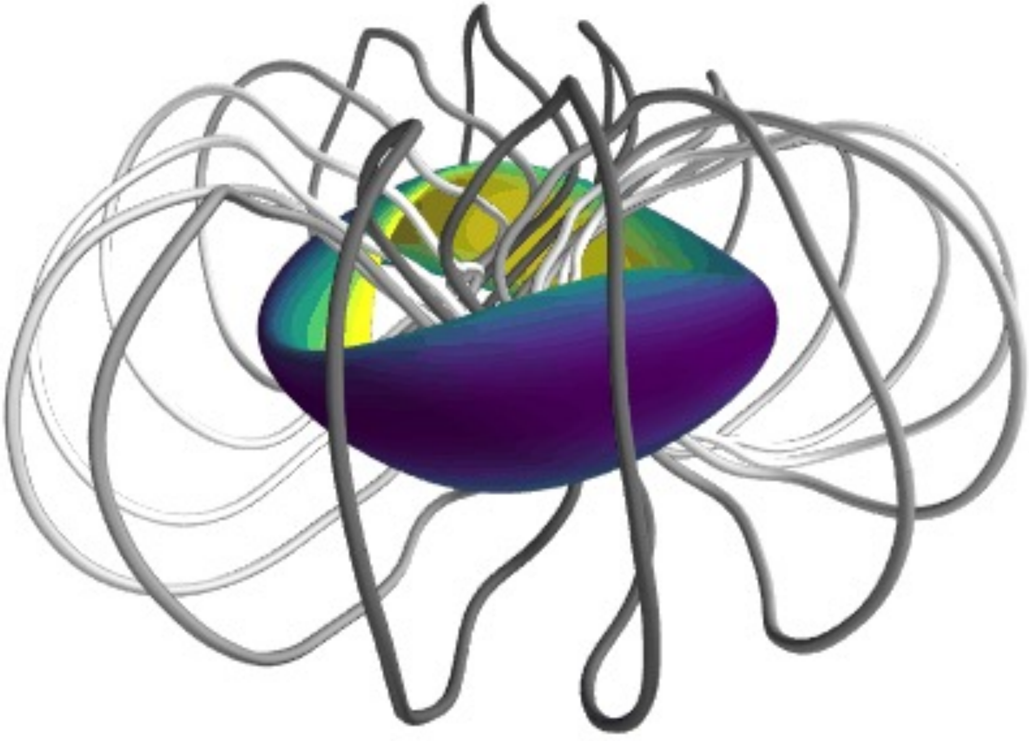
Inverse magnetostatics
& its application to stellarator optimization



Matt Landreman[1], Alan Kaptanoglu[2], John Kappel[1], Gabriel Langlois[2], Dhairya Malhotra[3]
1. University of Maryland 2. NYU Courant 3. Flatiron Institute

Inverse magnetostatics:

We want a certain magnetic field $\boldsymbol{B}$ in a region Ω.

Find an arrangement of electric currents outside Ω that produce $\boldsymbol{B}$.

I.e., invert the Biot-Savart Law $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$

Inverse magnetostatics:

We want a certain magnetic field $\boldsymbol{B}$ in a region Ω.

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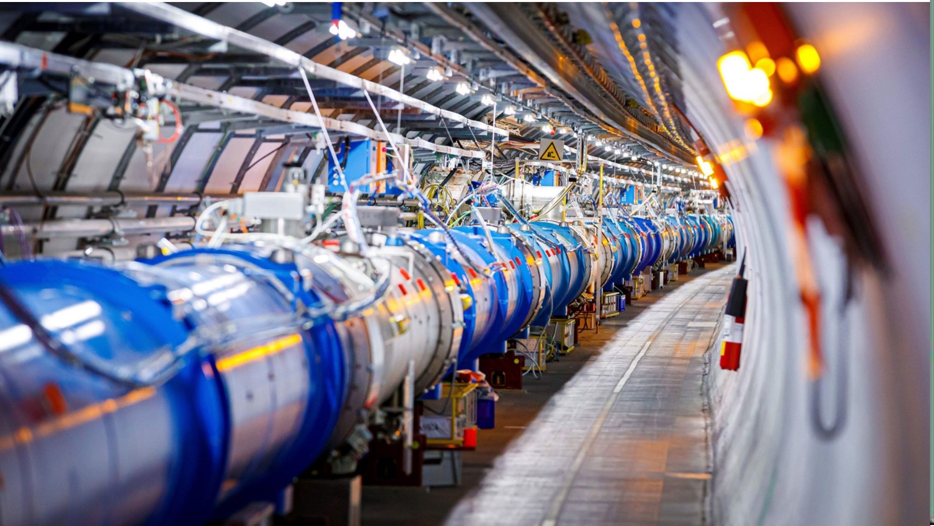
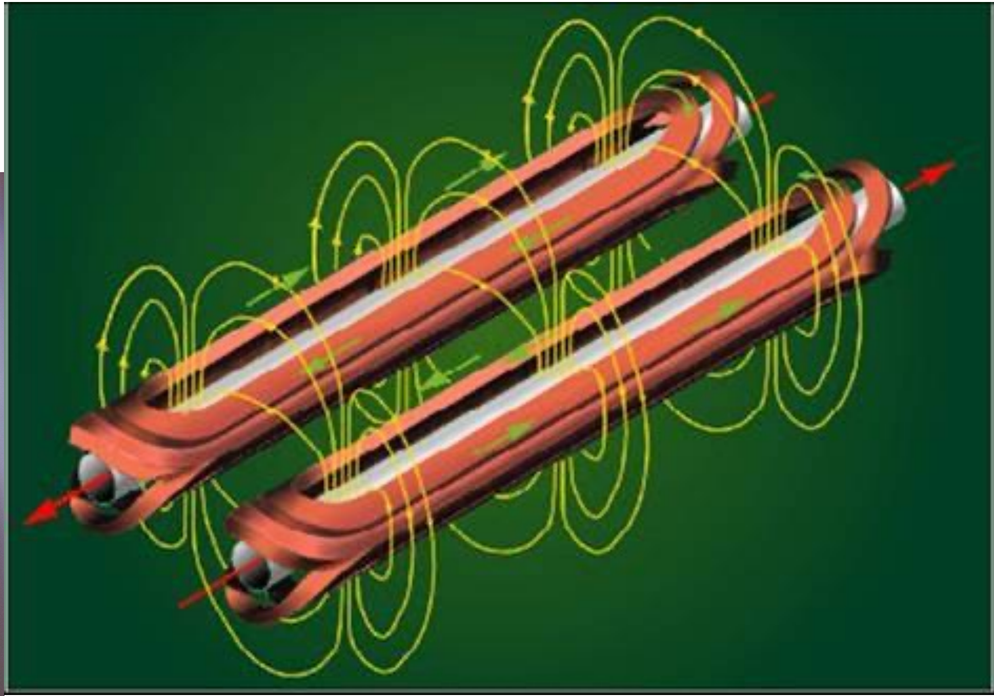
Example: Particle accelerators





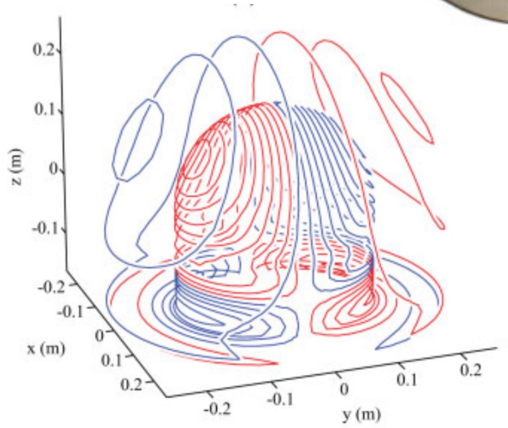
Inverse magnetostatics:

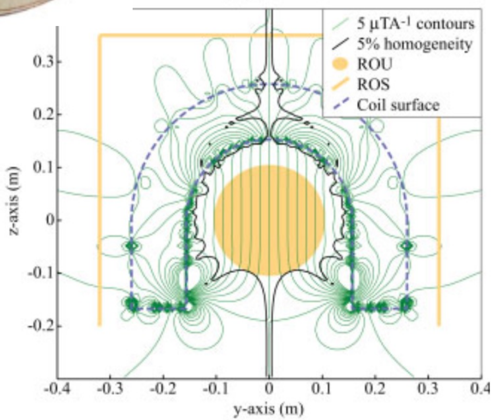
We want a certain magnetic field ***B*** in a region Ω.

Find an arrangement of electric currents outside Ω that produce ***B***.

Example: Magnetic resonance imaging

Poole & Bowtell (2007)



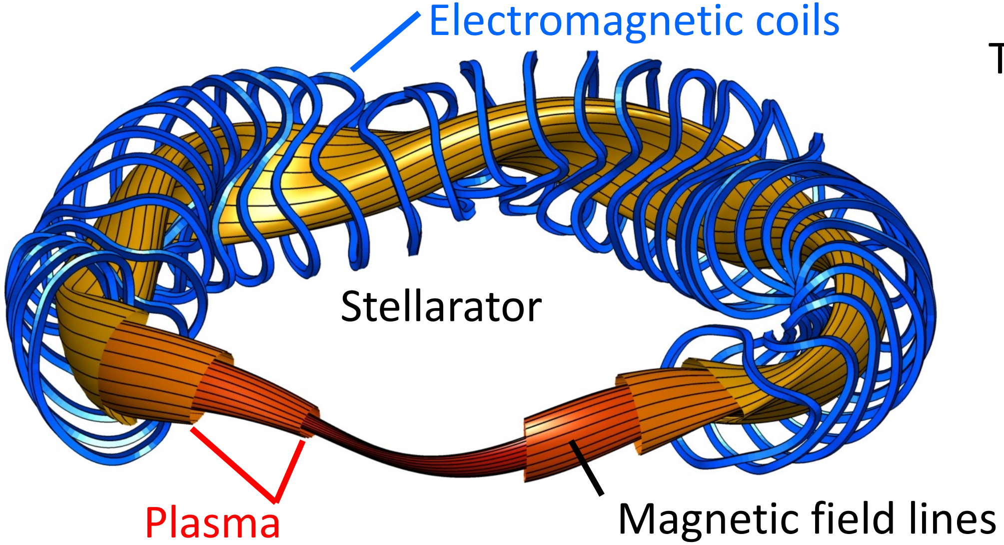


Inverse magnetostatics:

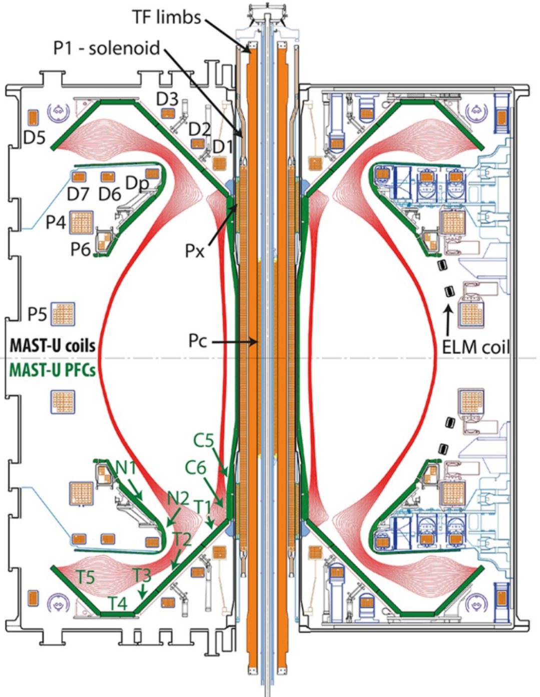
We want a certain magnetic field ***B*** in a region Ω.

Find an arrangement of electric currents outside Ω that produce ***B***.

