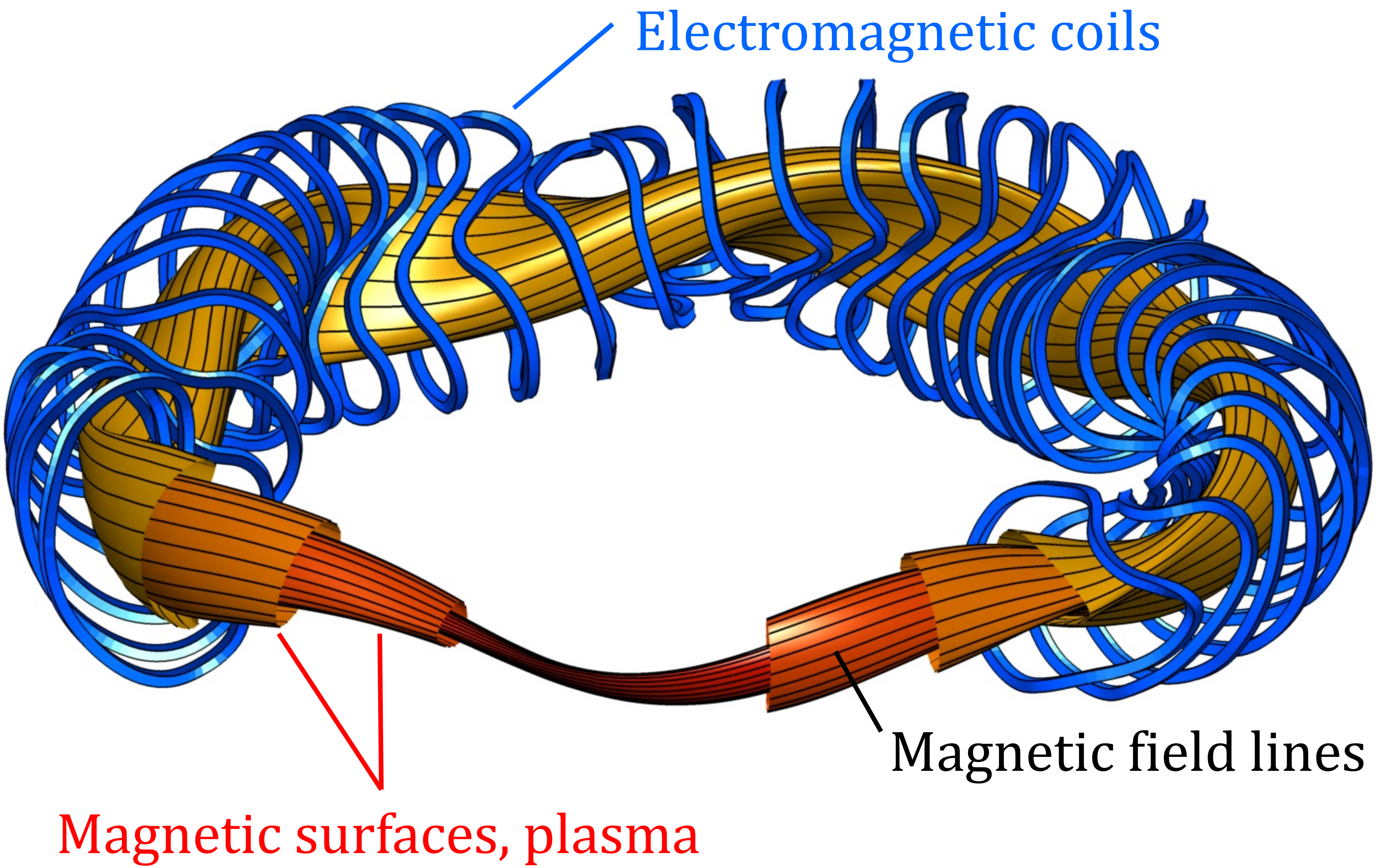


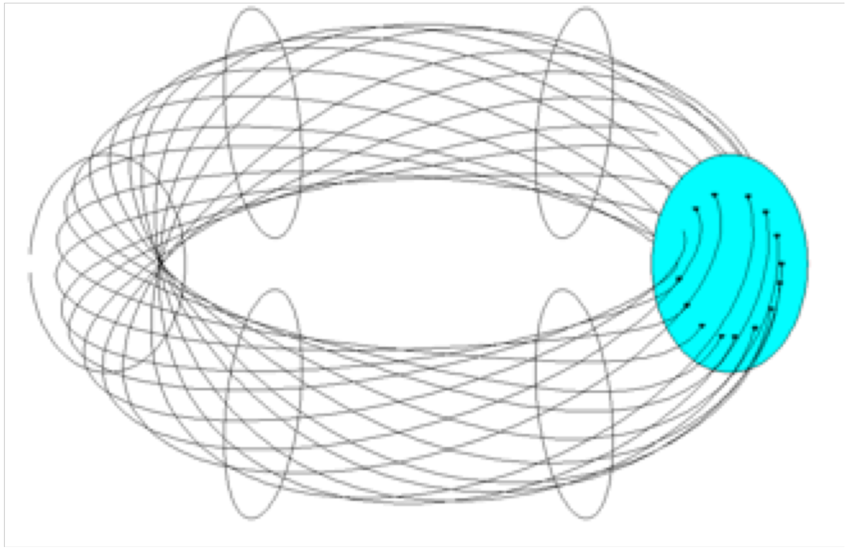


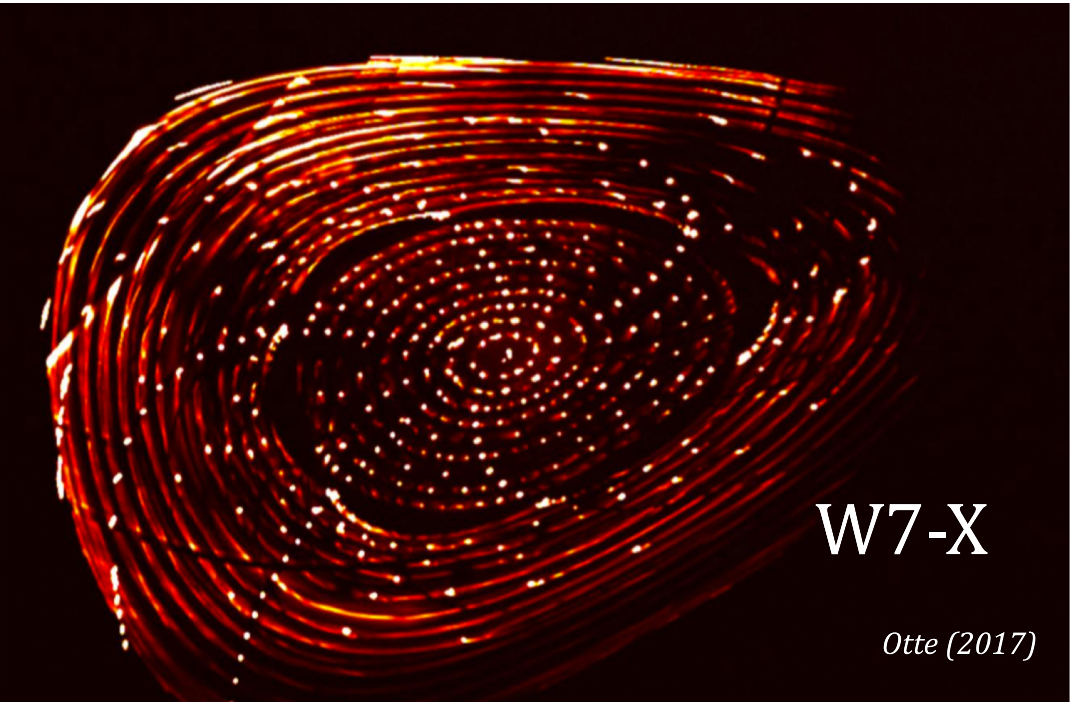
A magnetic confinement device with a non-axisymmetric but integrable magnetic field is a "stellarator"

E.g. Wendelstein 7-X (Germany):



