Confining charged particle orbits using hidden symmetry

Matt Landreman, UMD IREAP
Outline

• Magnetic confinement, & pros/cons of axisymmetry
• Integrability of non-axisymmetric $\mathbf{B}$ fields
• Quasi-symmetry
• Finding quasi-symmetric fields
Charged particles can be confined by magnetic fields in many contexts.

Planetary dipole fields

Particle traps for basic physics:

Hot laboratory plasmas, fusion
Confining charged particles with a magnetic field is tricky.

Uniform straight $\mathbf{B}$: confinement $\perp$ to $\mathbf{B}$, but end losses.
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But if field lines are bent, particles drift off them.
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Uniform straight $\mathbf{B}$: confinement $\perp$ to $\mathbf{B}$, but end losses.

But if field lines are bent, particles drift off them.

\[
(\text{Drift velocity}) \sim (\text{Particle speed}) \frac{(\text{Larmor radius})}{(\text{Scale length of } \mathbf{B})} \ll 1
\]
To confine particles, we can constrain their position with a conservation law.

Noether’s theorem:

For each continuous symmetry of a system*, there is a corresponding conserved quantity.

* For this talk: Lagrangian is independent of a coordinate.
Continuous rotational symmetry $\Rightarrow$ Canonical angular momentum is conserved.

$$L_\phi = mv_\phi R + qA_\phi R = \text{constant}$$

vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$
Continuous rotational symmetry  ⇒  Canonical angular momentum is conserved.

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vector potential:  \( \mathbf{B} = \nabla \times \mathbf{A} \)

Strong \( \mathbf{B} \) limit  ⇒  \( |m v_\phi| \ll |q A_\phi| \)  ⇒  Particles stuck to constant-\( A_\phi R \) surfaces.
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Vector potential: \( \mathbf{B} = \nabla \times \mathbf{A} \)

Strong \( \mathbf{B} \) limit ⇒ \( |m v_\phi| \ll |q A_\phi| \) ⇒ Particles stuck to constant-\( A_\phi R \) surfaces.

If \( A_\phi R \) surfaces are bounded like this, then particles will be confined:
In axisymmetry, particles are confined (close) to $A_\phi R$ surfaces, despite complicated orbits.

Particle orbits

$ma = qv \times B$
In axisymmetry, particles are confined (close) to $A_{\phi}R$ surfaces, despite complicated orbits.

\[ m\mathbf{a} = q\mathbf{v} \times \mathbf{B} \]
Particles are actually confined this way in nature and in the laboratory.

Van Allen (1959)
But, axisymmetric confinement has a big problem: requires an internal current.

\[ \nabla \times (\nabla \times A) = \nabla \times B = \mu_0 J \] so nested \( A_\phi R \) surfaces require a \( J_\phi \).
But, axisymmetric confinement has a big problem: requires an internal current.

\[ \nabla \times (\nabla \times A) = \nabla \times B = \mu_0 J \] so nested \( A_\phi R \) surfaces require a \( J_\phi \).

- Sustaining this current in steady-state is hard.
- This current drives instabilities.
- Not possible for low plasma density.

Can we achieve similar confinement without axisymmetry to avoid these problems?
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When axisymmetry is broken, we want field lines to still lie on surfaces.

**BAD:** Particle motion along $\mathbf{B}$ allows inside & outside to mix even without cross-$\mathbf{B}$ drift.

**GOOD:** $\mathbf{B}$ is “integrable”
When axisymmetry is broken, we want field lines to still lie on surfaces.

**BAD:** Particle motion along $B$ allows inside & outside to mix even without cross-$B$ drift.

**GOOD:** $B$ is “integrable”

Example: W7-X Stellarator

Hosoda, PRE (2009)

Pedersen, Nature Comm (2016)
A magnetic confinement device with a non-axisymmetric but integrable magnetic field is a "stellarator".

E.g. Wendelstein 7-X (Germany):

- Electromagnetic coils
- Magnetic field lines
- Magnetic surfaces, plasma
Integrability of magnetic fields can be viewed using Poincare plots

W7-X

Otte (2017)
Integrability of magnetic fields can be viewed using Poincare plots.