Example: Magnetic confinement fusion



Tokamak→

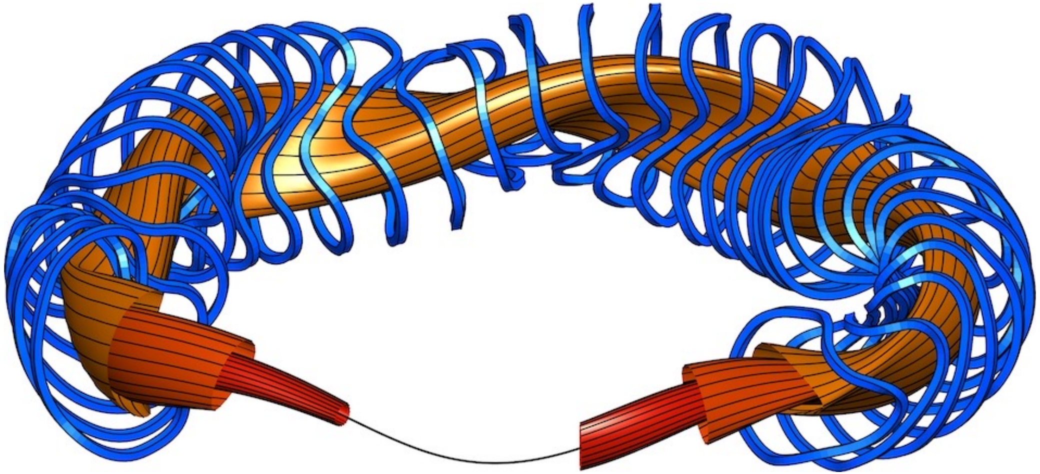


Outline

- Background & previous approaches
- Current voxels: topology optimization for electromagnets

  A A Kaptanoglu, G P Langlois, & ML, Comp Meth Appl Mech Engr (2023)
- Bounding the distance to the coils

  J Kappel, ML, & D Malhotra, arXiv:2309.11342 (2023)



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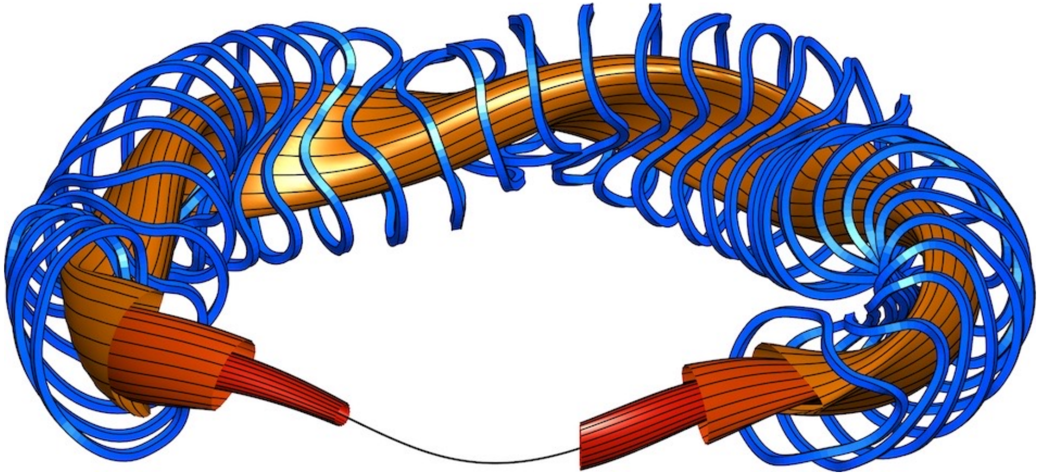
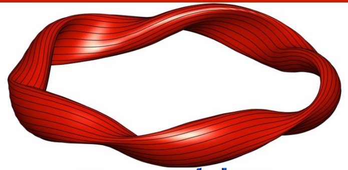
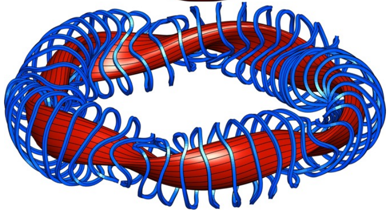


Most transport-optimized stellarators have used 2 sequential optimization stages



1. Parameters = shape of boundary toroidal surface. Objective = physics (confinement, stability, etc.)



2. Parameters = coil shapes.
   Objective = error in **B** on boundary shape from stage 1.

Shape of a toroidal boundary surface (+ pressure & current vs *r* inside, & total **B** flux) determines **B** everywhere inside:

$$\text{Consider a low-pressure plasma so } 0 \approx \mathbf{J} = \nabla \times \mathbf{B} \quad \Rightarrow \quad \mathbf{B} = \nabla \Phi.$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \nabla^2 \Phi = 0.$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \text{ on boundary} \quad \Rightarrow \quad \mathbf{n} \cdot \nabla \Phi = 0.$$

$$\Rightarrow \quad \text{Laplace's eq with Neuman condition.}$$

$$\Rightarrow \quad \text{Unique solution up to scale factor + constant.}$$

Most transport-optimized stellarators have used 2 sequential optimization stages

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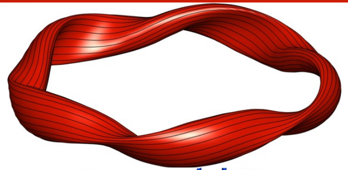

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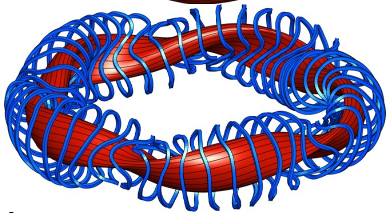
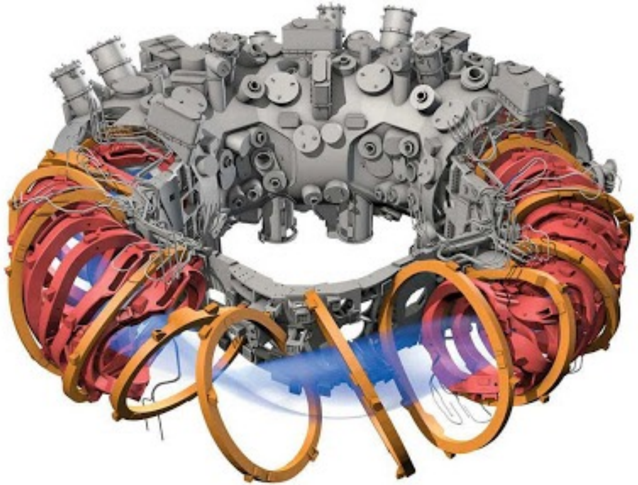
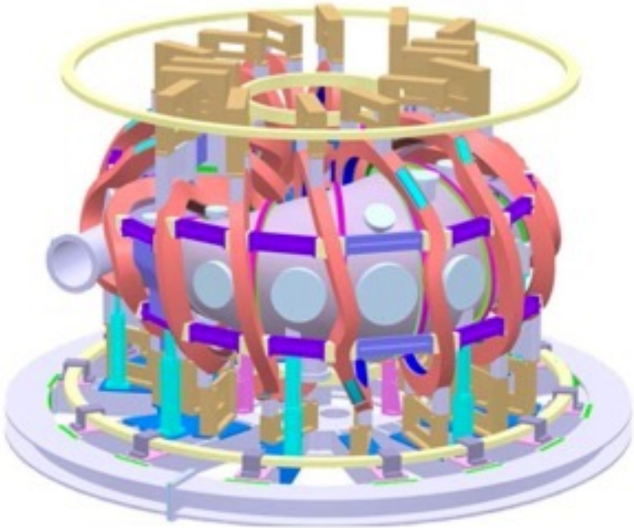


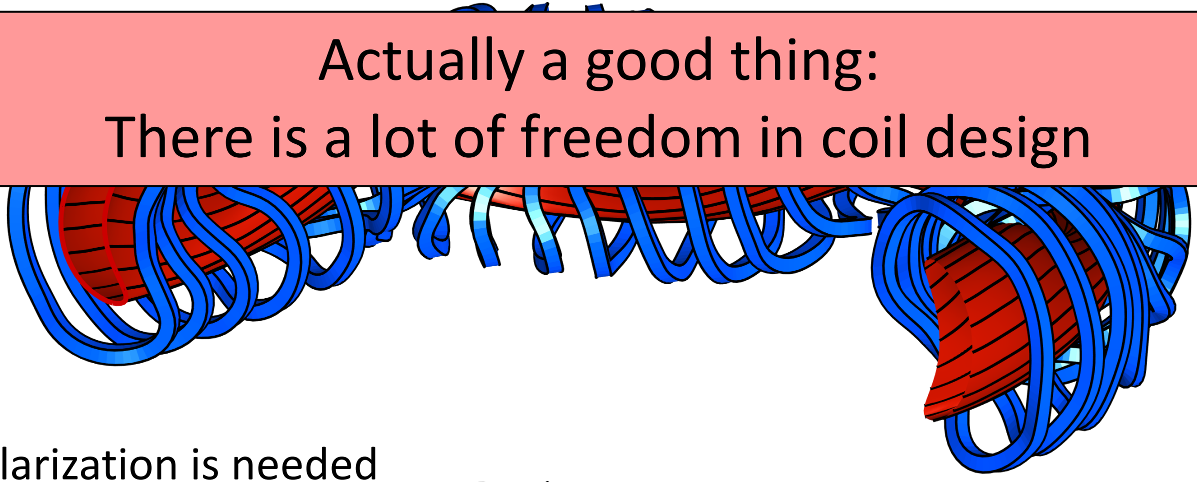
W7-X (Germany)



CFQS (China), under construction



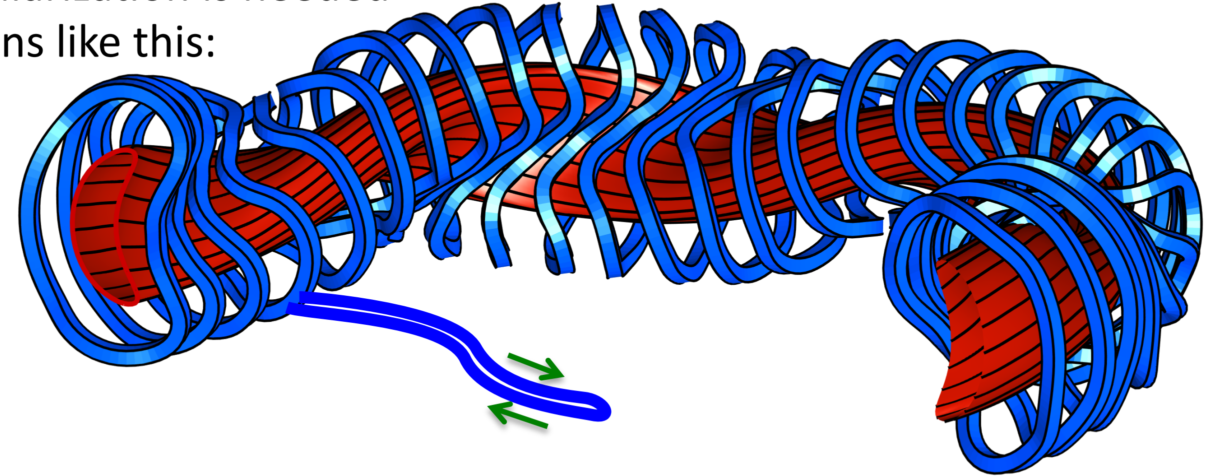
Calculating the currents that produce a given B is an ill-posed inverse problem: solution is not unique.



Actually a good thing:

There is a lot of freedom in coil design

Some kind of regularization is needed to exclude solutions like this:



Current potential methods: REGCOIL

Regcoil: Consider sheet current on a "coil winding surface"

ML, Nuclear Fusion (2017).

$$\boldsymbol{K} = \boldsymbol{n} \times \nabla \phi$$

Surface current

"current potential"

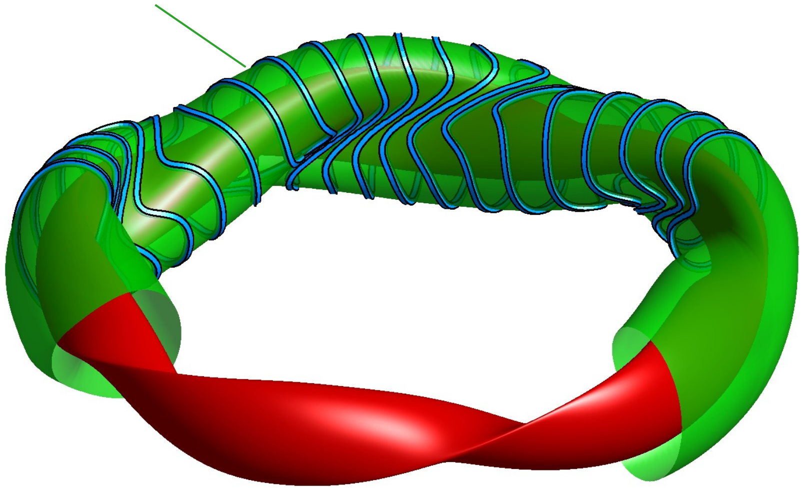
Normal to winding surface

$$\min_{\phi} \left(\int_{\text{Plasma surface}} d^2x \left[\left(\boldsymbol{B} - \boldsymbol{B}_{\text{target}} \right) \cdot \boldsymbol{n} \right]^2 + \lambda \int_{\text{Coil surface}} d^2x \, |\boldsymbol{K}|^2 \right)$$

B field error

Coil complexity

Regularization parameter



ϕ contours = coils

$K = |\boldsymbol{K}| \propto 1/\text{distance between coils}$

Pros:

- *Linear* least-squares: no local optima besides the global one.
- Only 2 parameters to vary: coil-to-plasma distance and λ.

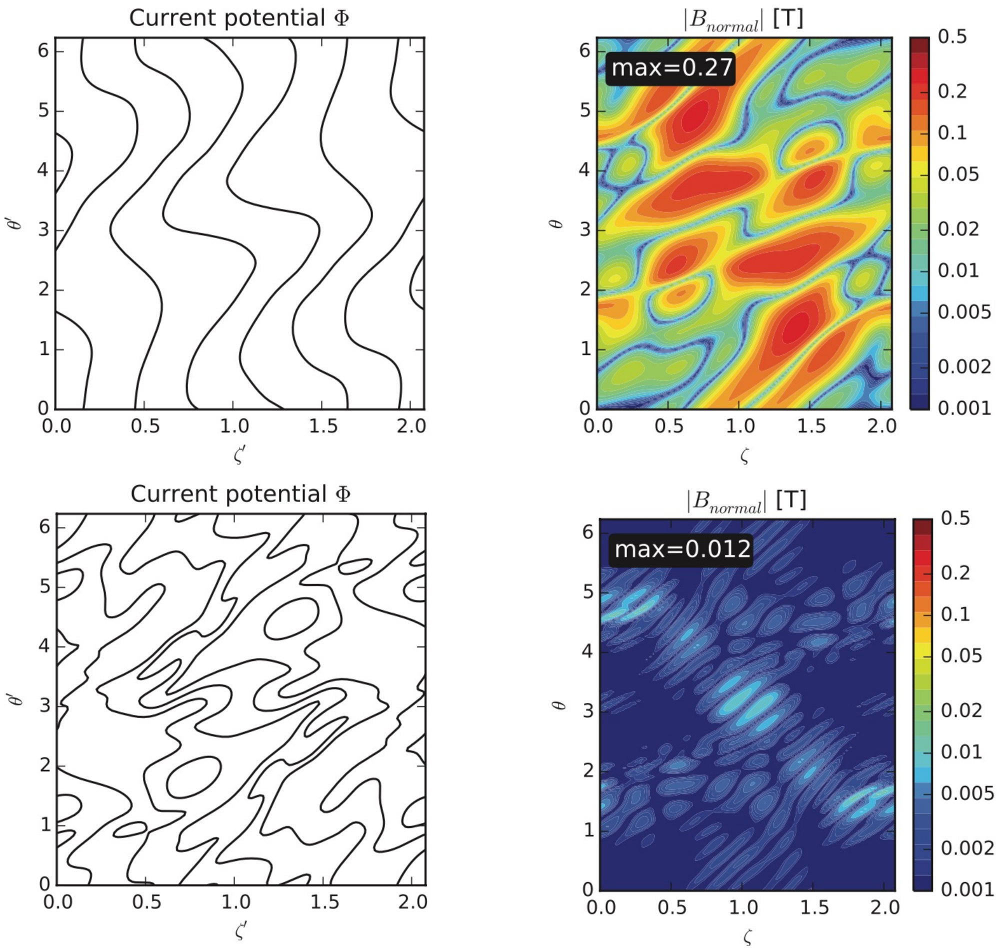
Cons:

- Neglects ripple from discrete coils.
- Limited flexibility in 3rd dimension.