Integrability of magnetic fields can be viewed using Poincare plots





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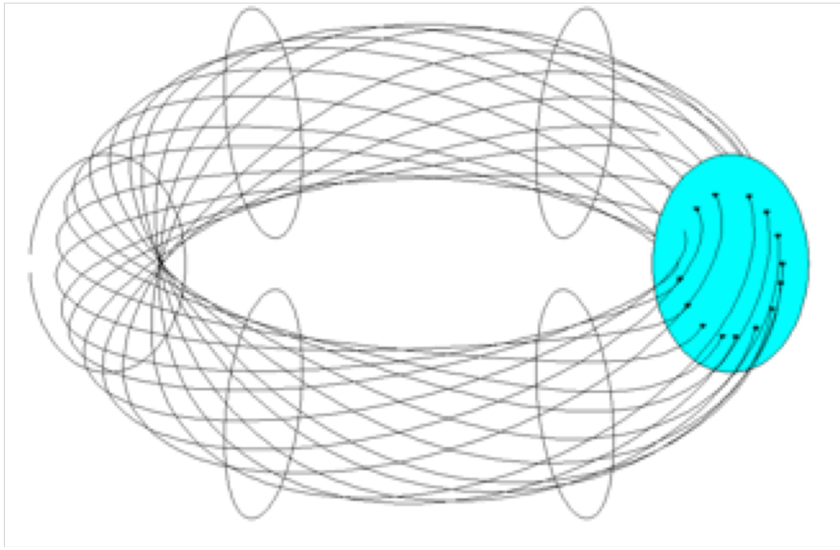
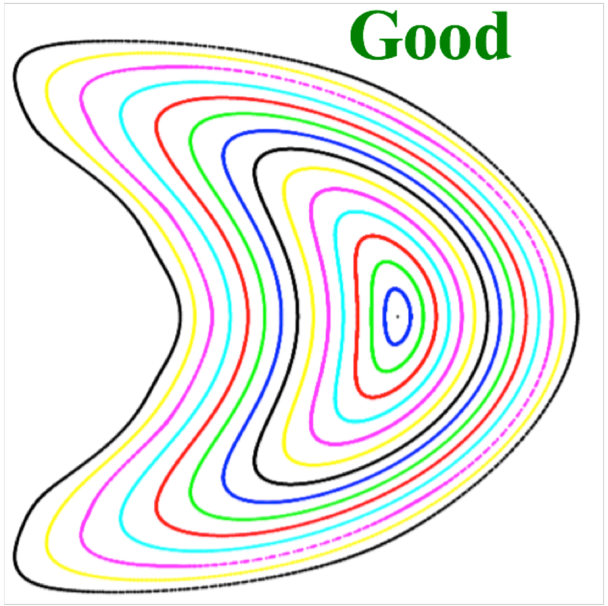
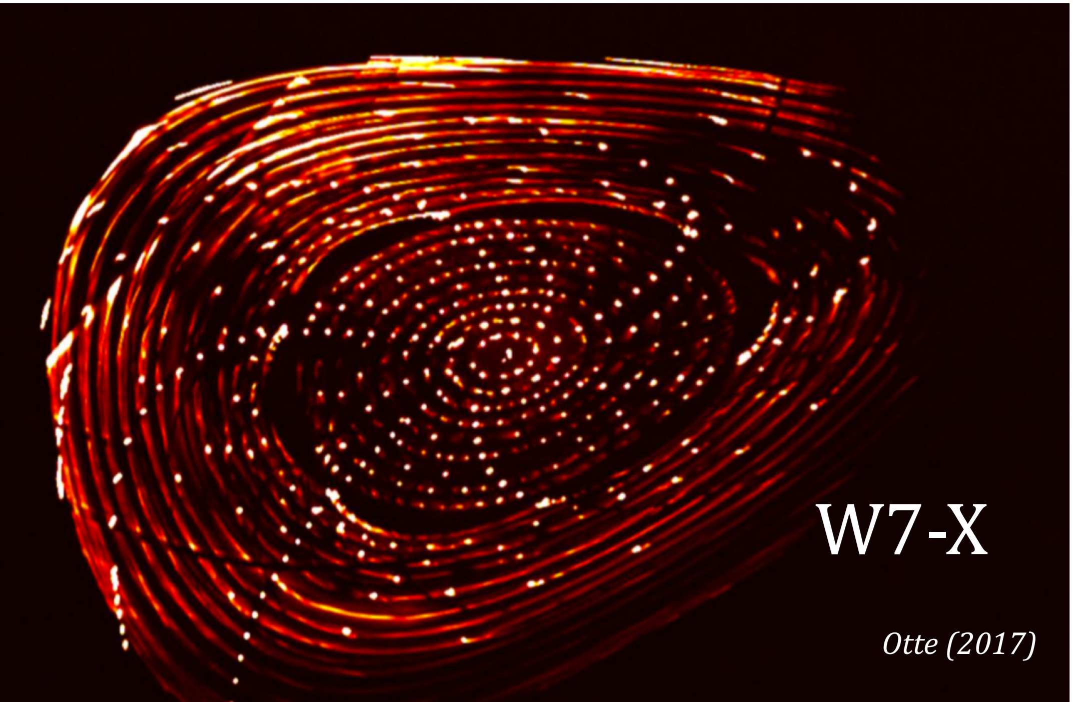
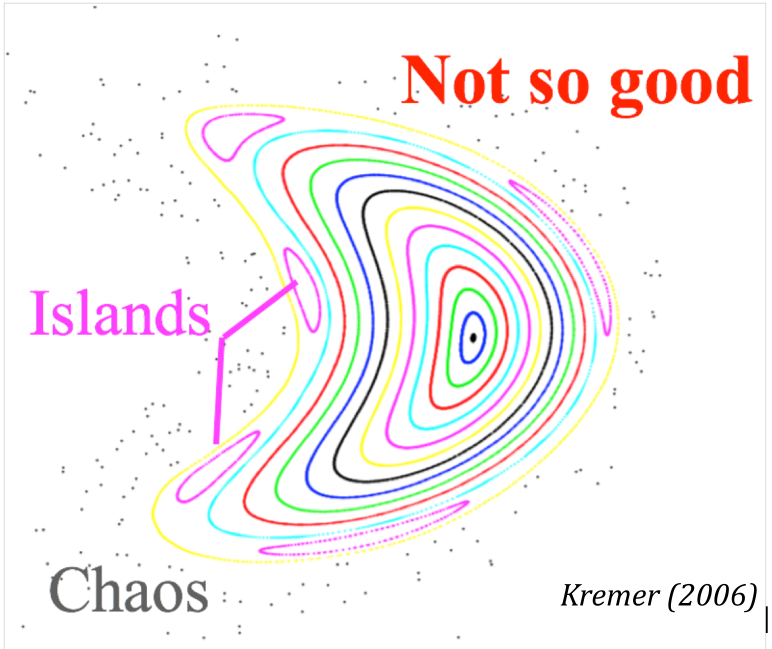
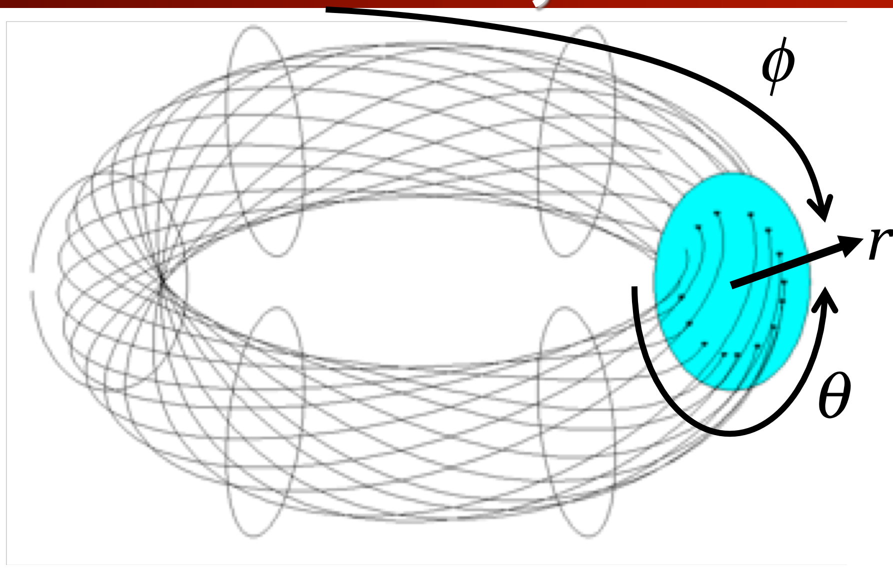


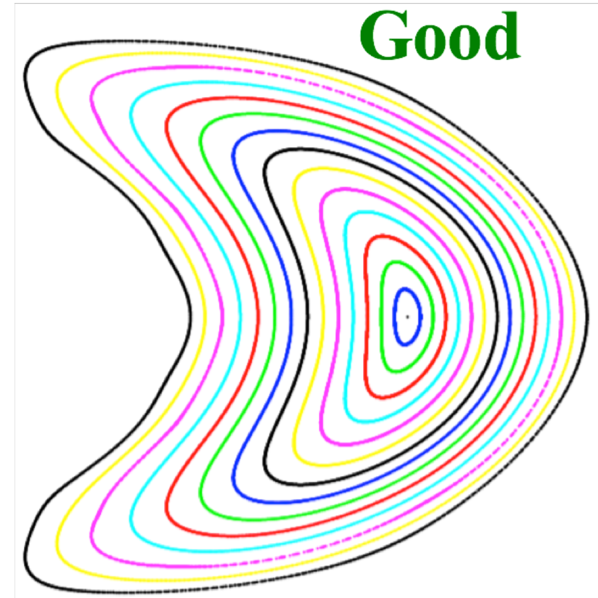






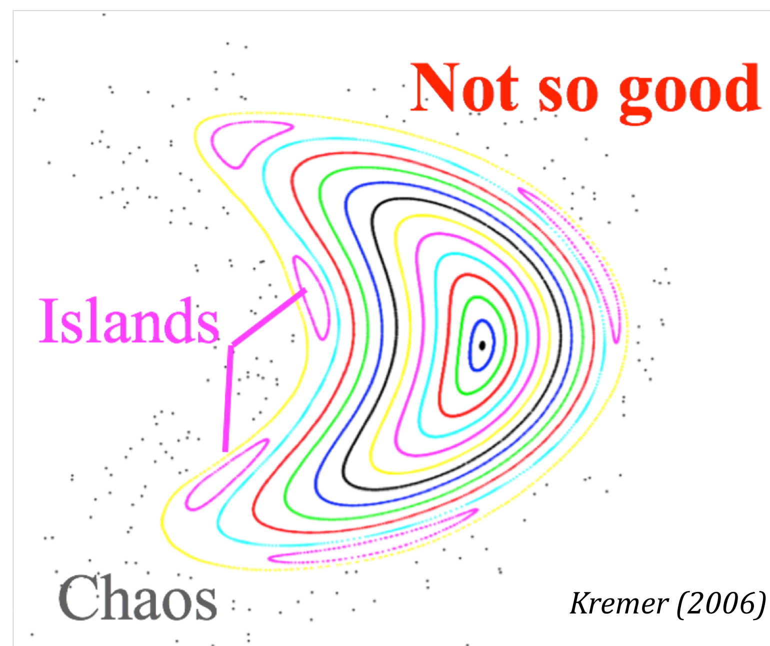
Magnetic field lines can be described by a Hamiltonian, where "time" is ϕ





$$\frac{d\theta}{d\phi}=\frac{\partial H}{\partial r}, \quad \frac{dr}{d\phi}=-\frac{\partial H}{\partial \theta}$$

So tools from Hamiltonian systems like KAM apply.



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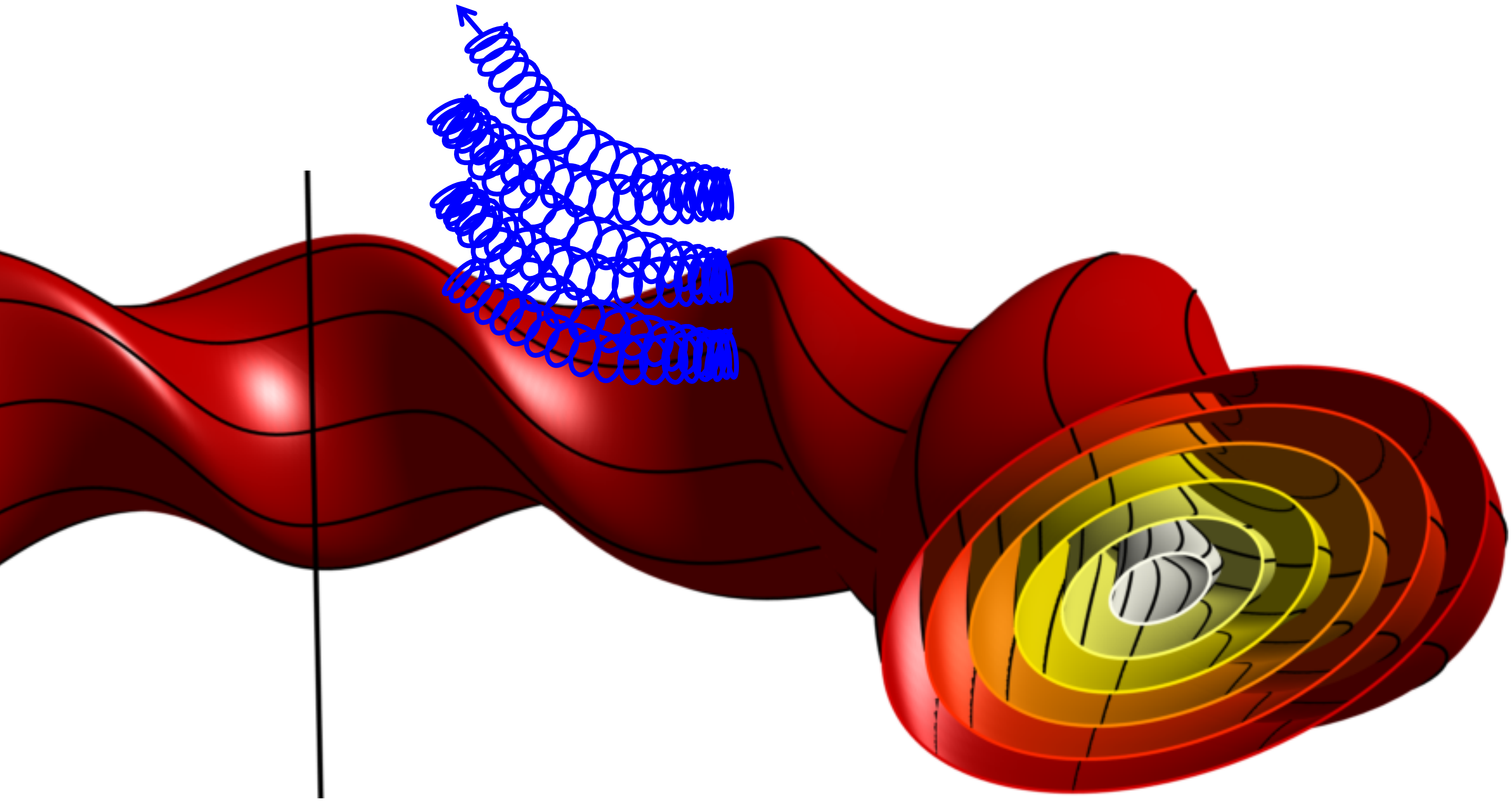
Even with magnetic surfaces, confinement in non-axisymmetric fields is poor due to cross-B drift.