Good

Not so good

Islands

Chaos

W7-X

Otte (2017)

Kremer (2006)
Magnetic field lines can be described by a Hamiltonian, where “time” is $\phi$

\[
\frac{d\theta}{d\phi} = \frac{\partial H}{\partial r}, \quad \frac{dr}{d\phi} = -\frac{\partial H}{\partial \theta}
\]

So tools from Hamiltonian systems like KAM apply.
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Even with magnetic surfaces, confinement in non-axisymmetric fields is poor due to cross-B drift.

\[
(\text{Drift velocity}) \sim (\text{Particle speed}) \left( \frac{\text{Larmor radius}}{\text{Scale length of } B} \right) \ll 1
\]
Even with magnetic surfaces, confinement in non-axisymmetric fields is poor due to cross-B drift.

“Quasi-symmetry” provides a solution.
Quasi-symmetry is a continuous symmetry in $|B|$ (not vector $B$) that implies confinement.

- When the Lagrangian is (1) expanded for large $B=|\mathbf{B}|$ and (2) written in a special coordinate system ("Boozer angles"), it depends on position only through the surface and $B$. 
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- Therefore a symmetry in $B$ implies a conserved quantity, even if $\mathbf{B}$ has no obvious symmetry.
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- When the Lagrangian is (1) expanded for large $B=|\mathbf{B}|$ and (2) written in a special coordinate system (“Boozer angles”), it depends on position only through the surface and $B$.

- Therefore a symmetry in $B$ implies a conserved quantity, even if $\mathbf{B}$ has no obvious symmetry.

- This conserved quantity resembles canonical angular momentum, so it implies confinement just as in axisymmetry.
Averaging over gyration, Lagrangian depends on $B$ only through surface and $|B|$. 

Lagrangian for particle in magnetic field: $\mathcal{L} = qA \cdot \dot{x} + \frac{m}{2} |\dot{x}|^2$ (Neglect $E$)
Averaging over gyration, Lagrangian depends on $B$ only through surface and $|B|$.

Lagrangian for particle in magnetic field:

$$\mathcal{L} = qA \cdot \vec{x} + \frac{m}{2} \left| \dot{\vec{x}} \right|^2$$  
(Neglect $E$)

Introduce cylindrical velocity coordinates: $v_\| = \frac{\vec{v} \cdot \vec{B}}{B}, \quad \mu = \frac{mv_\perp^2}{2B}$, gyroangle $\alpha$. 


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Lagrangian for particle in magnetic field:
\[
\mathcal{L} = q A \cdot \dot{x} + \frac{m}{2} |\dot{x}|^2
\]  
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Introduce cylindrical velocity coordinates:
\[\nu_\parallel = \frac{v \cdot B}{B}, \quad \mu = \frac{mv_\perp^2}{2B},\]  
gyroangle $\alpha$.

Expand in \(\left(\text{Larmor radius}\right)/\left(\text{Scale length of } B\right) \ll 1\), i.e. $|B| \to \infty$.

\[
\mathcal{L}\left(\mathbf{x}, \nu_\parallel, \mu, \varphi, \dot{x}, \dot{\nu}_\parallel, \dot{\mu}, \varphi\right) = q A \cdot \dot{x} + \frac{m \nu_\parallel}{B} B \cdot \dot{x} + \frac{m}{q} \mu \dot{\alpha} - \frac{mv_\perp^2}{2} - \mu B
\]
Averaging over gyration, Lagrangian depends on $B$ only through surface and $|B|$. 