In stage-2 coil optimization, there is a trade-off between field accuracy and coil simplicity

High regularization λ:
Simpler coils
but large field error

Low regularization λ:
Complicated coils
but small field error

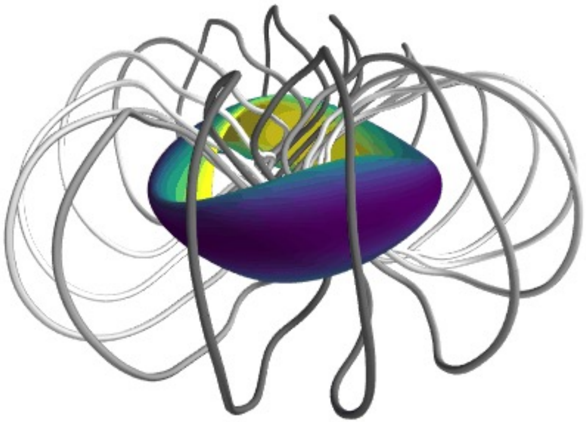


Filament coil optimization

Zhu, Hudson, et al, Nuclear Fusion (2018).

Coils represented as space curves.
Design variables: Fourier modes of Cartesian components.

$$x(t) = x_{c,0} + \sum_{n=1}^{N_F} \left[x_{c,n} \cos(nt) + x_{s,n} \sin(nt) \right]$$



Objective:

$$f = \underbrace{\int_{\substack{plasma \\ surf}} \left[\left(\boldsymbol{B} - \boldsymbol{B}_{target} \right) \cdot \boldsymbol{n} \right]^2}_{\text{Match target B}} + \underbrace{\lambda (length - target)^2 + \ldots}_{\text{Regularization}}$$

- Does account for *B* ripple from discreteness of coils.
- Non-convex, so there are multiple local minima. May need good initial guess.

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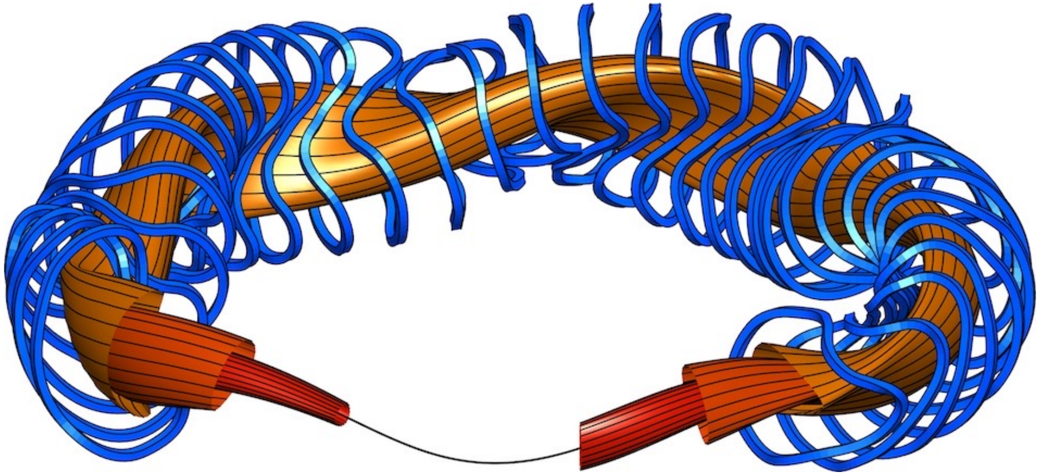
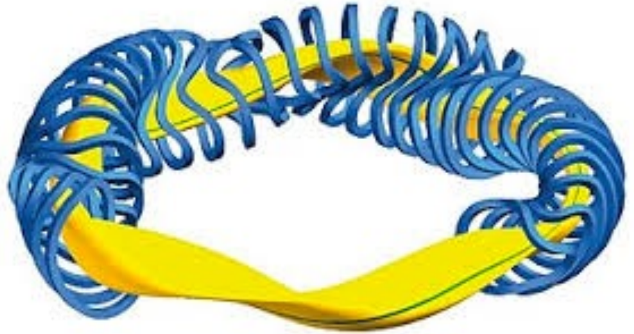
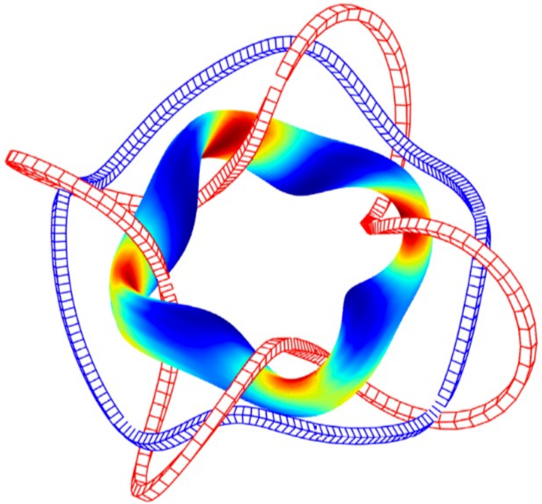


Current voxels: topology optimization for inverse magnetostatics

Kaptanoglu, Langlois, & ML, arXiv:2306.12555 (2023)

- Coil topology is an output rather than input.
- Generalize REGCOIL to lift restriction that currents lie on specified surface.
- Preserve REGCOIL advantages of linearity/convexity as far as possible.





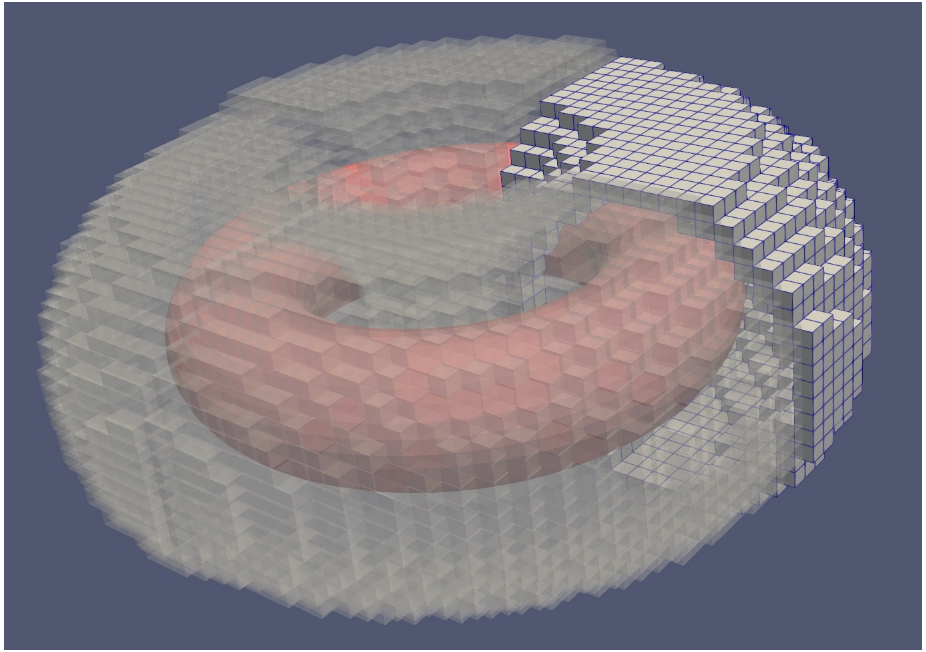
Yamaguchi (2019)

New topology optimization method for electromagnetic coils

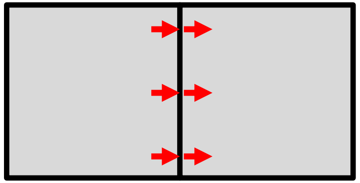
Pre-define grid of voxels where current might flow.

Current density ***J*** in each voxel represented by basis of 5 divergence-free functions. Amplitudes are the design variables.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}x\\-y\\0\end{pmatrix}, \begin{pmatrix}x\\0\\-z\end{pmatrix}$$



Charge conservation at each cell face gives linear equality constraints.



Alternative:
Design variables are the fluxes through faces. Linear constraints enforce 0 net flux in each volume.

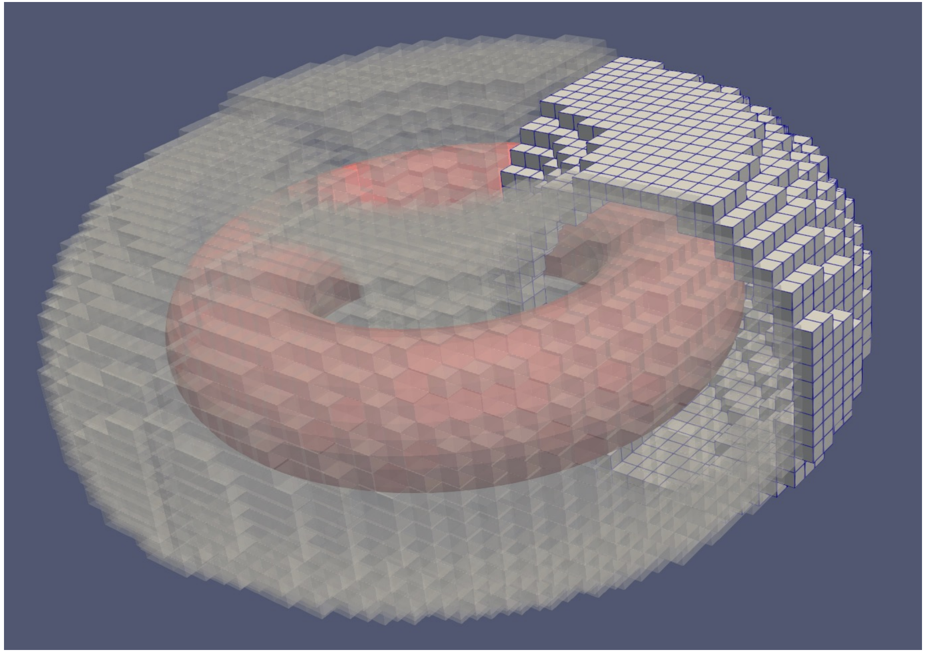
New topology optimization method for electromagnetic coils

Given basis functions, ***B*** can be computed by Biot-Savart Law:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

Minimize objective:

$$f = \underbrace{\int_{\substack{plasma\\surf}} (\boldsymbol{B}\cdot\boldsymbol{n})^2}_{\text{Match target B}} + \underbrace{\int_{voxels} \lambda\|\boldsymbol{J}\|_2^2 + \upsilon\|\boldsymbol{J}\|_1 + \eta\|\boldsymbol{J}\|_0}_{\text{Regularization}}$$



- Without sparsity-promoting terms, problem is linear.
- With ℓ_1 sparsity, problem remains convex.
- With ℓ_0 sparsity, good algorithms exist.