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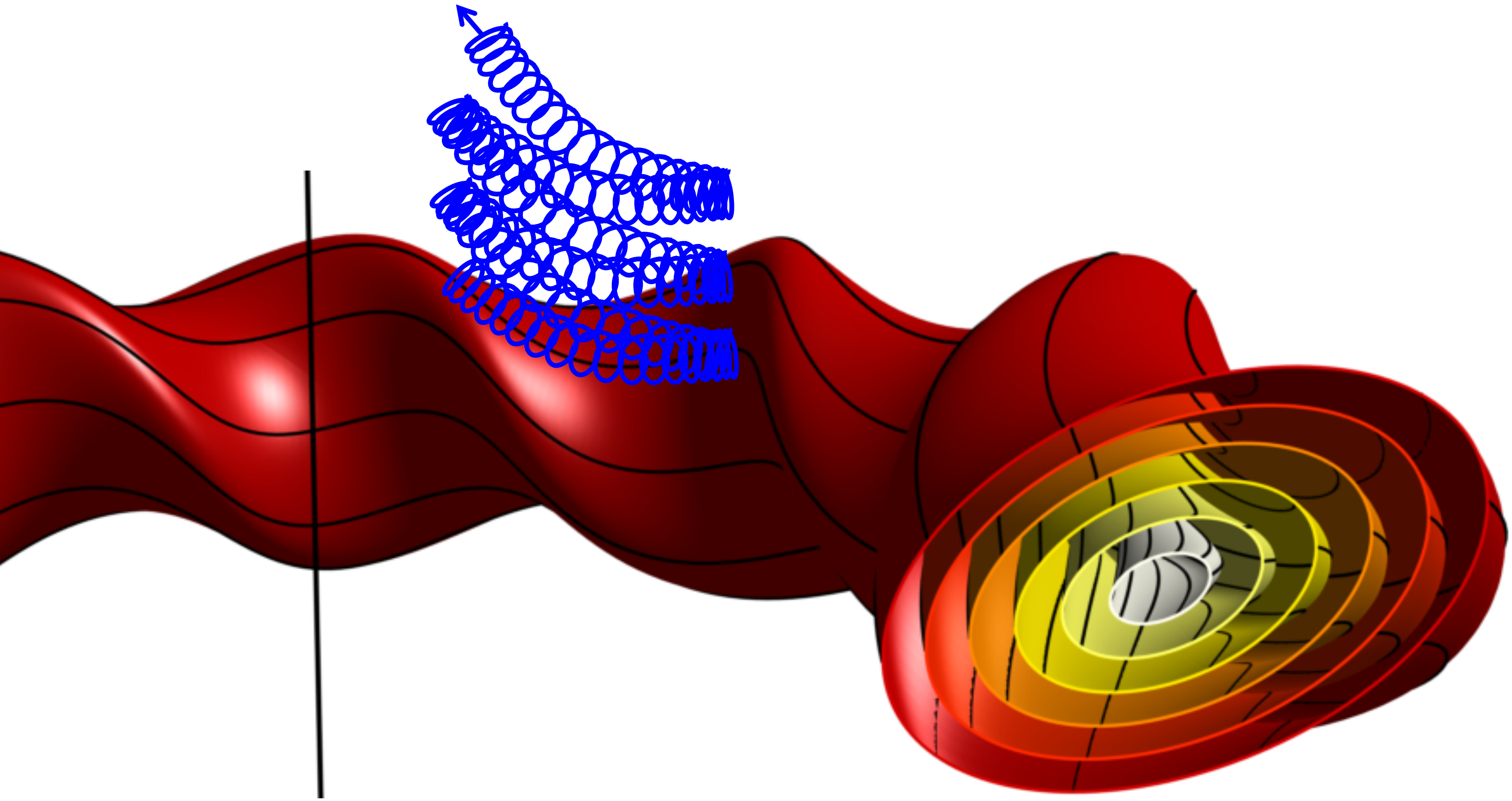

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“Quasi-symmetry” provides a solution.

Quasi-symmetry is a continuous symmetry in |B| (not vector B) that implies confinement.

- When the Lagrangian is (1) expanded for large B=|**B**| and (2) written in a special coordinate system ("Boozer angles"), it depends on position only through the surface and B.

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- Therefore a symmetry in B implies a conserved quantity, even if **B** has no obvious symmetry.

- This conserved quantity resembles canonical angular momentum, so it implies confinement just as in axisymmetry.

Averaging over gyration, Lagrangian depends on B only through surface and |B|.

Lagrangian for particle in magnetic field: $\mathcal{L} = q\mathbf{A}\cdot\dot{\mathbf{x}} + \frac{m}{2}\left|\dot{\mathbf{x}}\right|^2$ (Neglect $\mathbf{E}$)

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Expand in $\left(\text{Larmor radius}\right)/\left(\text{Scale length of }\mathbf{B}\right) \ll 1$, i.e. $\left|\mathbf{B}\right| \to \infty$.

$$\mathcal{L}\left(\mathbf{x},\upsilon_{||},\mu,\varphi,\dot{\mathbf{x}},\dot{\upsilon}_{||},\dot{\mu},\varphi\right) = q\mathbf{A}\cdot\dot{\mathbf{x}} + \frac{m\upsilon_{||}}{B}\mathbf{B}\cdot\dot{\mathbf{x}} + \frac{m}{q}\mu\dot{\alpha} - \frac{m\upsilon_{||}^2}{2} - \mu B$$

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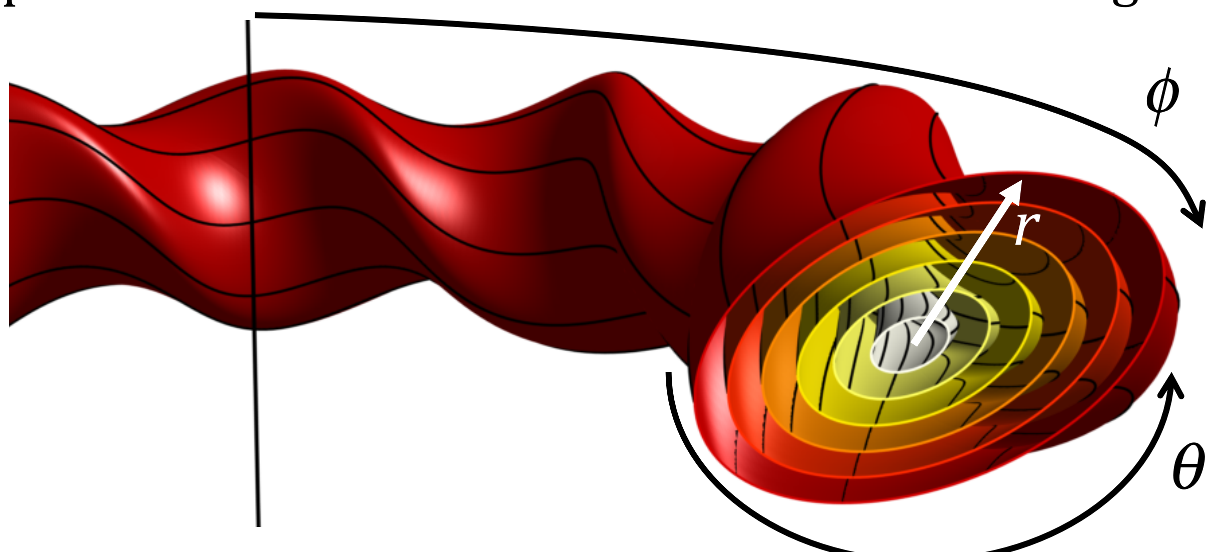
Spatial coordinates: surface label r & 'Boozer angles' (θ, ϕ):



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Depends only on r

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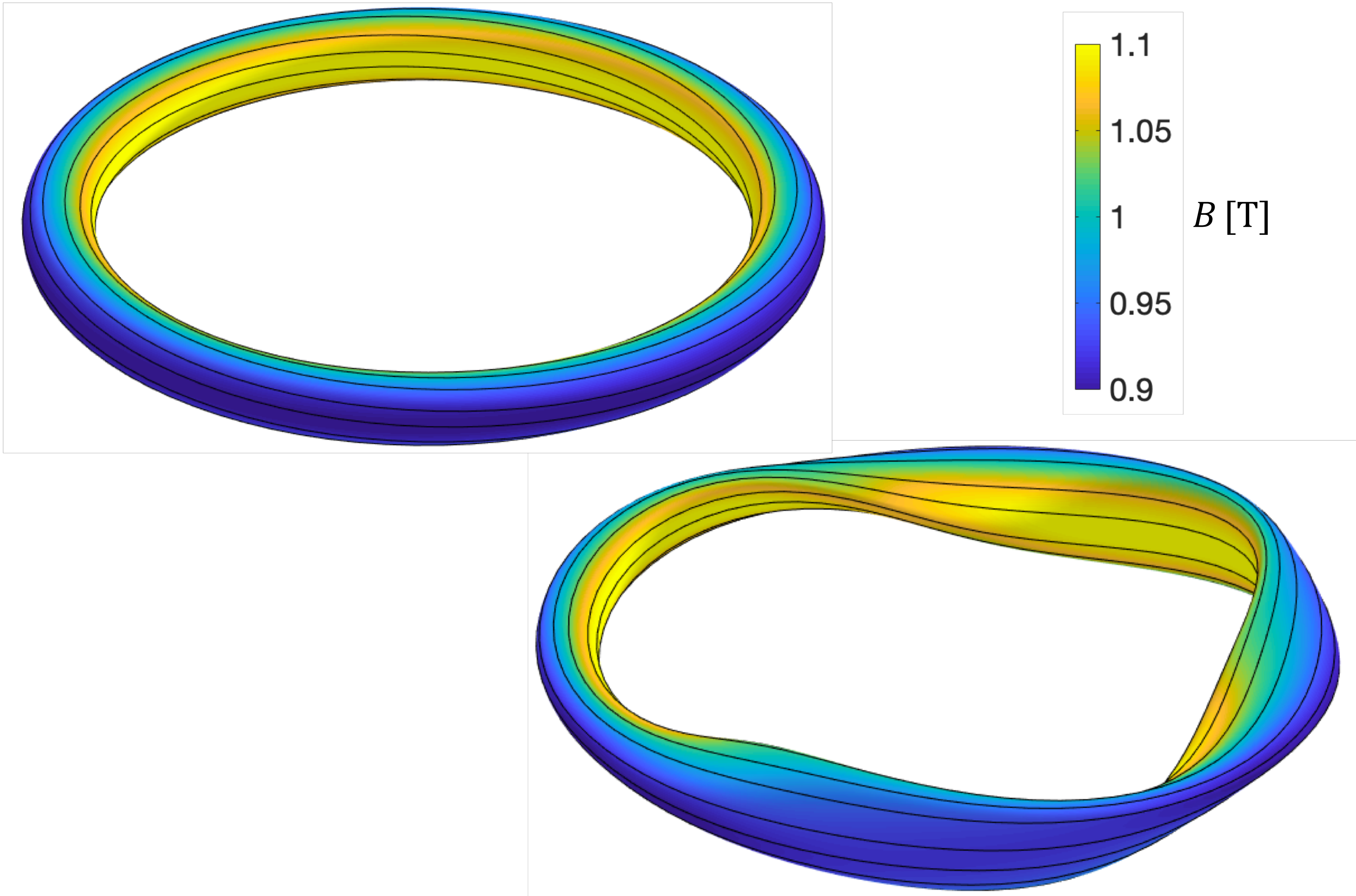
Depends only on r

$$\frac{\partial B}{\partial \phi} = 0 \quad \Rightarrow \quad \text{Conservation of } \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -q\psi_p + \frac{m\upsilon_{||}B_\phi}{B} \approx -q\psi_p \quad \Rightarrow \quad \text{Confinement!}$$

Due to quasisymmetry, different B fields can have isomorphic particle orbits.