Lagrangian for particle in magnetic field: 

$$\mathcal{L} = qA \cdot \dot{x} + \frac{m}{2} |\dot{x}|^2$$  \hspace{1cm} \text{(Neglect } E)$$

Introduce cylindrical velocity coordinates: 

$$\nu_\parallel = \frac{v \cdot B}{B}, \quad \mu = \frac{mv_\perp^2}{2B}, \quad \text{gyroangle } \alpha.$$ 

Expand in \(\left(\text{Larmor radius}\right)/\left(\text{Scale length of } B\right) \ll 1\), i.e. \(|B| \to \infty\).

$$\mathcal{L}\left(x, \nu_\parallel, \mu, \varphi, \dot{x}, \dot{\nu}_\parallel, \dot{\mu}, \varphi\right) = qA \cdot \dot{x} + \frac{mv_\parallel}{B} B \cdot \dot{x} + \frac{m}{q} \mu \dot{\alpha} - \frac{mv_\parallel^2}{2} - \mu B$$

Spatial coordinates: surface label $r$ & ‘Boozer angles’ $(\theta, \phi)$:
Averaging over gyration, Lagrangian depends on \( B \) only through surface and \(|B|\).

Lagrangian for particle in magnetic field: \[ \mathcal{L} = qA \cdot \dot{x} + \frac{m}{2} \left| \dot{x} \right|^2 \] (Neglect \( \mathbf{E} \))

Introduce cylindrical velocity coordinates: \( v_\| = \frac{\mathbf{v} \cdot \mathbf{B}}{B} \), \( \mu = \frac{mv_\bot^2}{2B} \), gyroangle \( \alpha \).

Expand in \((\text{Larmor radius})/(\text{Scale length of } \mathbf{B}) \ll 1\), i.e. \(|\mathbf{B}| \rightarrow \infty\).

\[ \mathcal{L}(\mathbf{x}, v_\|, \mu, \varphi, \dot{x}, \dot{v}_\|, \dot{\mu}, \dot{\varphi}) = qA \cdot \dot{x} + \frac{mv_\|}{B} \mathbf{B} \cdot \dot{x} + \frac{m}{q} \mu \dot{\alpha} - \frac{mv_\bot^2}{2} - \mu B \]

Spatial coordinates: surface label \( r \) & ‘Boozer angles’ \((\theta, \phi)\):

\[ \mathcal{L} = q\psi_t \dot{\theta} - q\psi_p \dot{\phi} + \frac{mv_\|}{B} \left[ \dot{r} B_\psi(r, B) + \dot{\theta} B_\theta + \dot{\phi} B_\phi \right] + \frac{m}{q} \mu \dot{\alpha} - \frac{mv_\bot^2}{2} - \mu B \]

Depends only on \( r \)
Lagrangian for particle in magnetic field:
\[ \mathcal{L} = q \mathbf{A} \cdot \dot{\mathbf{x}} + \frac{m}{2} |\dot{x}|^2 \] (Neglect \( E \))

Introduce cylindrical velocity coordinates:
\[ v_\| = \frac{\mathbf{v} \cdot \mathbf{B}}{B}, \quad \mu = \frac{mv^2_\perp}{2B}, \] gyroangle \( \alpha \).

Expand in \((\text{Larmor radius})/\left(\text{Scale length of} \ B\right) \ll 1, \text{i.e.} \ |B| \rightarrow \infty.\)

\[ \mathcal{L} \left( \mathbf{x}, v_\|, \mu, \phi, \dot{x}, \dot{v}_\|, \dot{\mu}, \dot{\phi} \right) = q \mathbf{A} \cdot \dot{\mathbf{x}} + \frac{mv_\|}{B} \mathbf{B} \cdot \dot{\mathbf{x}} + \frac{m}{q} \mu \dot{\alpha} - \frac{mv^2_\|}{2} - \mu B \]

Spatial coordinates: surface label \( r \) & ‘Boozer angles’ \((\theta, \phi)\):

\[ \mathcal{L} = q \psi_t \dot{\theta} - q \psi_p \dot{\phi} + \frac{mv_\|}{B} \left[ \dot{r} B_\psi (r, B) + \dot{\theta} B_\theta + \dot{\phi} B_\phi \right] + \frac{m}{q} \mu \dot{\alpha} - \frac{mv^2_\|}{2} - \mu B \]

\[ \frac{\partial B}{\partial \phi} = 0 \quad \Rightarrow \quad \text{Conservation of} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -q \psi_p + \frac{mv_\| B_\phi}{B} \approx -q \psi_p \quad \Rightarrow \quad \text{Confinement!} \]
Due to quasisymmetry, different B fields can have isomorphic particle orbits.
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Several quasi-symmetric confinement experiments have been designed using optimization.