The ℓ_0-regularized problem is solved using the "relax & split" method

Original problem: $$\min_{\alpha}\left[\frac{1}{2}\|A\alpha-b\|_2^2+\lambda\|\alpha\|_0^G\right] \text{ s.t. } C\alpha=0.$$

Relaxed problem: $$\min_{\beta}\left\{\min_{\alpha}\left[\frac{1}{2}\|A\alpha-b\|_2^2+\frac{1}{2\nu}\|\alpha-\beta\|_2^2\right]+\lambda\|\beta\|_0^G\right\} \text{ s.t. } C\alpha=0.$$

Iterate: $$\alpha^{(k)}=\operatorname*{argmin}_{\alpha}\left[\frac{1}{2}\|A\alpha-b\|_2^2+\frac{1}{2\nu}\left\|\alpha-\beta^{(k-1)}\right\|_2^2\right] \text{ s.t. } C\alpha=0.$$

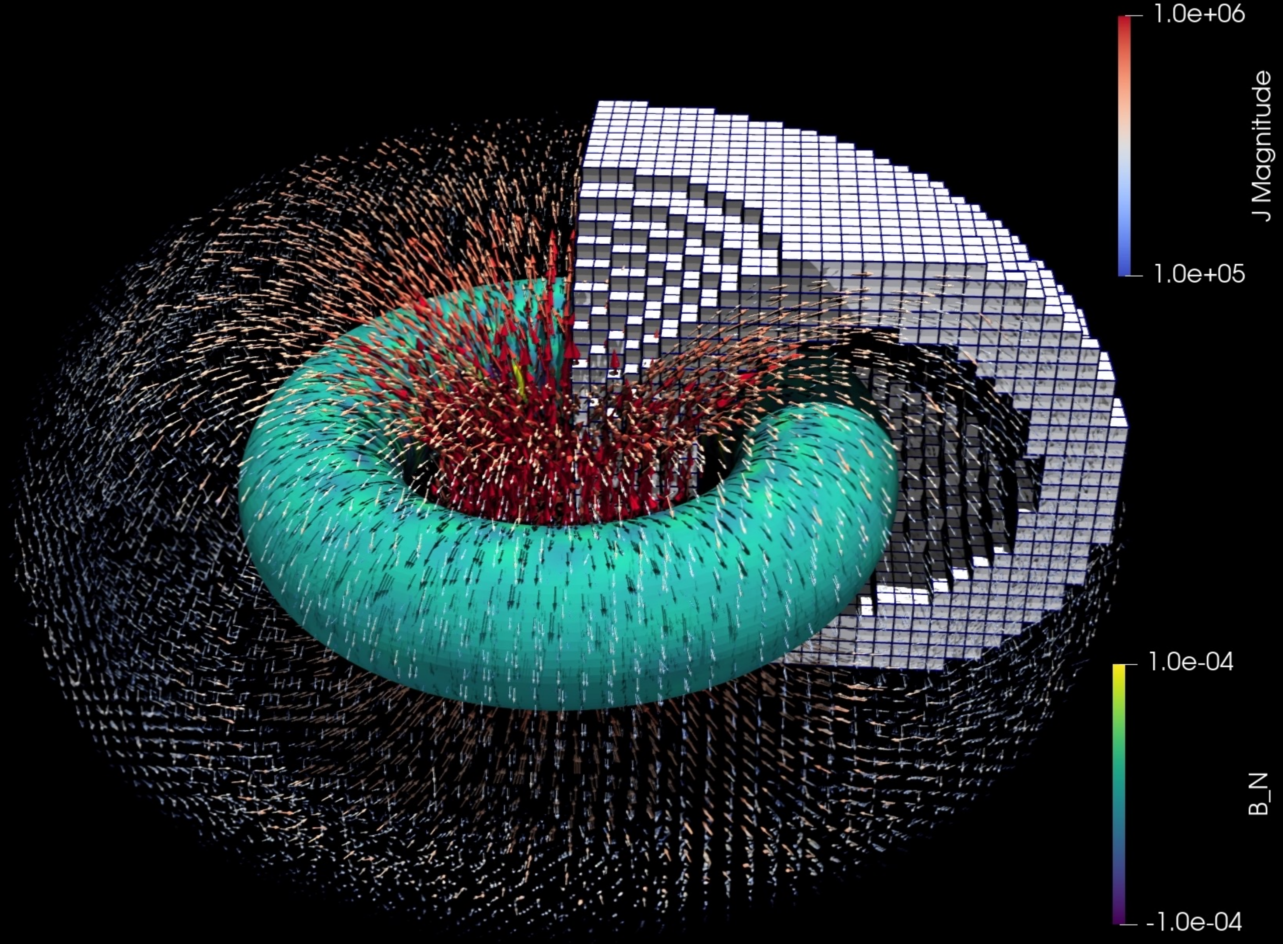
Linear least-squares with linear constraints.
Solved with MINRES + approximate Schur complement preconditioner.

$$\beta^{(k)}=\operatorname*{argmin}_{\beta}\left\{\frac{1}{2\nu}\left\|\alpha^{(k)}-\beta\right\|_2^2+\lambda\|\beta\|_0^G\right\}$$

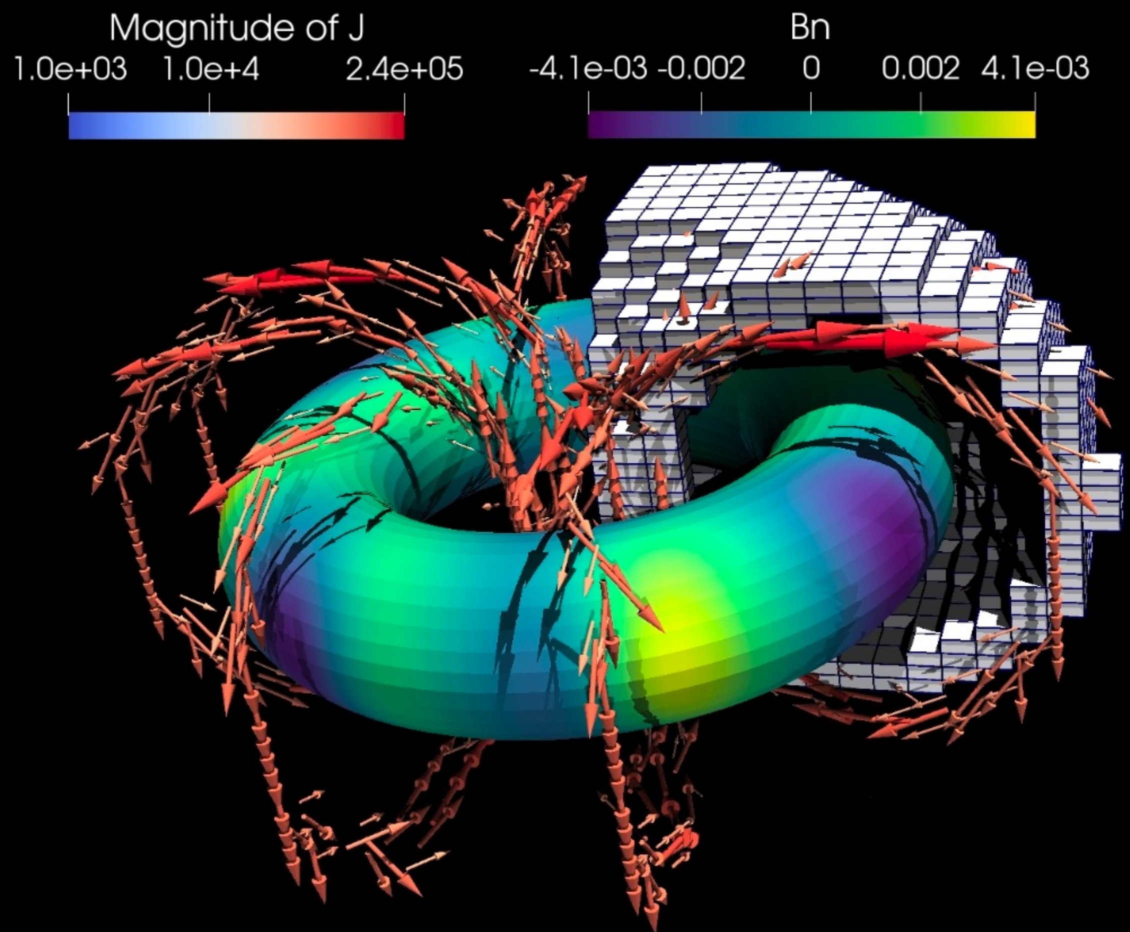
Equivalent to a proximal operator.
Solved exactly by $\beta^{(k)}=\alpha^{(k)}$ but with small entries set to 0.

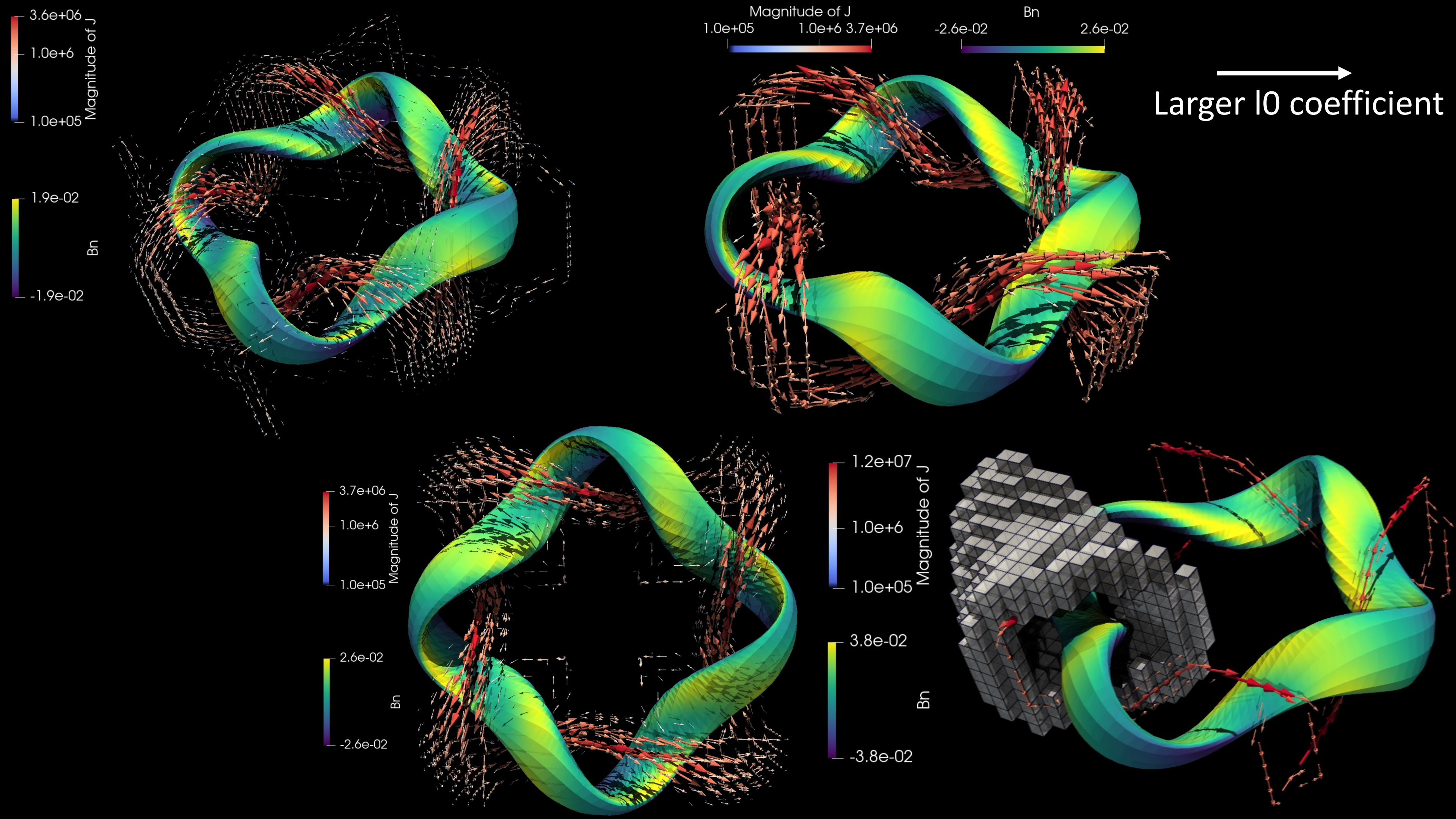
Zheng, Askham, Brunton, Kutz, Aravkin, IEEE Access (2019), Champion, Zheng, Aravkin, Brunton, Kutz, IEEE Access (2020)

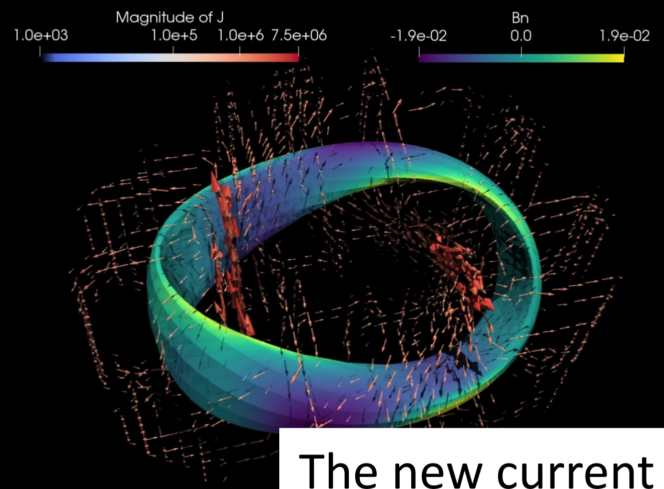
In axisymmetry, without sparsity terms, expected currents are recovered

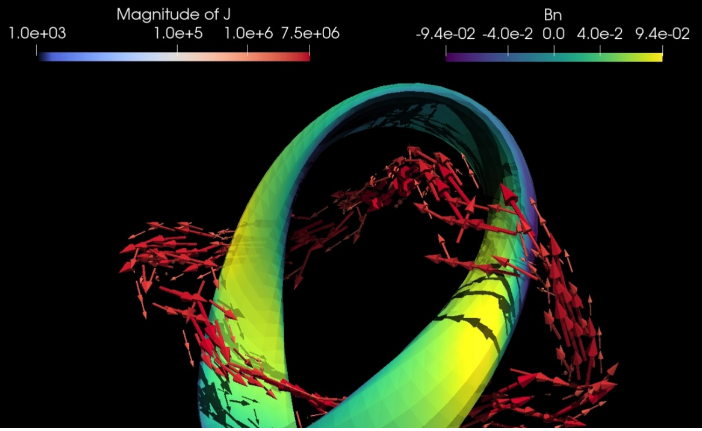


With sparsity objective included, currents coalesce into discrete coils



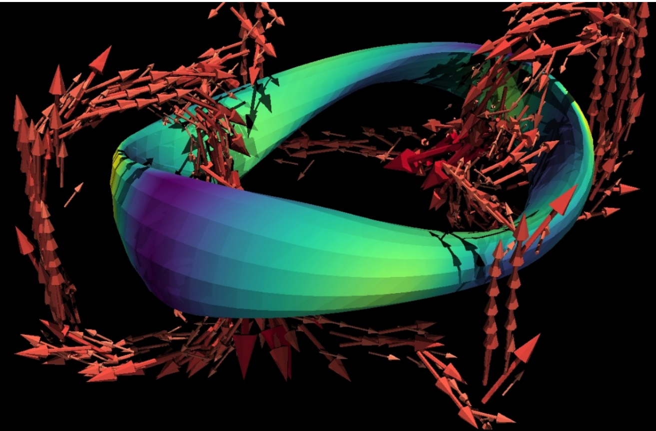

Larger l0 coefficient
Magnitude of J
3.6e+06
1.0e+6
1.0e+05
Bn
1.9e-02
-1.9e-02
Magnitude of J
1.0e+05
1.0e+6 3.7e+06
Bn
-2.6e-02
2.6e-02
Magnitude of J
3.7e+06
1.0e+6
1.0e+05
Bn
2.6e-02
-2.6e-02
Magnitude of J
1.2e+07
1.0e+6
1.0e+05
Bn
3.8e-02
-3.8e-02