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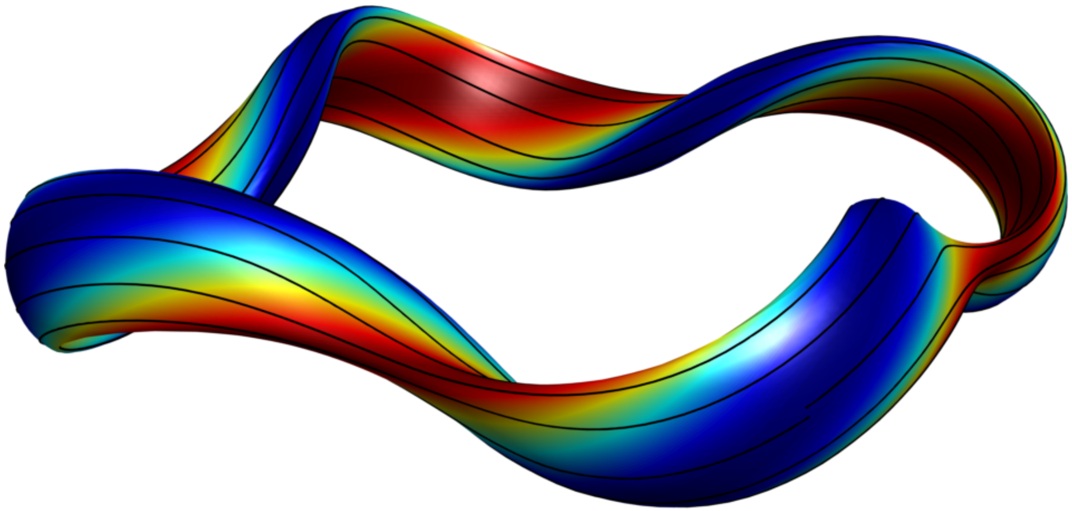
Several quasi-symmetric confinement experiments have been designed using optimization.

Boundary shape varied to minimize symmetry-breaking in |B|.



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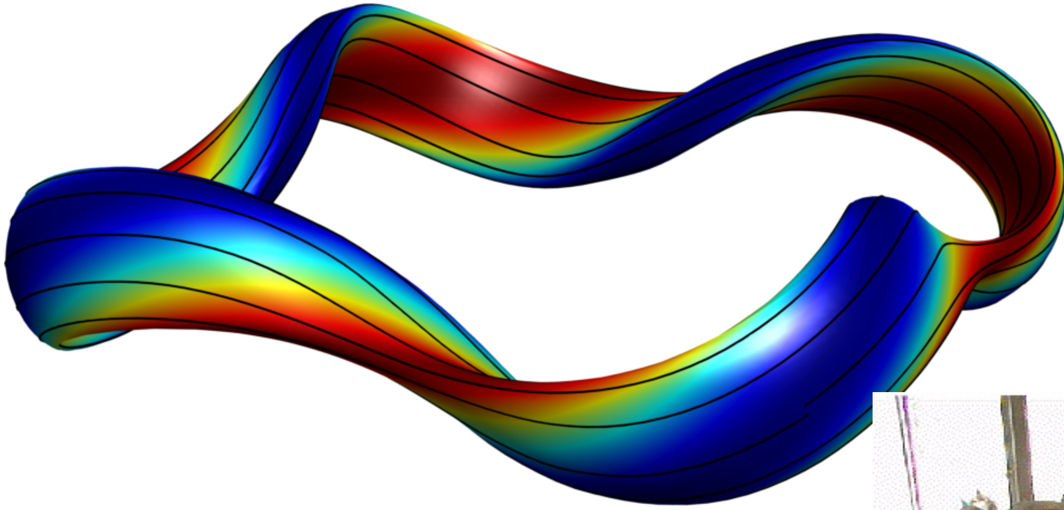
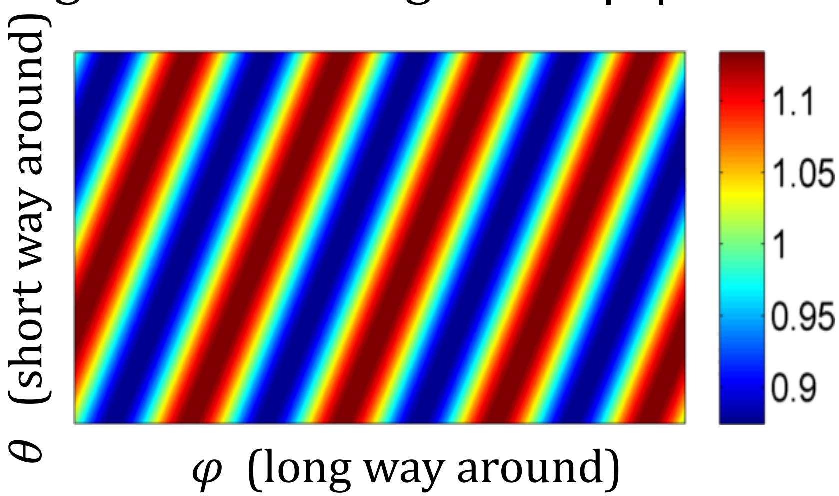
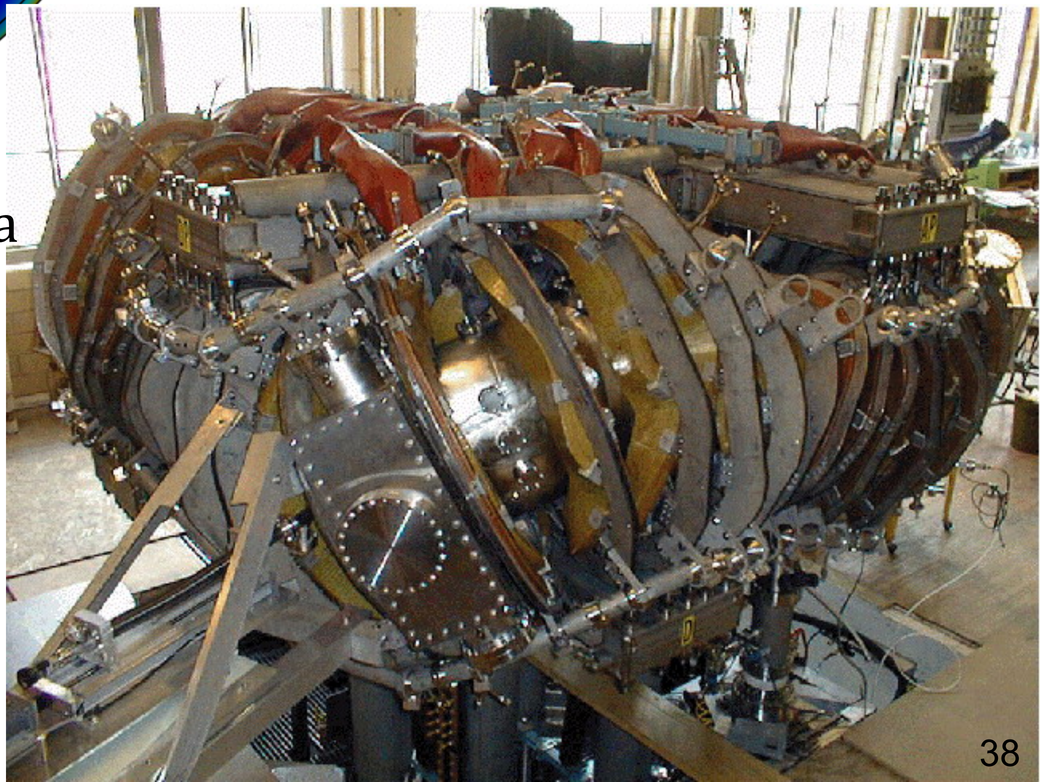


HSX:
Helically Symmetric eXperiment
(Univ. Wisconsin)

Magnetic field magnitude |B| in Tesla





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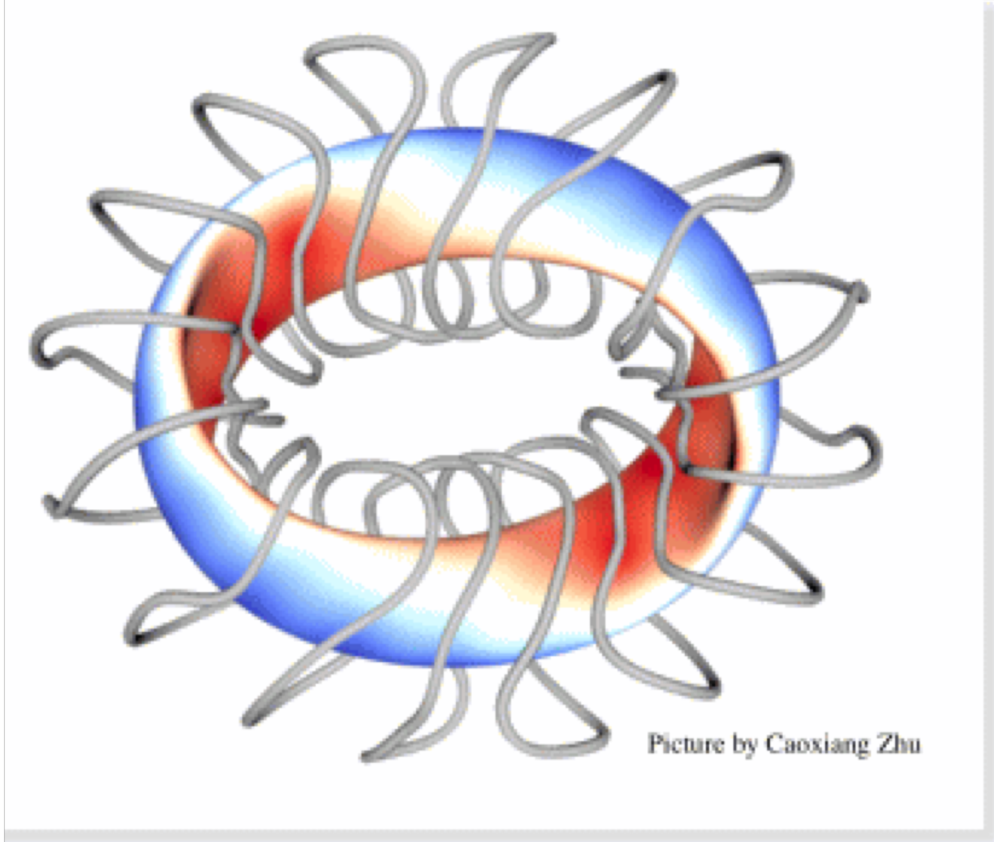
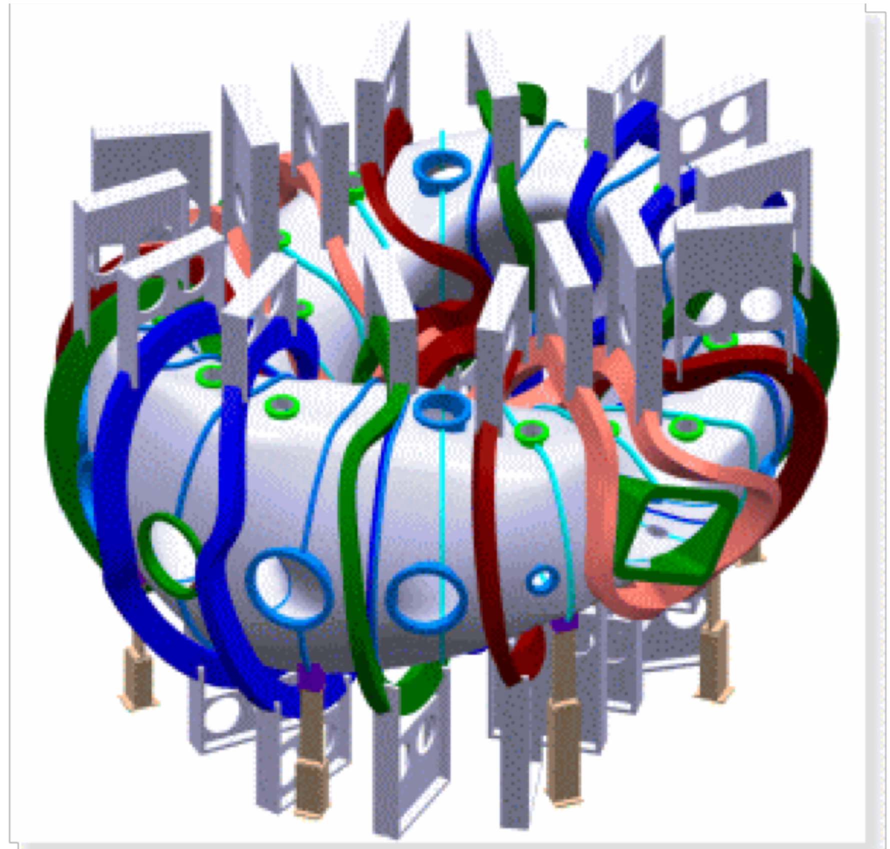
CFQS (Chinese First Quasi-symmetric Stellarator),
Under construction