Boundary shape varied to minimize symmetry-breaking in $|B|$. 
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Boundary shape varied to minimize symmetry-breaking in $|B|$.

HSX:
Helically Symmetric eXperiment
(Univ. Wisconsin)

Magnetic field magnitude $|B|$ in Tesla

θ (short way around)

φ (long way around)
Several quasi-symmetric confinement experiments have been designed using optimization.

Boundary shape varied to minimize symmetry-breaking in |B|.

CFQS (Chinese First Quasi-symmetric Stellarator),
Under construction

CFQS modular coil shape and plasma

CFQS, coils and vacuum chamber
We have developed a new procedure to construct quasi-symmetric configurations.


Directly solve equations for magnetohydrodynamic equilibrium & \( \partial B / \partial \phi = 0 \), expanding in aspect ratio. \( >10^6 \times \) faster!

**Input:**
Shape of the magnetic axis, & few other constants.

**Output:**
We have developed a new procedure to construct quasi-symmetric configurations.


Realization with coils:

Fourier amplitudes $B_{m,n}$

- $m \neq 0, n = 0$ (Quasiaxisymmetric)
- $m = 0, n \neq 0$ (Mirror)
- $m \neq 0, n \neq 0$ (Helical)

Symmetric
Non symmetric

Minor radius (normalized): $r / r_{\text{max}}$
Quasisymmetry can be achieved more accurately at high “aspect ratio”

Aspect = 10

Aspect = 80

Fourier amplitudes $B_{m,n}$

Symmetric

Non symmetric

Symmetric

Non symmetric
Conclusions: Symmetry is important in magnetic confinement

- Magnetic confinement and symmetry are connected via canonical momentum conservation.
- Axisymmetry can yield robust confinement but requires an internal current $J$. With nonaxisymmetric shaping you don’t need $J$ but confinement is not automatic.
- Without axisymmetry, integrability of $B$ is not automatic.
- For large $|B|$, a “hidden” symmetry can yield an approximate conserved quantity that implies particle confinement.
There are important outstanding questions about quasi-symmetry.

- Is there a coordinate-free way to see quasi-symmetry in the Lagrangian?
- Are there phenomena like quasi-symmetry in other physical systems?
- Can quasi-symmetric fields be produced with simply shaped coils far from the plasma?
Extra slides
Tokamak:
- Axisymmetric
- Robust confinement
- Requires $J_{\phi}$ in plasma: HUGE problem!

Stellarator:
- Nonaxisymmetric
- Requires careful shaping to get confinement
- No $\mathbf{J}$ required in plasma
Example of very nonaxisymmetric magnetic confinement: W7-X (Germany)

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Good-enough particle confinement, but not perfect - Not quasisymmetric.
Quasi-symmetry can also be helical.

HSX: Helically Symmetric eXperiment (Univ. Wisconsin)

No J required in plasma ⇒ Very stable.
In axisymmetry, particles are confined (close) to $A_\phi R$ surfaces, despite complicated orbits.

“Magnetic surface”: Traced out by a magnetic field line
No plasma is perfectly axisymmetric.

Turbulence & waves break symmetry:
Why not make the nested \( A_\phi R \) surfaces spherical instead of toroidal?

\( A_\phi R \) would need to depend on \( Z \) along the symmetry axis.

\[
B_R = - \frac{\partial A_\phi}{\partial Z} = - \frac{1}{R} \frac{\partial (A_\phi R)}{\partial Z}
\]

So \( B_R \) would diverge \(( \propto 1/R)\) along the symmetry axis.
Curl in cylindrical coordinates, assuming axisymmetry

\[ B_R = -\frac{\partial A_\phi}{\partial Z} = -\frac{1}{R} \frac{\partial (A_\phi R)}{\partial Z} \]

\[ B_\phi = \frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R} \]

\[ B_Z = \frac{1}{R} \frac{\partial (A_\phi R)}{\partial R} \]