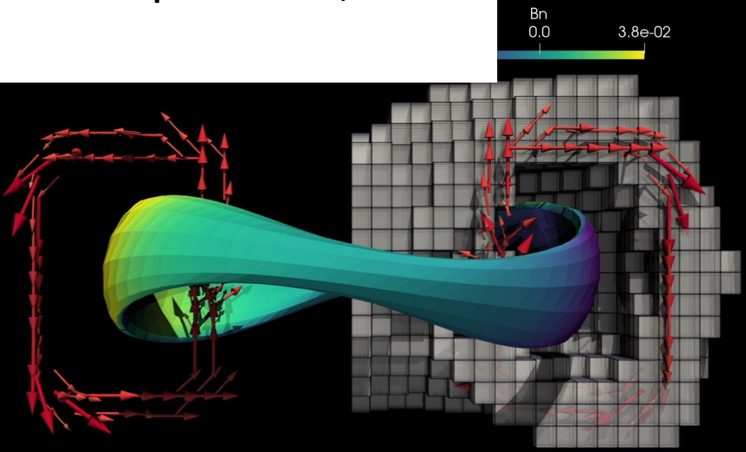




Larger l0 coefficient

The new current voxel for inverse magnetostatics enables topologically unconstrained 3D solutions with quadratic, convex, or structured objectives.





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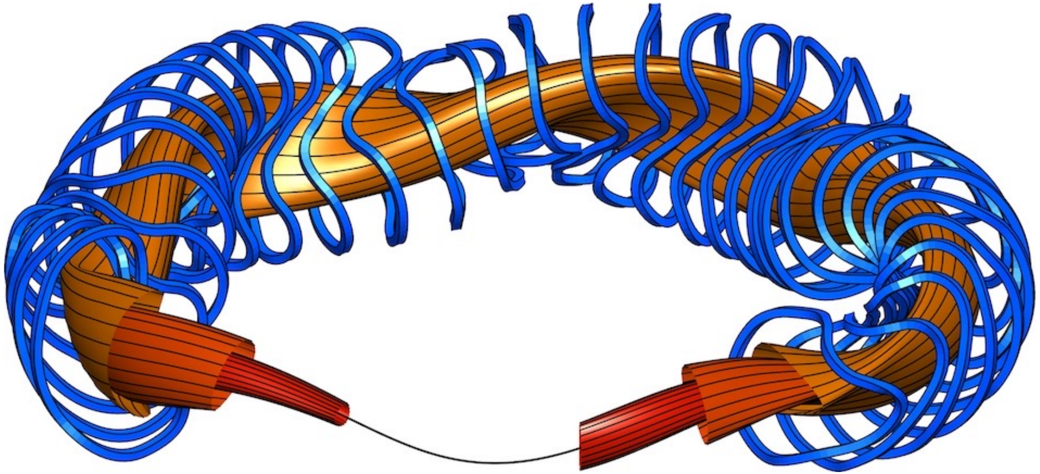
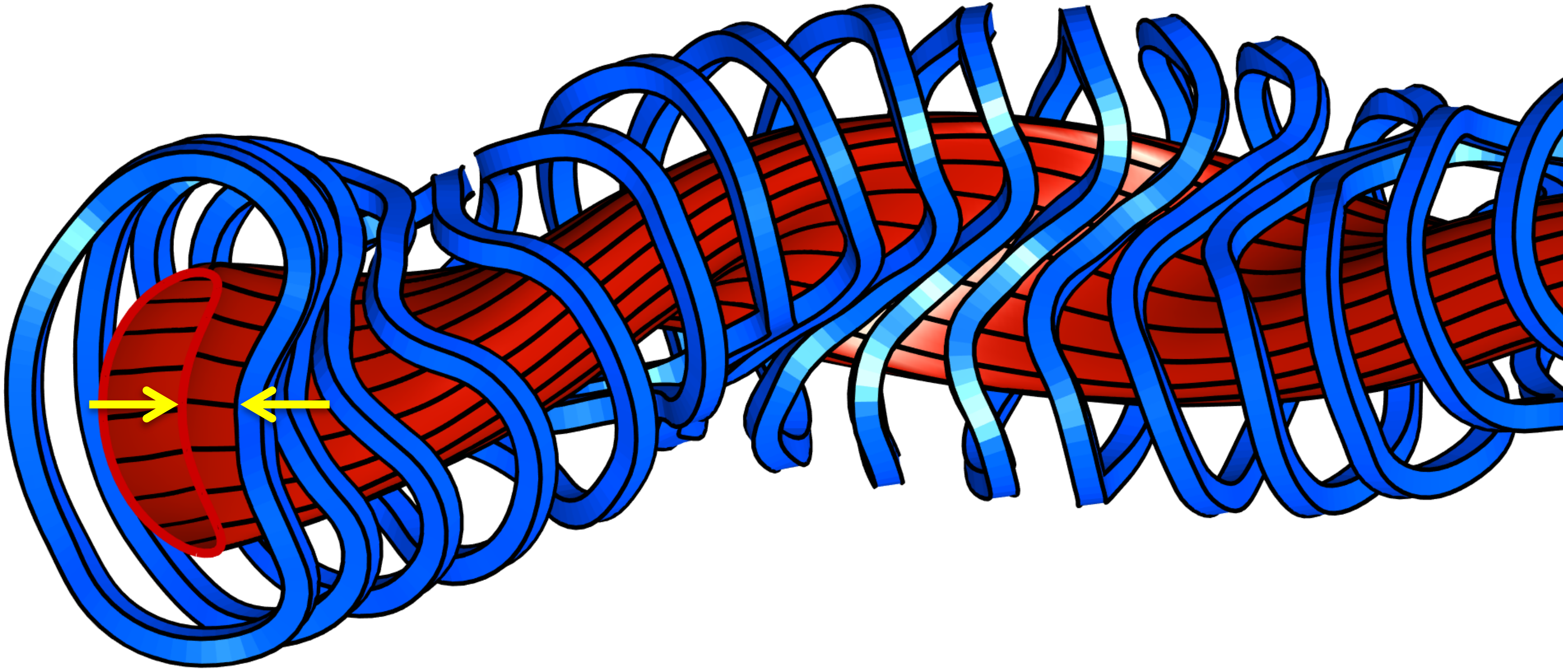


How far away can the magnets be?



The small coil-to-plasma separation in stellarators is a headache for engineering

W7-X

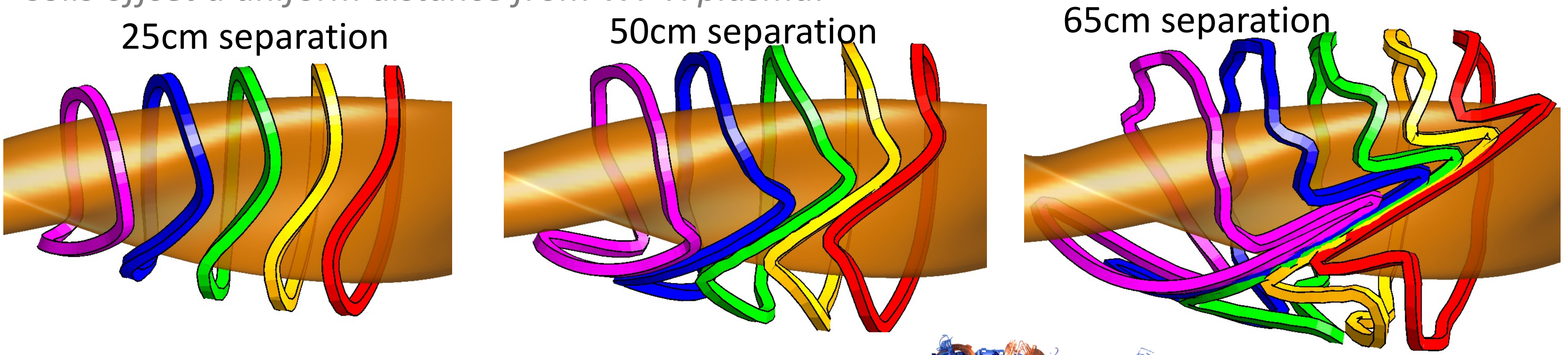
"Lesson 1: A lack of generous margins, clearances and reasonable tolerance levels implies an unnecessary increase of the complexity and leads to late design changes. This has a strong impact on schedule, budget, man-power and potentially sours the relationship to funding bodies."

Klinger et al, Fusion Engineering & Design (2013)

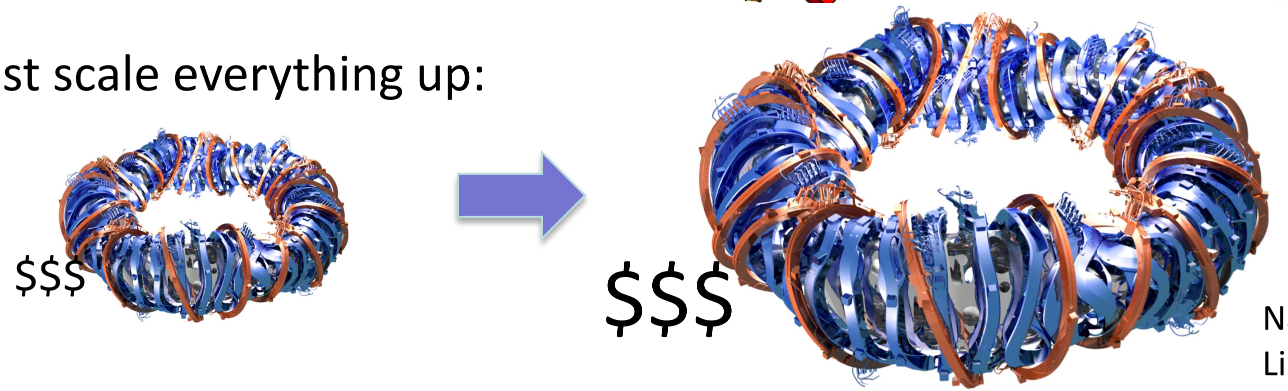
In a reactor, must fit ~ 1.5m "blanket" between plasma and coils to absorb neutrons

But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

Coils offset a uniform distance from W7-X plasma:



So we must scale everything up:



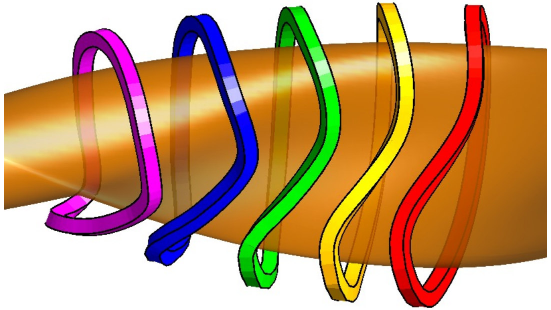
Najmabadi et al (2008), Lion et al (2021)

In a reactor, must fit ~ 1.5m "blanket" between plasma and coils to absorb neutrons

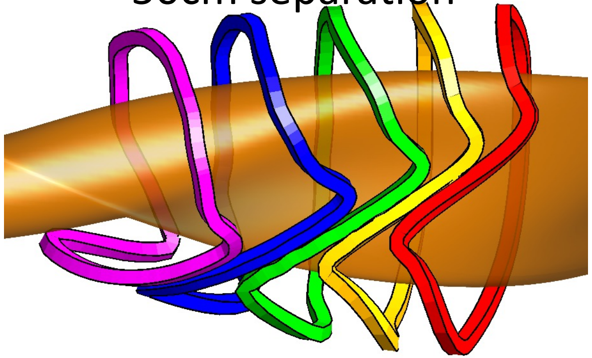
But at fixed plasma shape & size, coils shapes become impractical if they are too far away:

Coils offset a uniform distance from W7-X plasma:

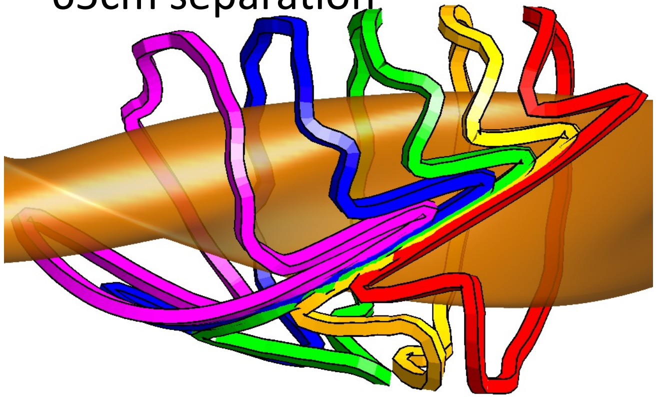
25cm separation



50cm separation



65cm separation



Hypothesis:

The coil-to-plasma distance scale for which coils are feasible is ~ the **∇B** scale length

At any point, a magnetic field has multiple gradient length scales

∇B, $\nabla_{||} B$, $\nabla_{\perp} B$, $\boldsymbol{b} \cdot \nabla \boldsymbol{b}$, $||\nabla \boldsymbol{B}|| = \sqrt{\nabla \boldsymbol{B} : \nabla \boldsymbol{B}}$, eigenvalues of $\nabla \boldsymbol{B}$, $||\nabla\nabla \boldsymbol{B}||$...