CFQS modular coil shape and plasma



CFQS, coils and vacuum chamber

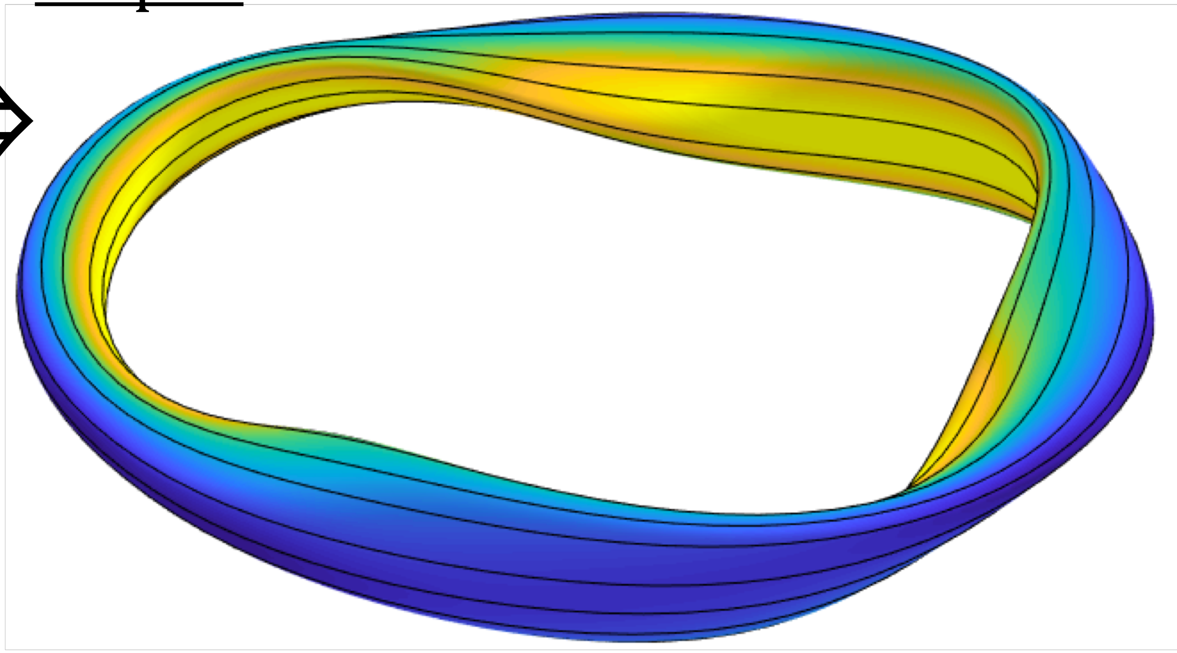
We have developed a new procedure to construct quasi-symmetric configurations.

Landreman, Sengupta, & Plunk, J. Plasma Physics (2018)

Directly solve equations for magnetohydrodynamic equilibrium & $\partial B / \partial \phi = 0$, expanding in aspect ratio. $>10^6 \times$ faster!

<u>Input:</u>
Shape of the magnetic axis, & few other constants. $\Rightarrow$

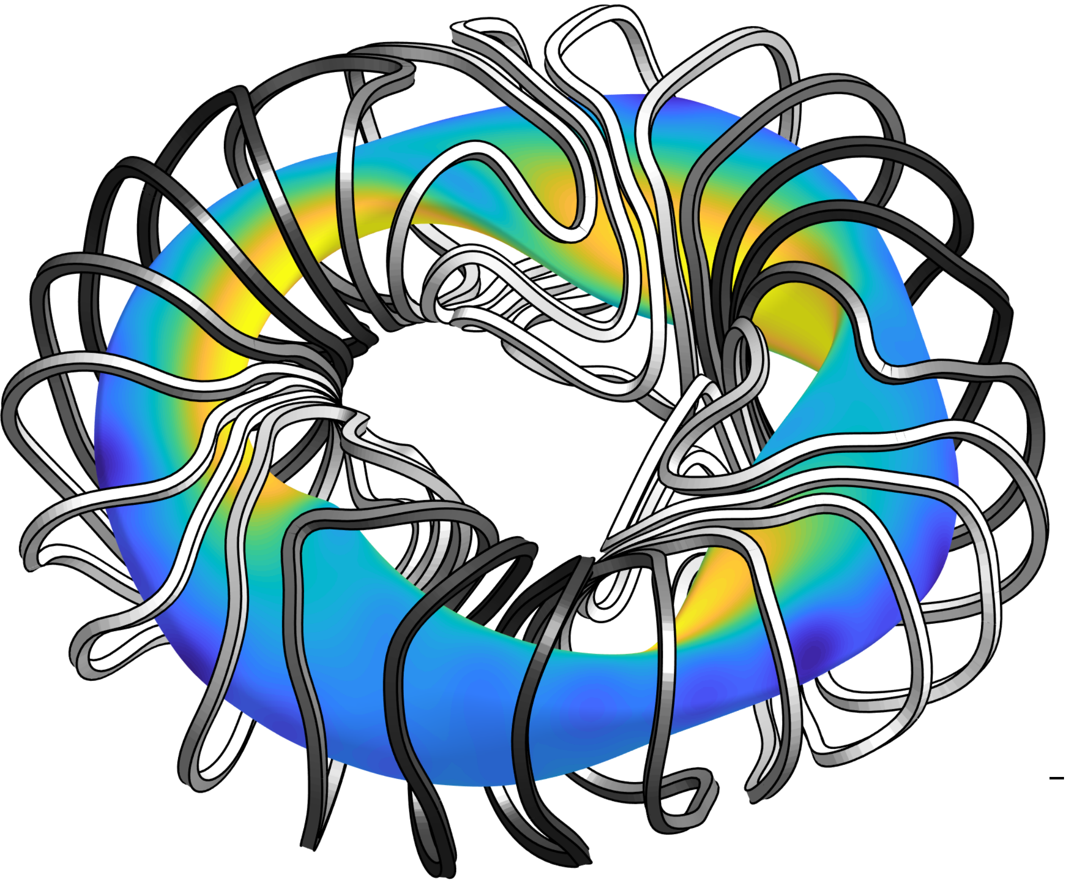
<u>Output:</u>



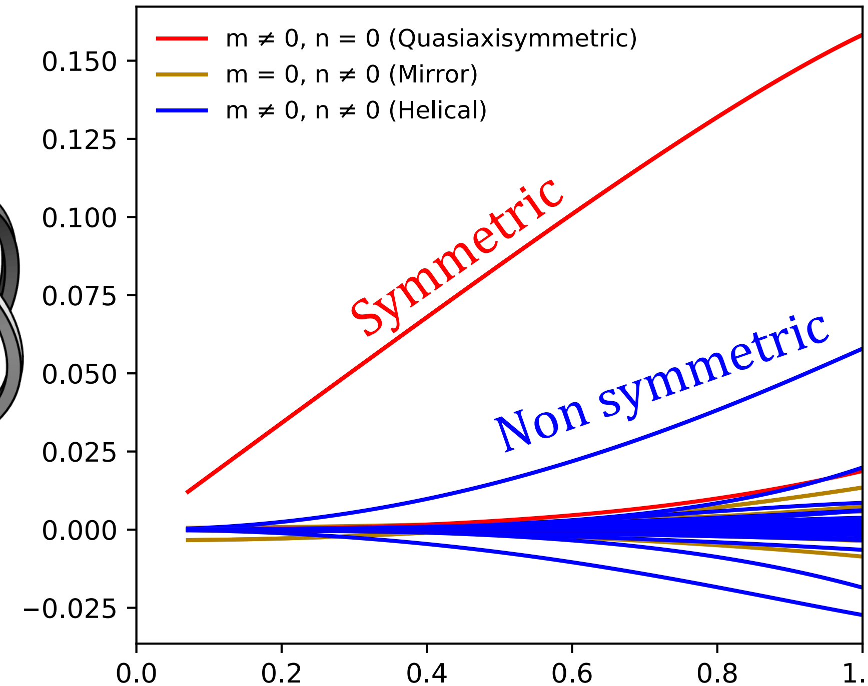
We have developed a new procedure to construct quasi-symmetric configurations.

Landreman, Sengupta, & Plunk, J. Plasma Physics (2018)

Realization with coils:

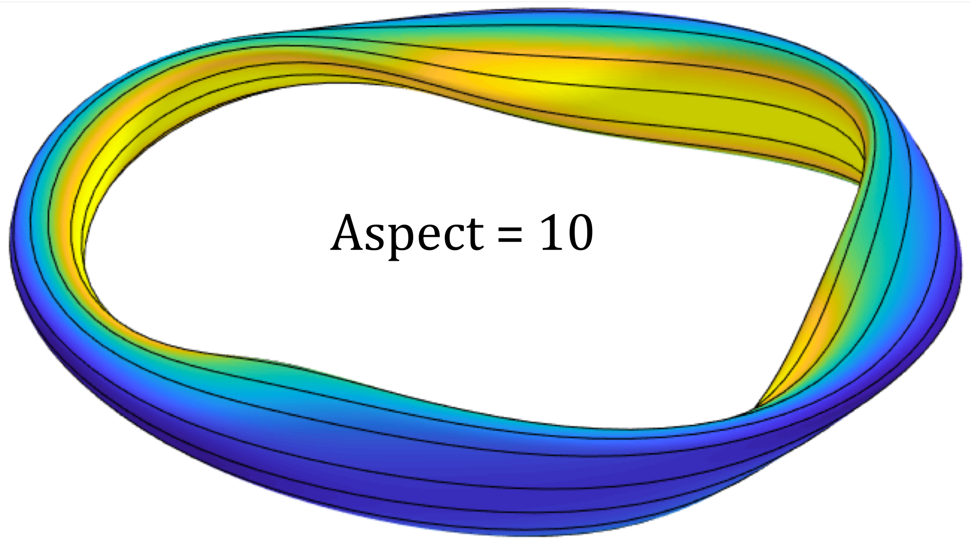


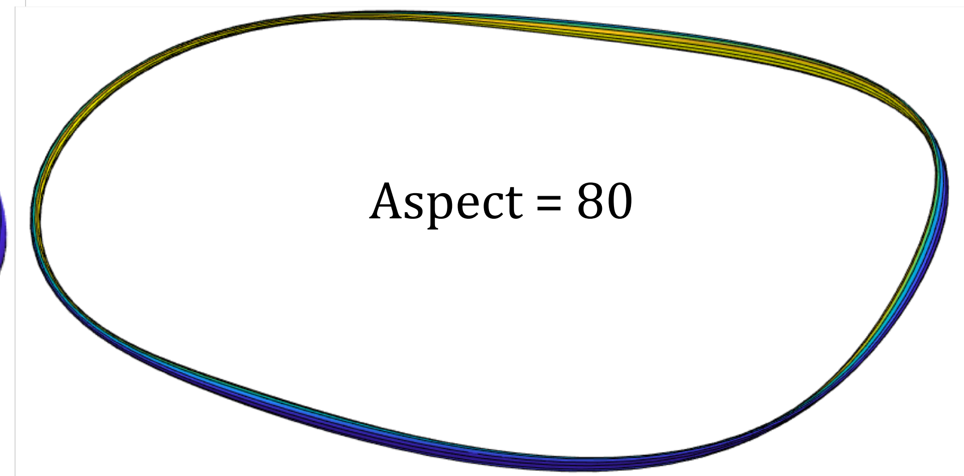
Fourier amplitudes $B_{m,n}$

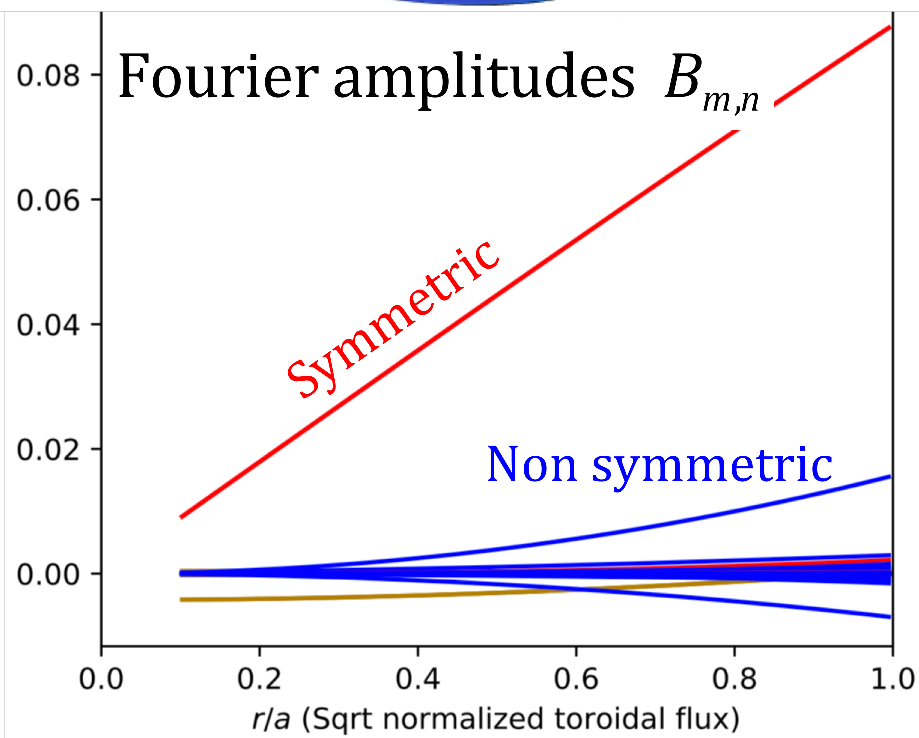


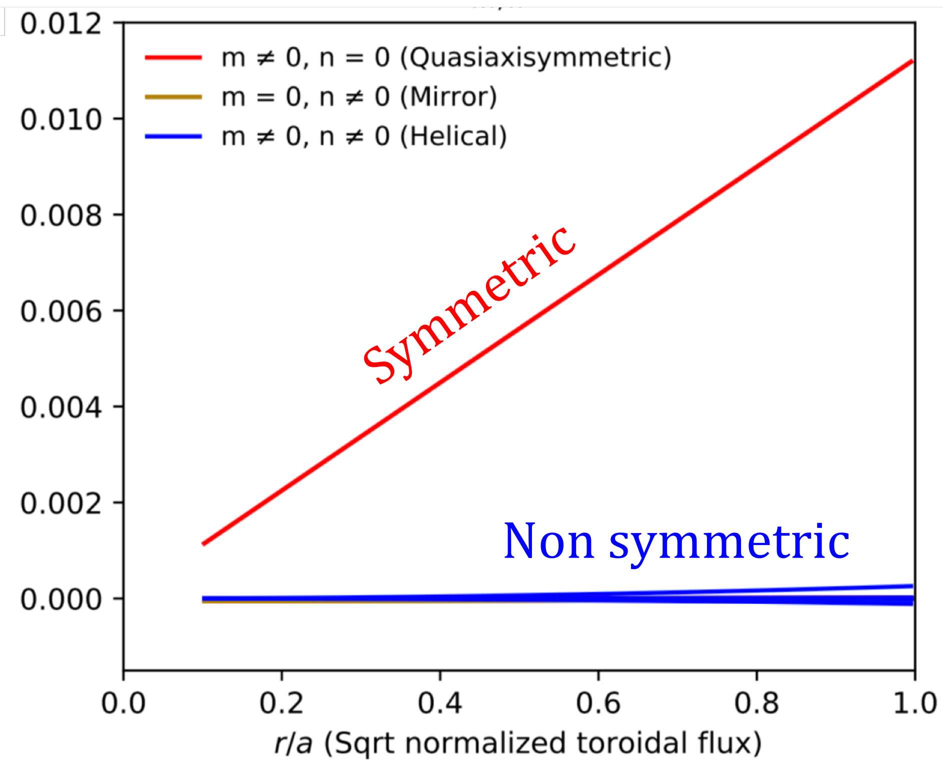
Minor radius $\left(\text{normalized}\right)$: $r\,/\,r_{\max}$

Quasisymmetry can be achieved more accurately at high "aspect ratio"







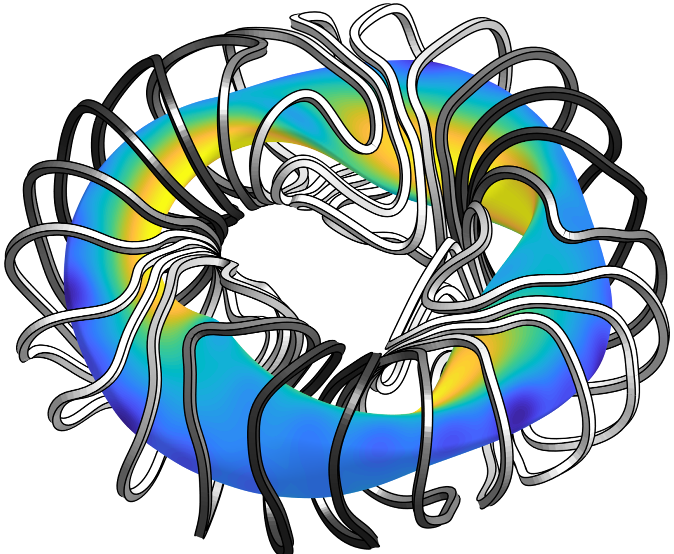


Conclusions:
Symmetry is important in magnetic confinement

- Magnetic confinement and symmetry are connected via canonical momentum conservation.
- Axisymmetry can yield robust confinement but requires an internal current **J**. With nonaxisymmetric shaping you don't need **J** but confinement is not automatic.
- Without axisymmetry, integrability of **B** is not automatic.
- For large |**B**|, a "hidden" symmetry can yield an approximate conserved quantity that implies particle confinement.

There are important outstanding questions about quasi-symmetry.

- Is there a coordinate-free way to see quasi-symmetry in the Lagrangian?
- Are there phenomena like quasi-symmetry in other physical systems?
- Can quasi-symmetric fields be produced with simply shaped coils far from the plasma?

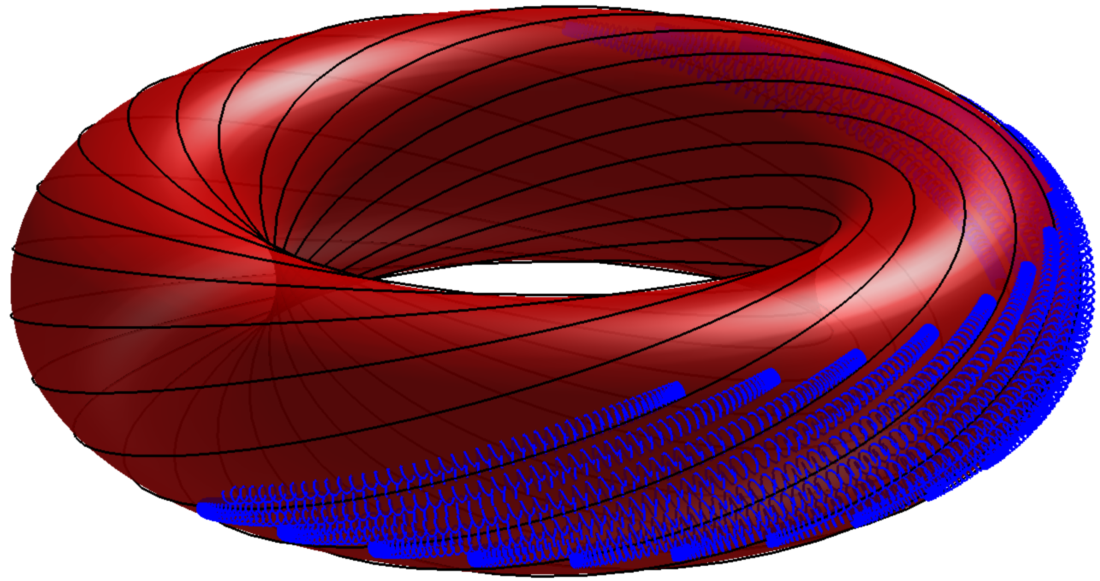


Extra slides

Tokamak vs stellarator

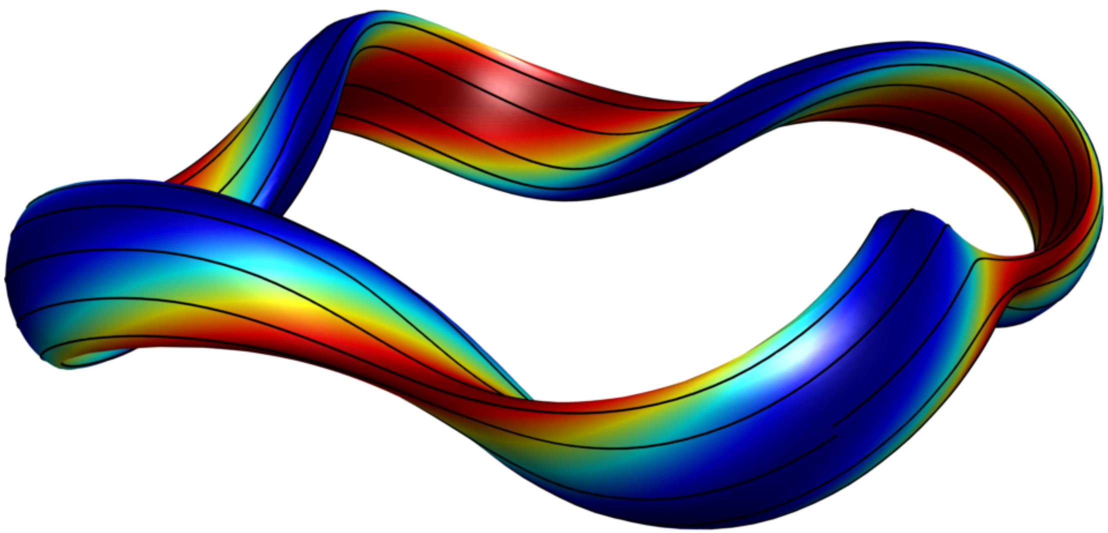
Tokamak:

- Axisymmetric
- Robust confinement
- Requires J_ϕ in plasma: HUGE problem!