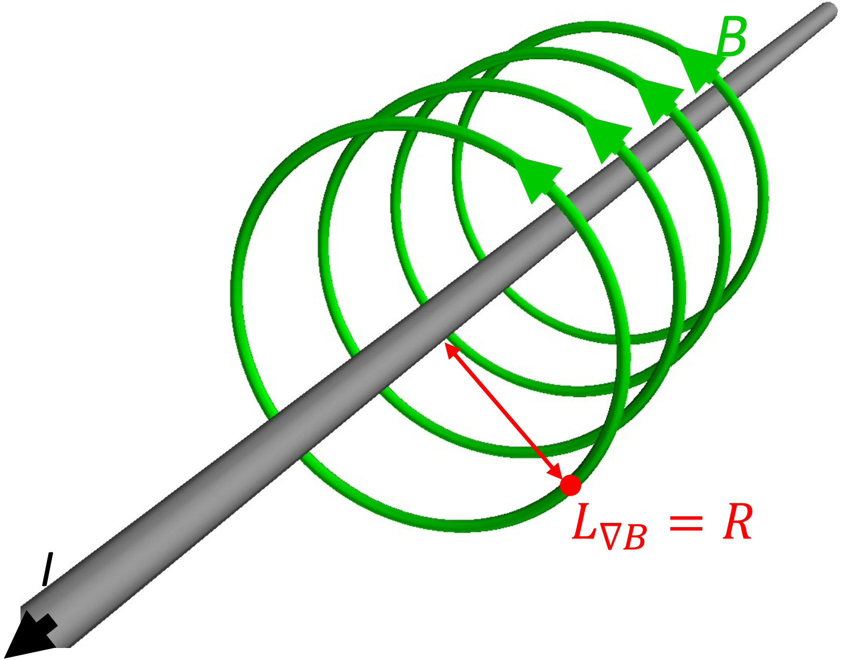
$(B = |\boldsymbol{B}|, \quad \boldsymbol{b} = \boldsymbol{B}/B)$

Frobenius norm

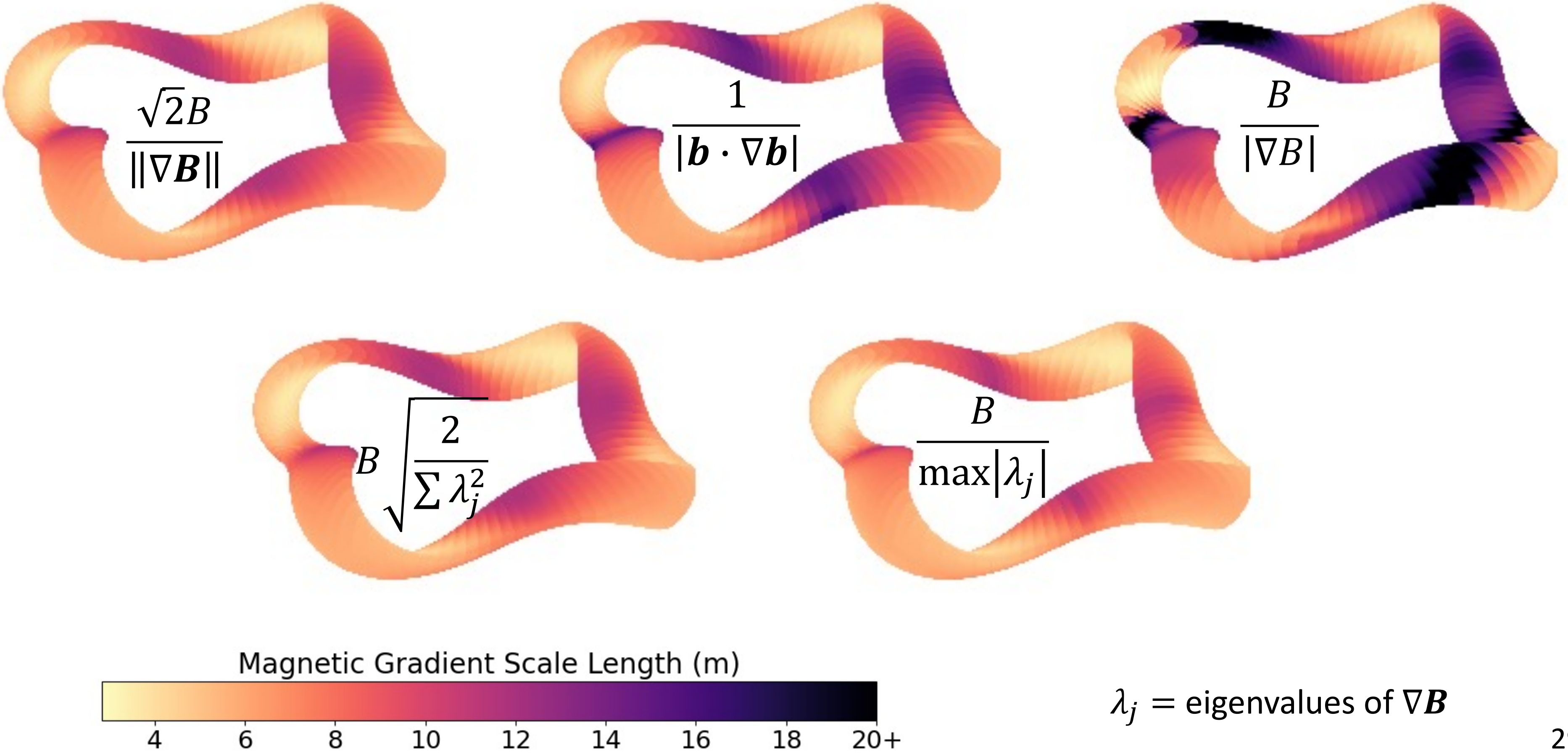
$||\nabla \boldsymbol{B}||$ smoothly captures largest gradient $\Longrightarrow$ shortest length scale

Normalize so scale length gives the distance to an infinite straight wire:

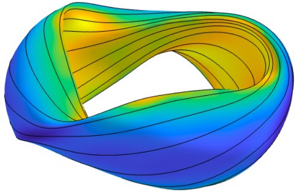
$$L_{\nabla \boldsymbol{B}} = \frac{\sqrt{2}B}{||\nabla \boldsymbol{B}||}$$

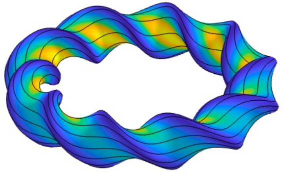


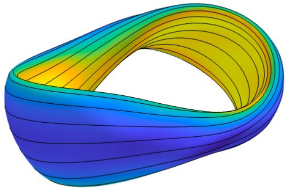
The different **B** scale lengths are not identical, but have similarities, e.g. all are small on the inside of concave regions

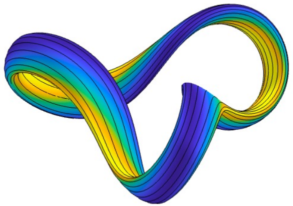


To test hypothesis that ∇B is related to coil-plasma distance, scale length will be compared to "real" coil designs for a diverse set of ~45 configurations



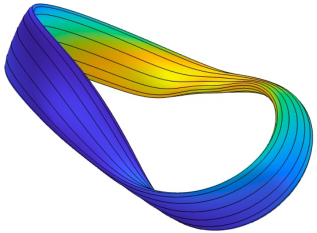


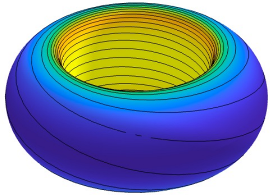


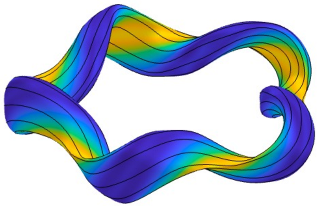


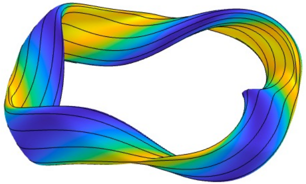
NCSX (li383 & c09r00)
ARIES-CS
HSX
W7-X (std, high-mirror, …)
LHD, R=3.5, 3.6, 3.75
CFQS
ML+Paul QA, QH
ML, Buller, Drevlak QA, QH
Near-axis QH
Jorge et al QI
Goodman et al QIs
ESTELL
ITER
CNT

CTH
TJ-II
QPS
ATF
CIEMAT-QI
Garabedian QA
Henneberg et al QA
Wistell-A, B
Wechsung et al QA
Giuliani et al QA
Ku & Boozer nfp=4 QH
Nuhrenberg & Zille QH
Drevlak QH
…





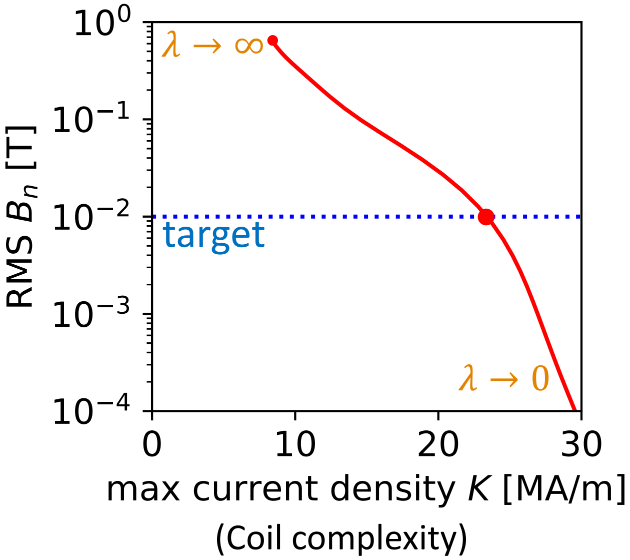




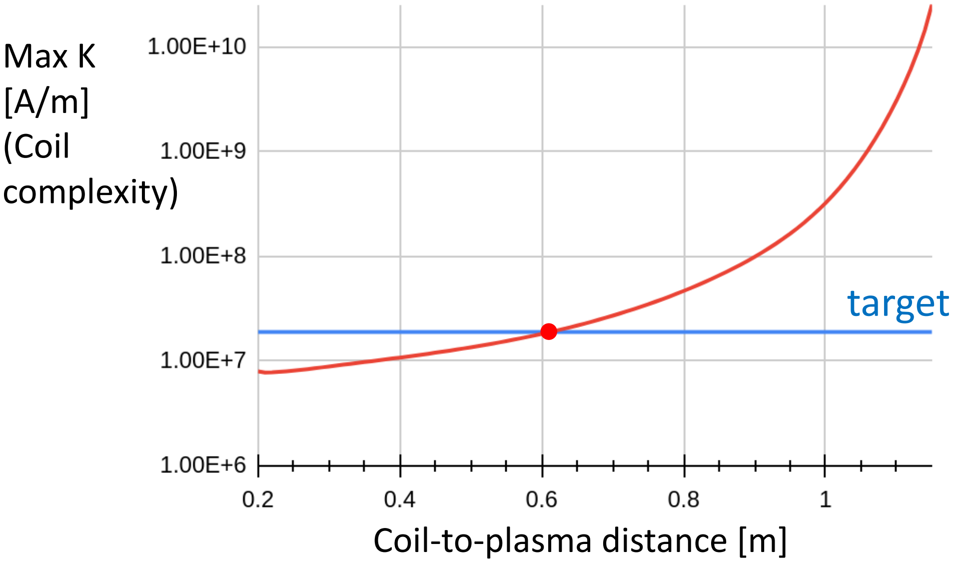
All scaled to same minor radius (1.7 m) and $\langle B \rangle$ = 5.9 T.

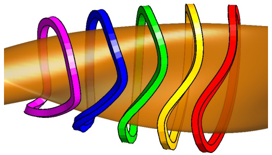
Methodology: Apply REGCOIL, adjust regularization λ and coil-to-plasma separation to match $\boldsymbol{B}$ error and coil current density between configurations

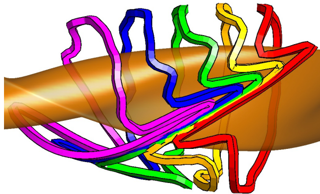
At fixed coil-to-plasma separation, λ trades off between $\boldsymbol{B}$ field error and coil complexity.



At the target $\boldsymbol{B}$ field error, coil complexity increases with coil-to-plasma separation

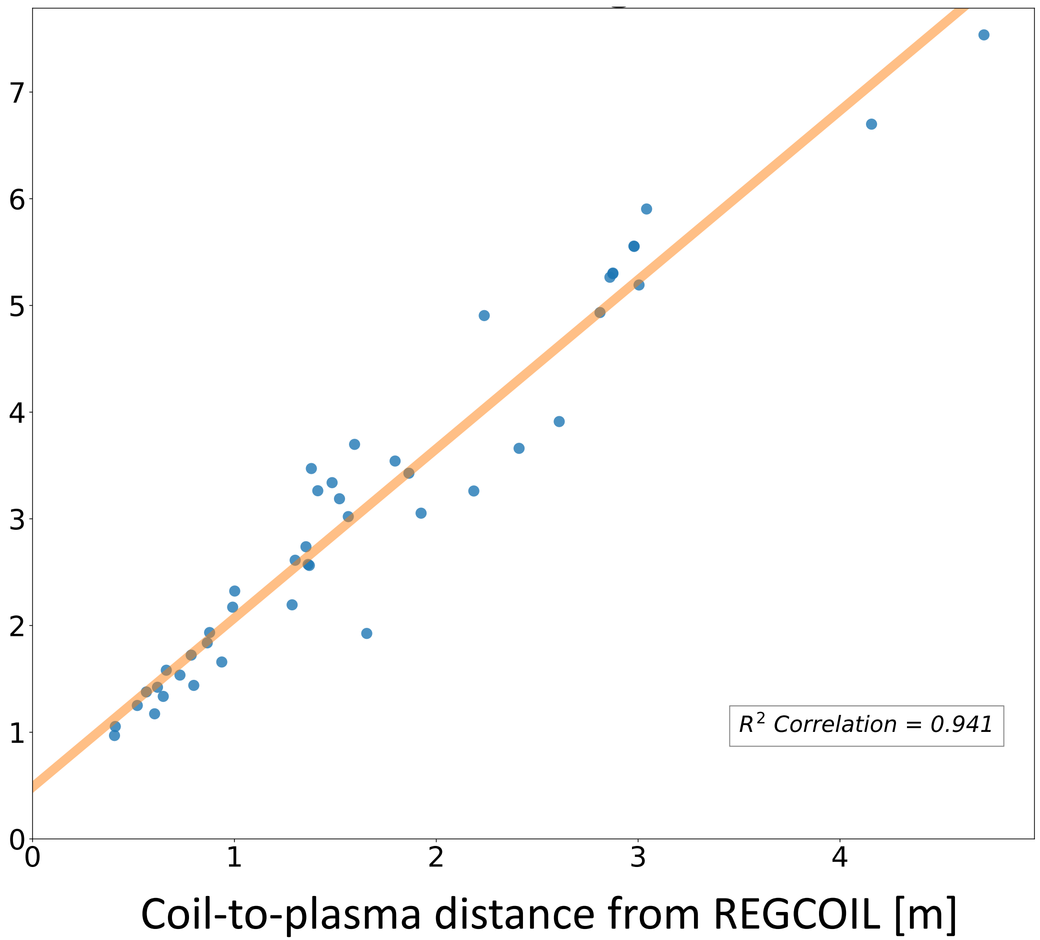






Main result: ∇B length is well correlated with real coil designs

$$\min \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|} = L_{\nabla \boldsymbol{B}}\ [\mathrm{m}]$$



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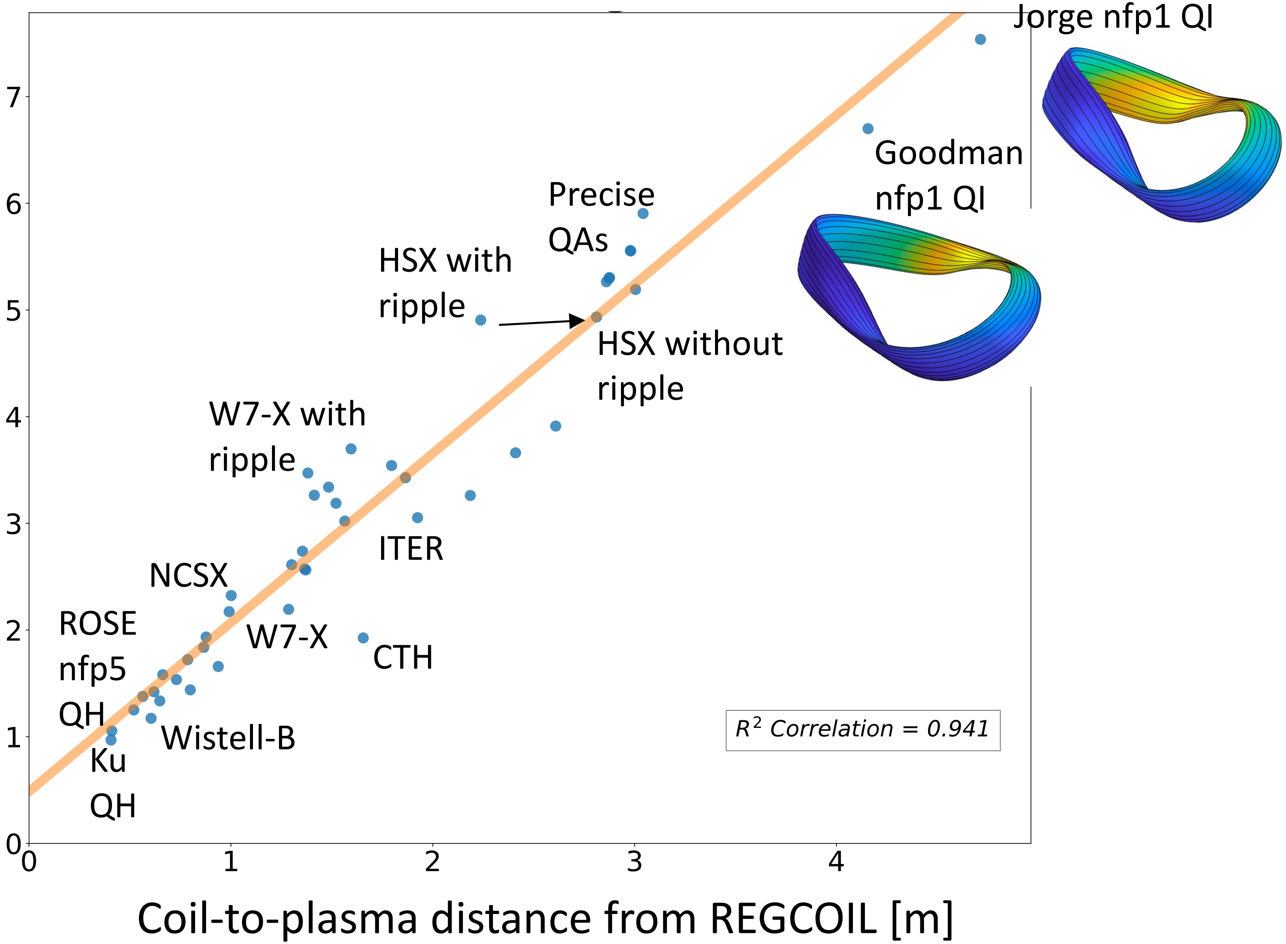
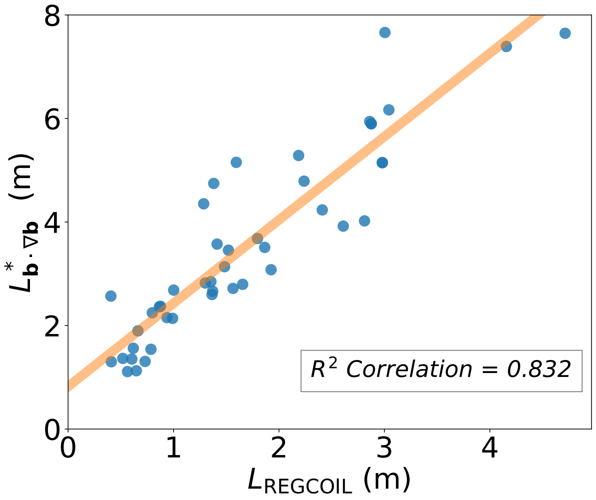
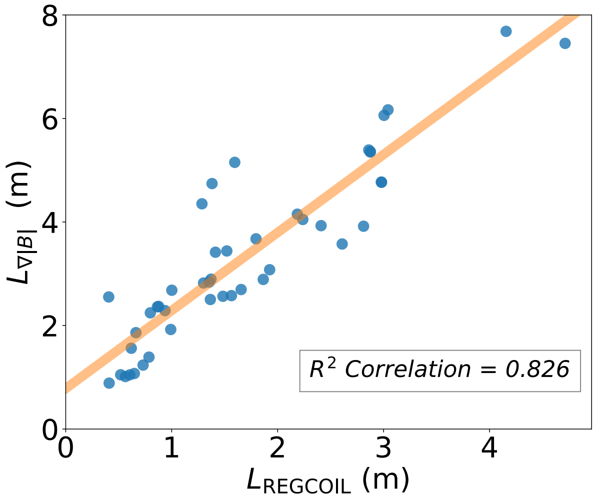


Other scale lengths can be reasonably well correlated as well

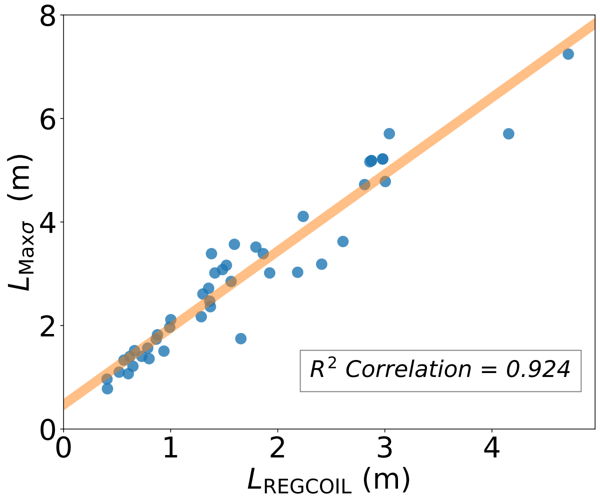
Field line radius of curvature



Gradient of scalar B

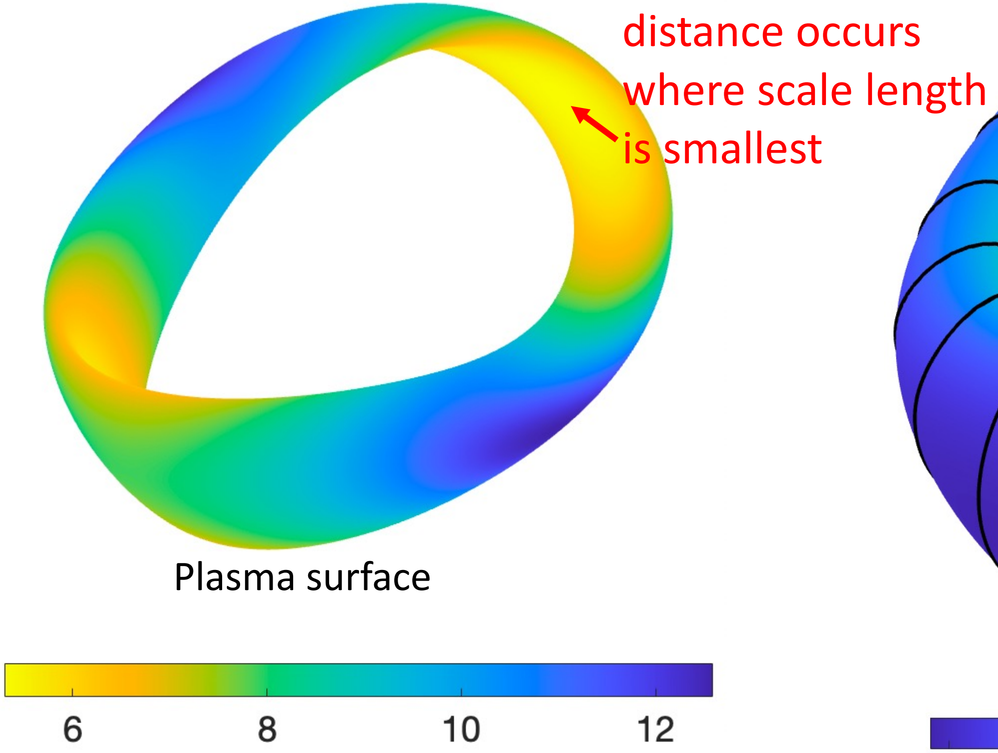


Largest eigenvalue



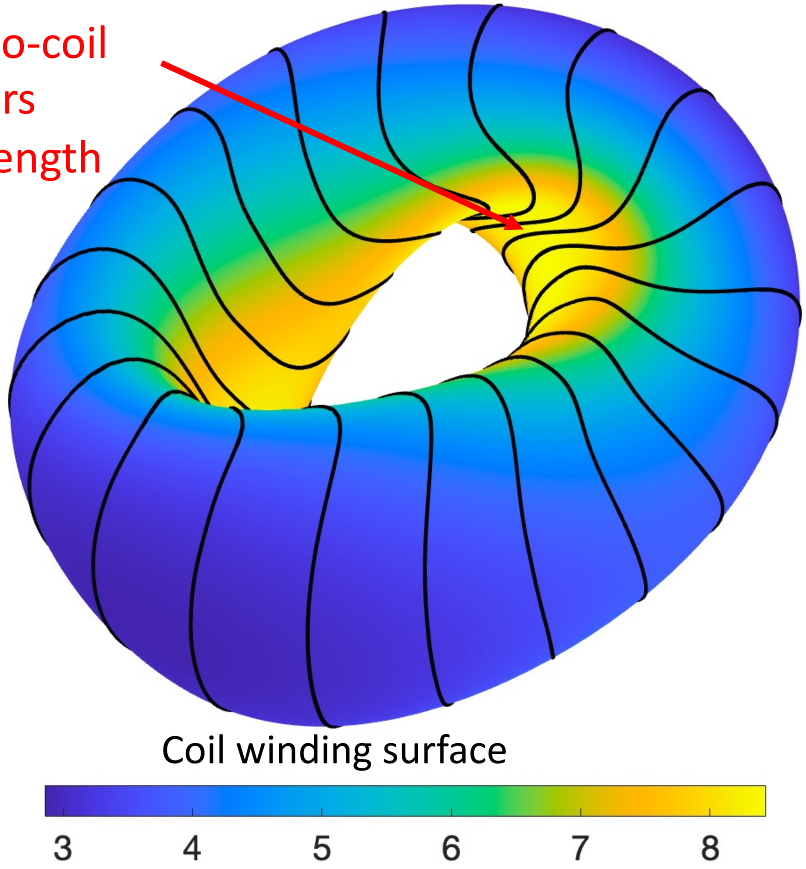
The location of limiting ∇B length and coil complexity are also correlated *spatially*

$$L_{\nabla \boldsymbol{B}} = \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|} \text{ [m]}$$