Stellarator:

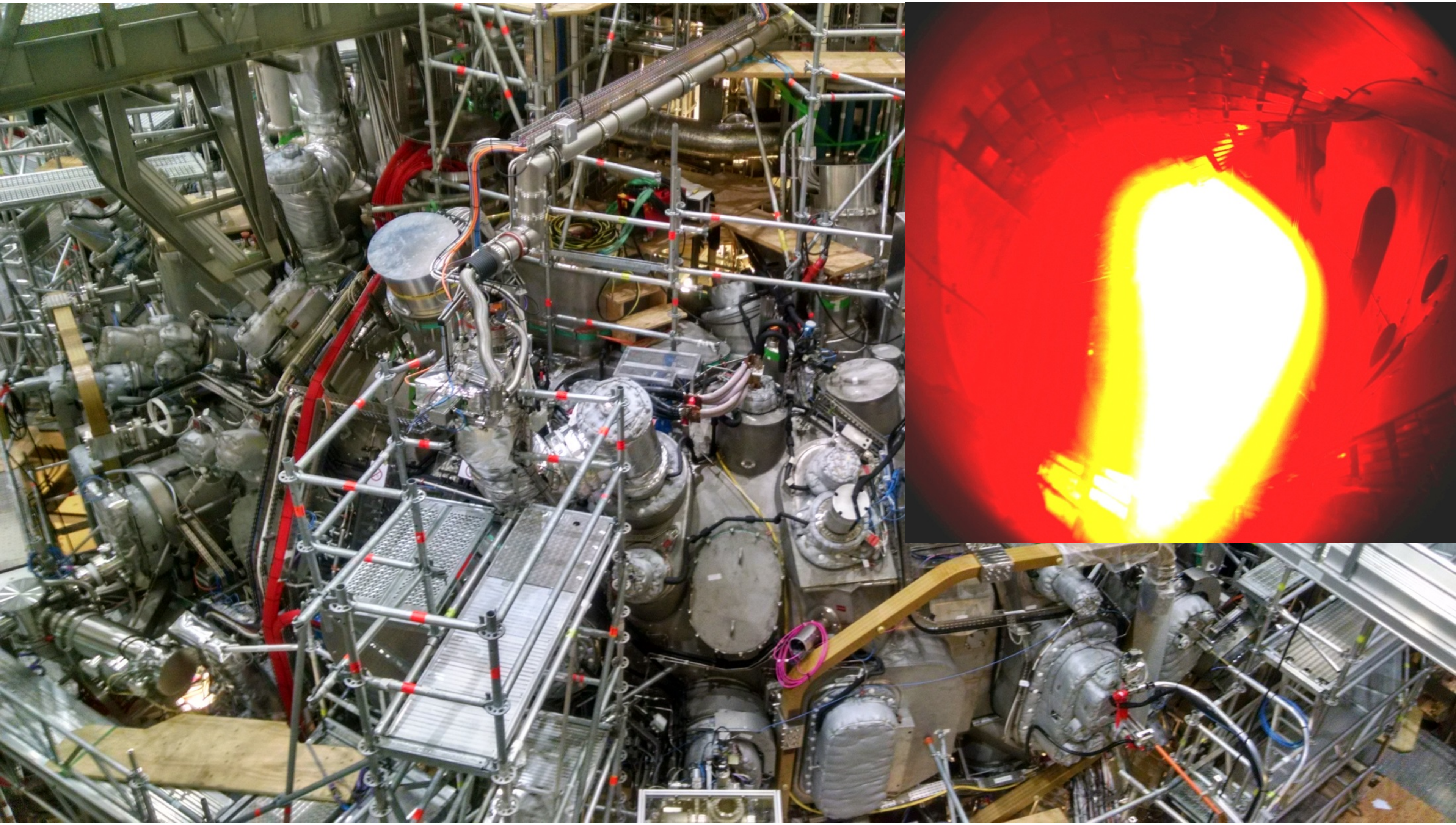
- Nonaxisymmetric
- Requires careful shaping to get confinement
- No **J** required in plasma



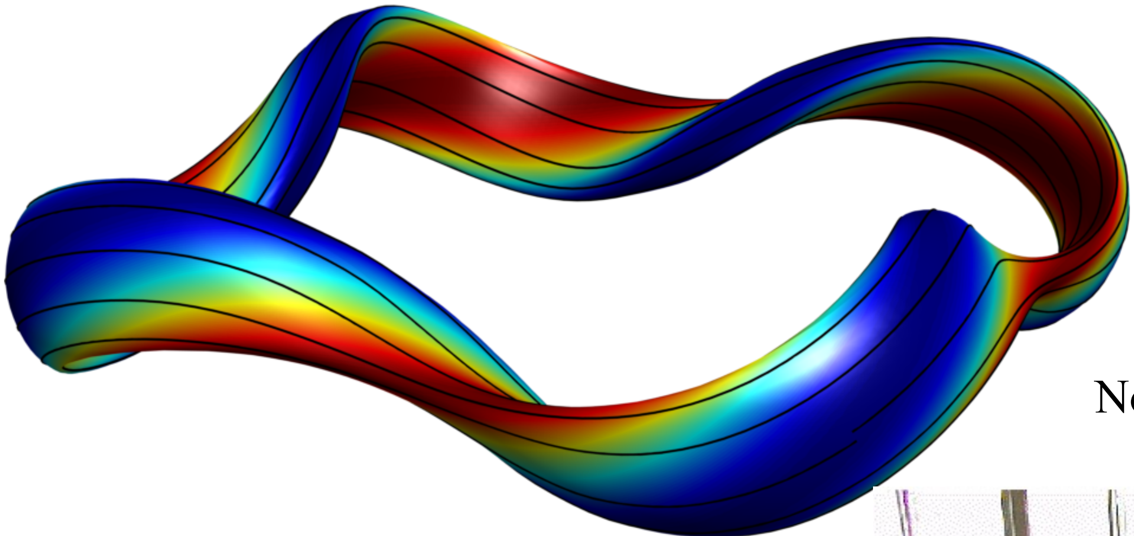
Example of very nonaxisymmetric magnetic confinement: W7-X (Germany)

Science Oct 21, 2015

Good-enough particle confinement, but not perfect -
Not quasisymmetric.



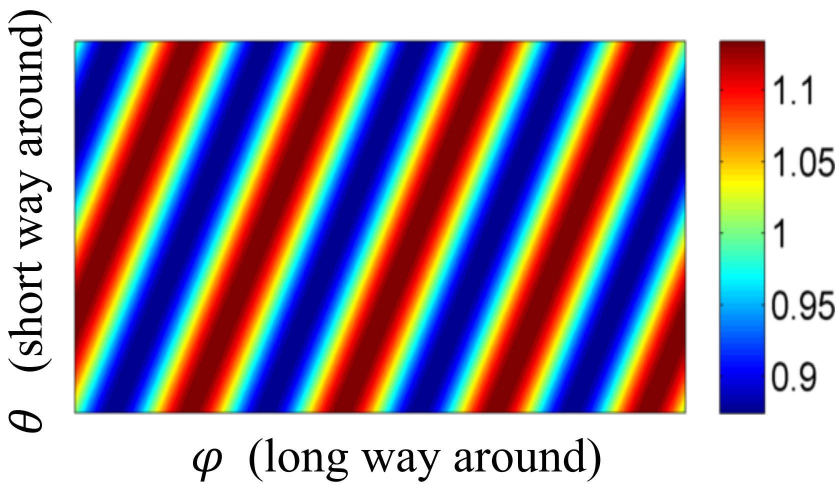
Quasi-symmetry can also be helical.

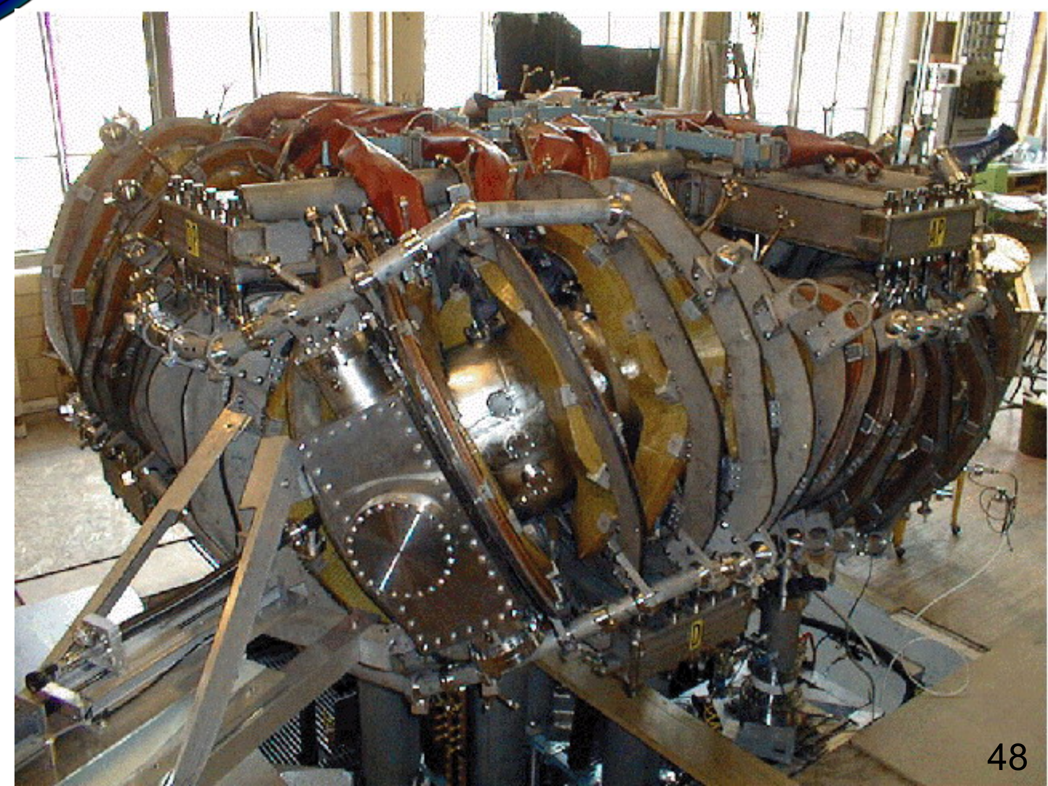


HSX:
Helically Symmetric eXperiment
(Univ. Wisconsin)

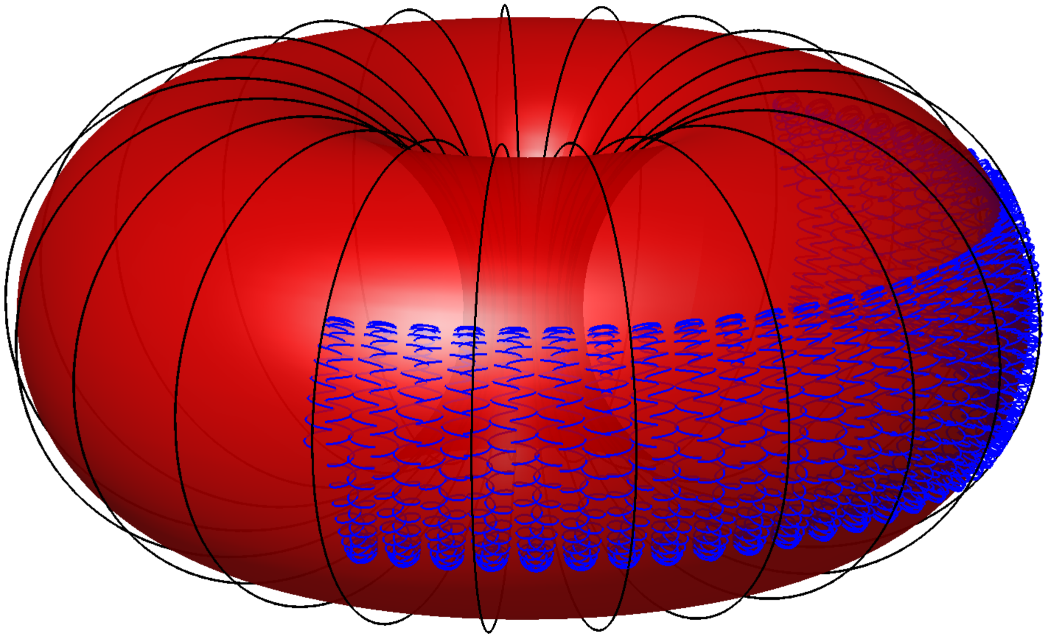
No **J** required in plasma $\Rightarrow$ Very stable.

Magnetic field magnitude |B| in Tesla

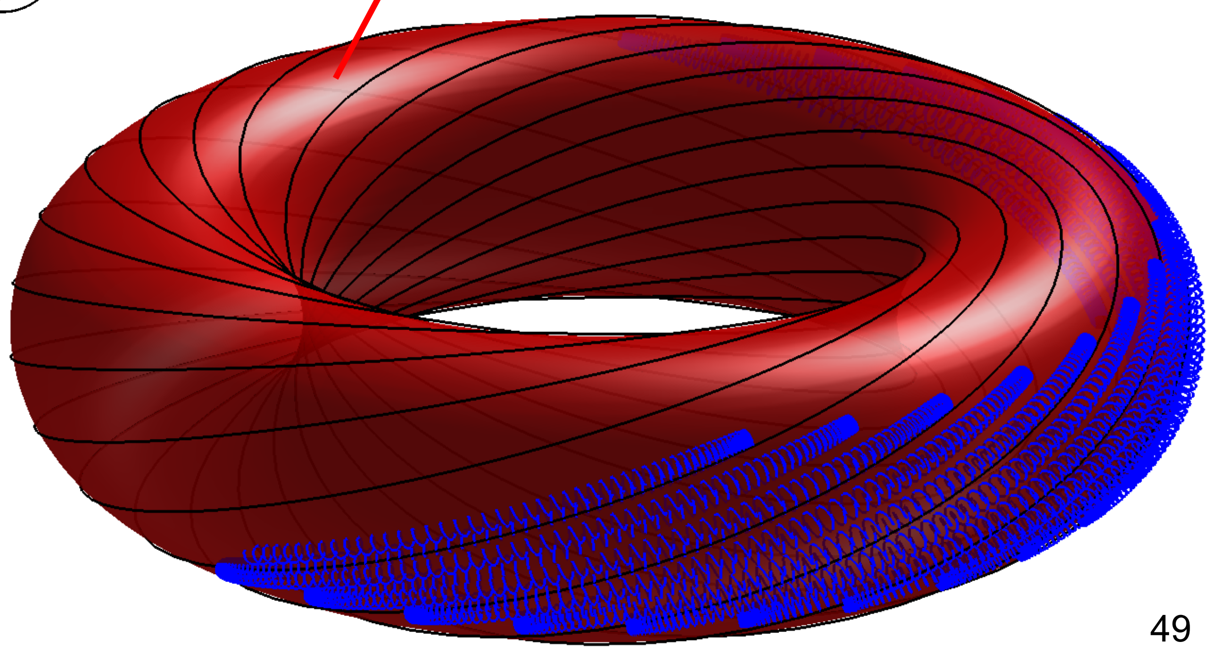




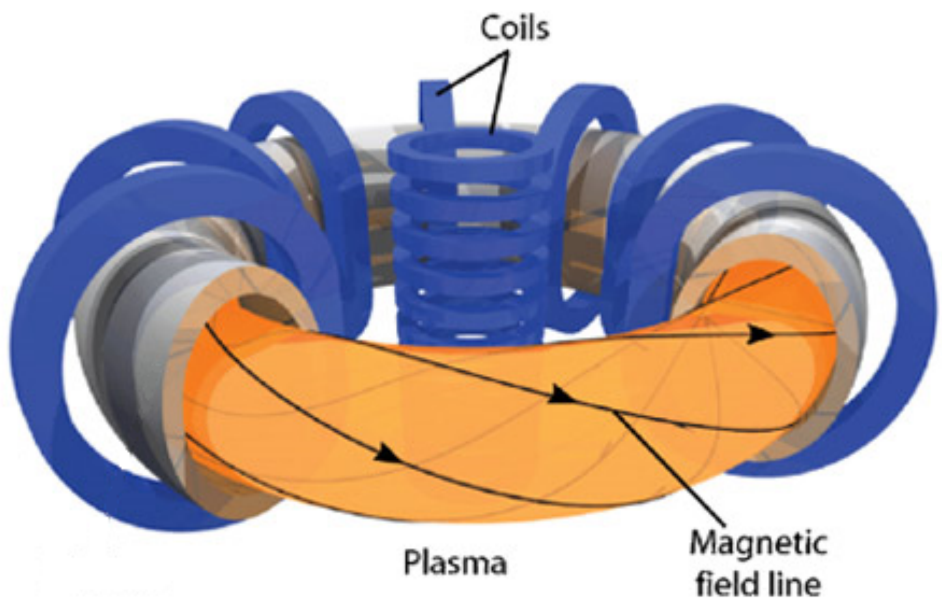
In axisymmetry, particles are confined (close) to $A_\varphi R$ surfaces, despite complicated orbits.

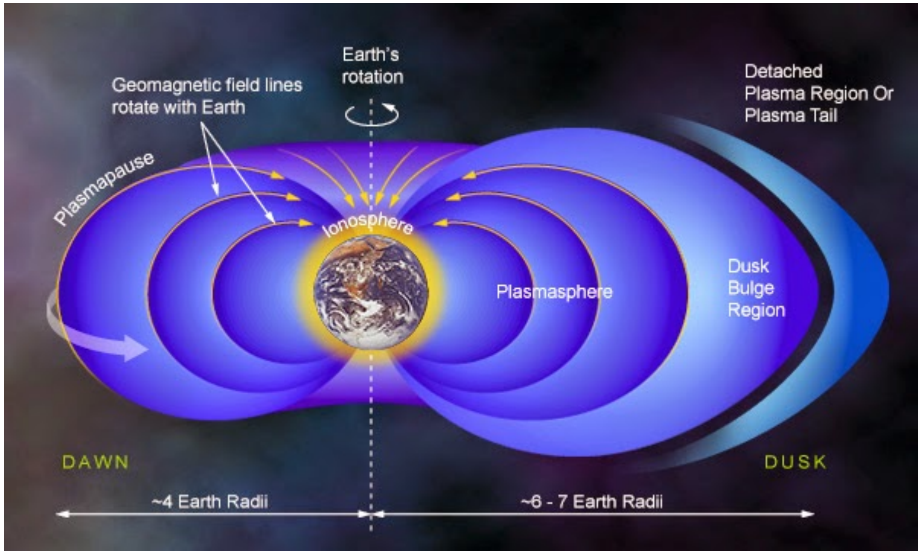


"Magnetic surface": Traced out by a magnetic field line

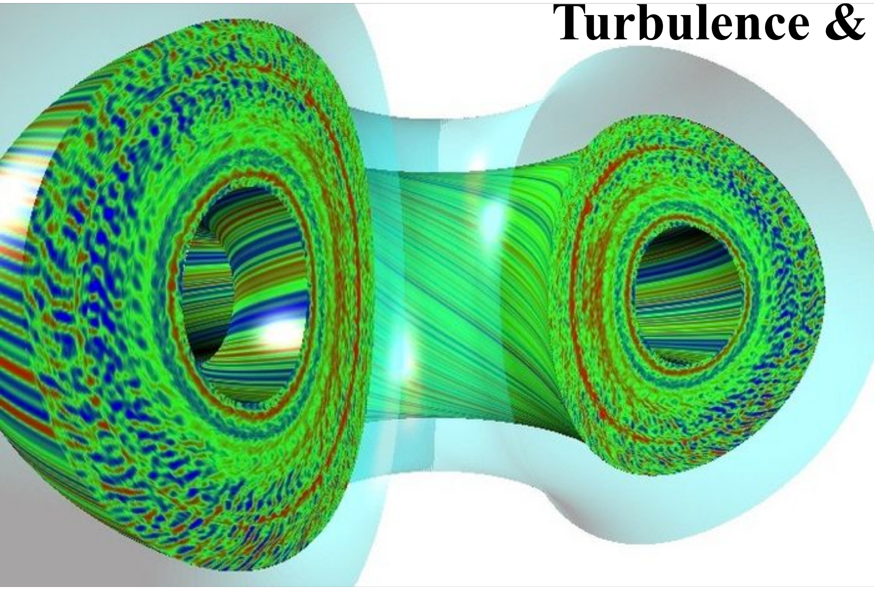


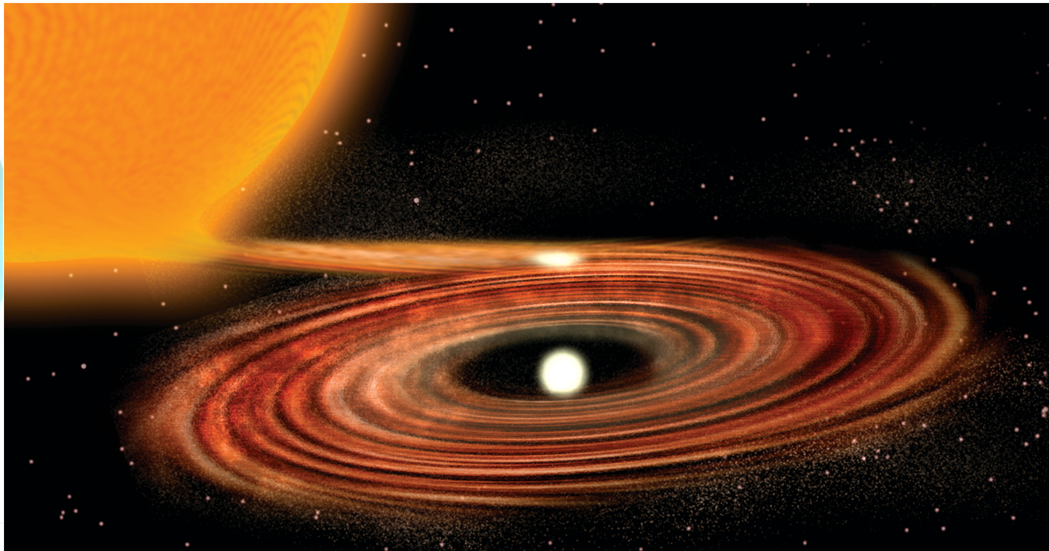
No plasma is perfectly axisymmetric.





Turbulence & waves break symmetry:





Why not make the nested $A_\varphi R$ surfaces spherical instead of toroidal?

$A_\phi R$ would need to depend on Z along the symmetry axis.

$$B_R = -\frac{\partial A_\phi}{\partial Z} = -\frac{1}{R}\frac{\partial\left(A_\phi R\right)}{\partial Z}$$

So B_R would diverge $\left(\propto 1/R\right)$ along the symmetry axis.

Curl in cylindrical coordinates, assuming axisymmetry

$$B_R = -\frac{\partial A_\phi}{\partial Z} = -\frac{1}{R}\frac{\partial\left(A_\phi R\right)}{\partial Z}$$

$$B_\phi = \frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R}$$

$$B_Z = \frac{1}{R}\frac{\partial\left(A_\phi R\right)}{\partial R}$$