Plasma surface

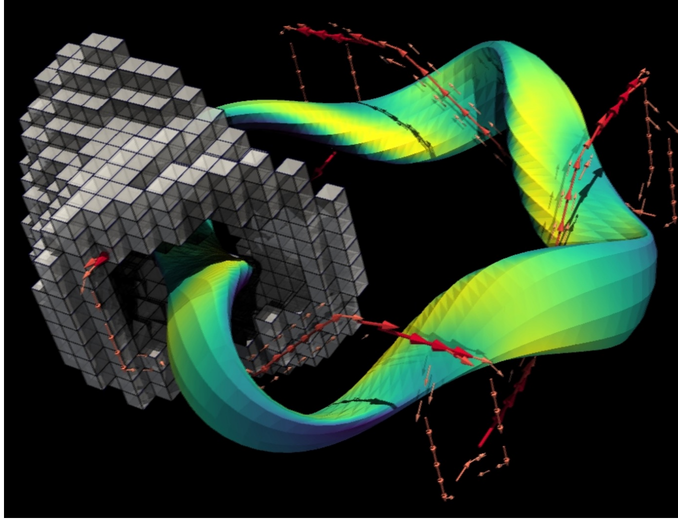
Current density K [MA/m]



Coil winding surface

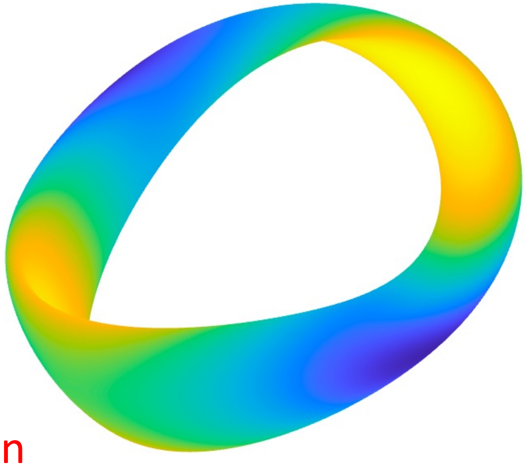
Conclusions:

- The new current voxel for inverse magnetostatics enables topologically unconstrained 3D solutions with quadratic, convex, or structured objectives.
- The maximum coil separation can be understood from the scale length $L_{\nabla \boldsymbol{B}}$.



Questions:

- Is there a more efficient solver for the current voxel system?
- Can the current voxel method be generalized for more flexible geometry (tetrahedra?)
- Is $L_{\nabla \boldsymbol{B}}$ a useful objective function to enable larger coil separation?
- Besides $L_{\nabla \boldsymbol{B}}$, are there other metrics to predict coil separation?

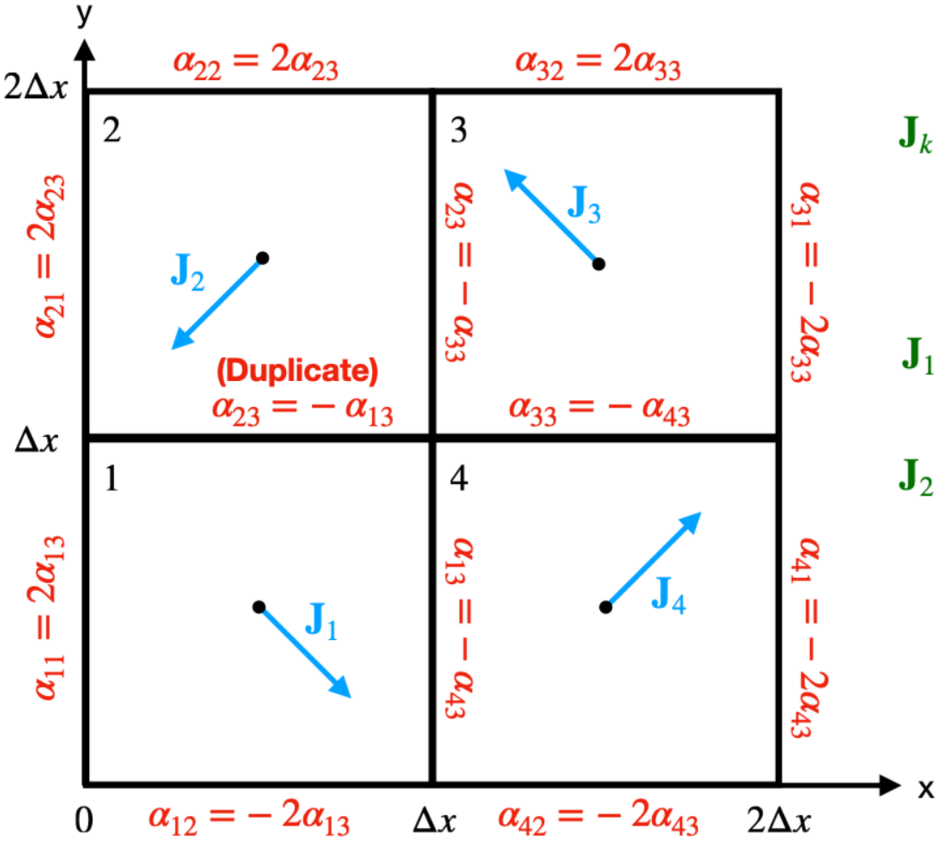


Improvements in algorithms for inverse magnetostatics can translate to significant benefits for fusion, accelerators, MRI, etc.

Extra slides

2D example

3 basis functions per cell, 12 degrees of freedom, 11 unique constraints



Basis expansion

$$\mathbf{J}_k = \alpha_{k1}\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_{k2}\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_{k3}\begin{bmatrix}X_k\\-Y_k\\0\end{bmatrix}$$

Solution

$$\mathbf{J}_1 = \varphi\begin{bmatrix}2+X_k\\-2-Y_k\end{bmatrix} \quad \mathbf{J}_3 = \varphi\begin{bmatrix}-2+X_k\\2-Y_k\end{bmatrix}$$

$$\mathbf{J}_2 = -\varphi\begin{bmatrix}2+X_k\\2-Y_k\end{bmatrix} \quad \mathbf{J}_4 = -\varphi\begin{bmatrix}-2+X_k\\-2-Y_k\end{bmatrix}$$

Solution at midpoints $(X_k, Y_k) = (0,0)$

$$\mathbf{J}_1 = 2\varphi\begin{bmatrix}1\\-1\end{bmatrix} \quad \mathbf{J}_3 = 2\varphi\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\mathbf{J}_2 = 2\varphi\begin{bmatrix}-1\\-1\end{bmatrix} \quad \mathbf{J}_4 = 2\varphi\begin{bmatrix}1\\1\end{bmatrix}$$

At any point, a magnetic field has multiple gradient length scales

∇B, $\nabla_{||} B$, $\nabla_{\perp} B$, $\boldsymbol{b} \cdot \nabla \boldsymbol{b}$, $||\nabla \boldsymbol{B}|| = \sqrt{\nabla \boldsymbol{B} : \nabla \boldsymbol{B}}$, Frobenius norm, eigenvalues of $\nabla \boldsymbol{B}$, $||\nabla\nabla \boldsymbol{B}||$…

$(B = |\boldsymbol{B}|, \quad \boldsymbol{b} = \boldsymbol{B}/B)$

$||\nabla \boldsymbol{B}||$ captures largest gradient $\Longrightarrow$ shortest length scale

We can get some insights by considering vacuum fields:

$\boldsymbol{B} = \nabla \Phi$ so $\nabla \boldsymbol{B} = \nabla\nabla\Phi$ is a symmetric 3×3 matrix $\Longrightarrow$ 6 degrees of freedom.

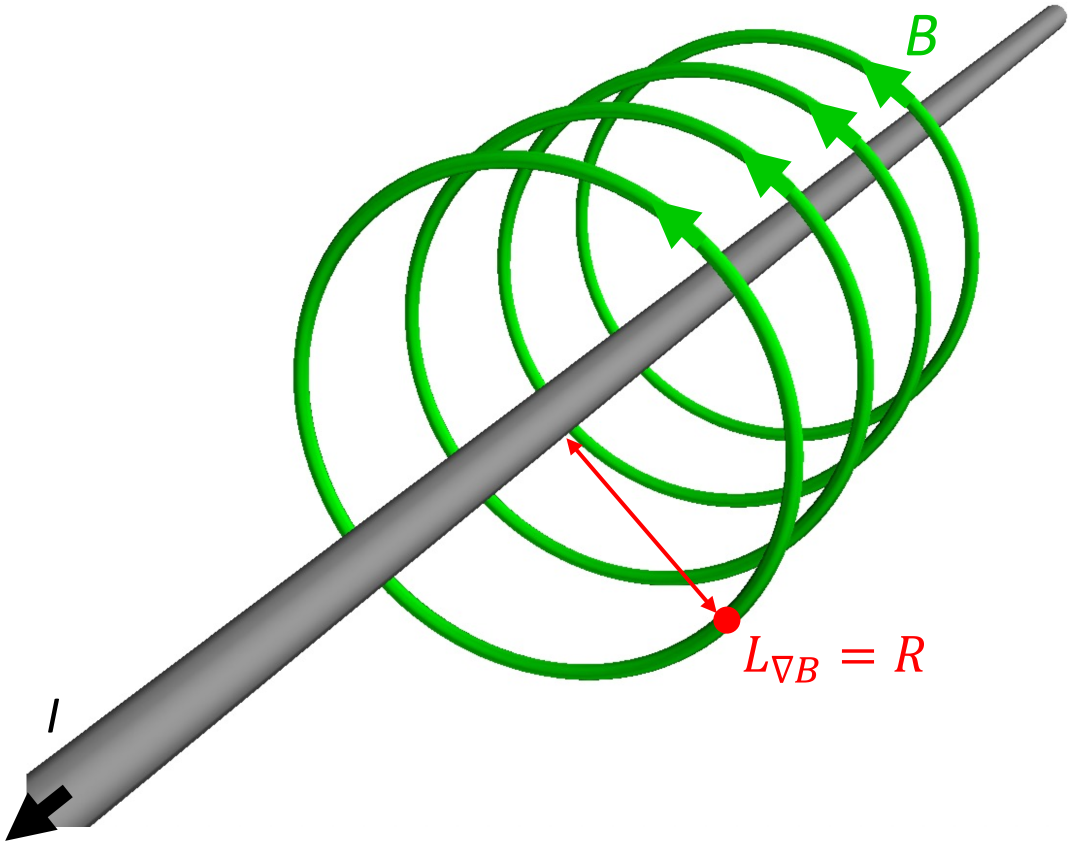
$$\nabla \boldsymbol{B} = \begin{pmatrix} \partial_{xx}\Phi & \partial_{xy}\Phi & \partial_{xz}\Phi \\ \partial_{yx}\Phi & \partial_{yy}\Phi & \partial_{yz}\Phi \\ \partial_{zx}\Phi & \partial_{zy}\Phi & \partial_{zz}\Phi \end{pmatrix}$$

-1 degree of freedom since $\nabla \cdot \boldsymbol{B} = 0$.

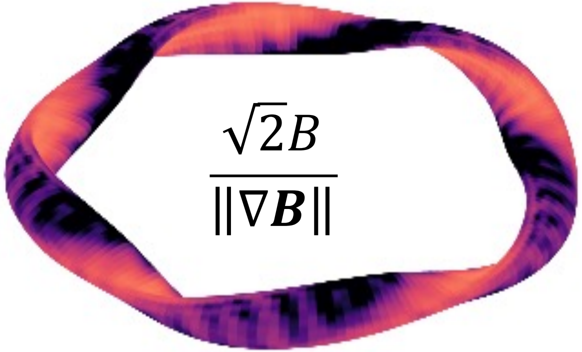
Some entries can be made to vanish by rotating the coordinate system.

The ∇B scale lengths can be normalized so that in the case of an infinite straight wire, they give the distance to the wire

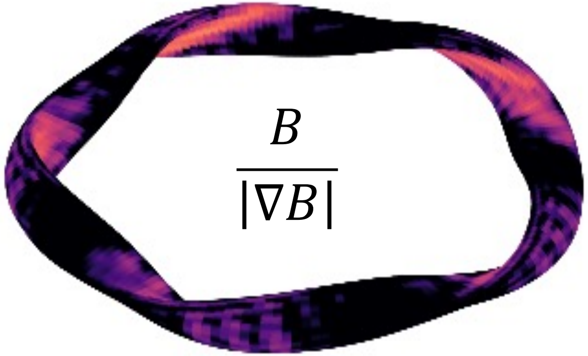
$$L_{\nabla B} = \frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|}$$



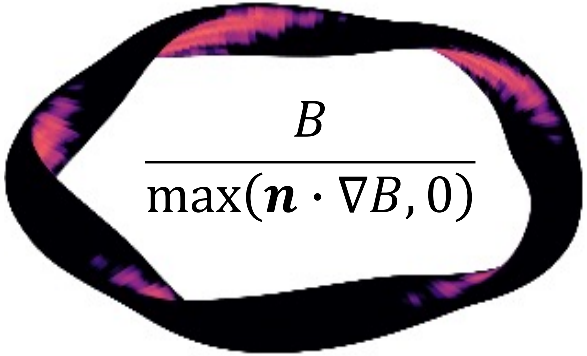
The different **B** scale lengths are not identical, but have similarities, e.g. all are small on the inside of concave regions



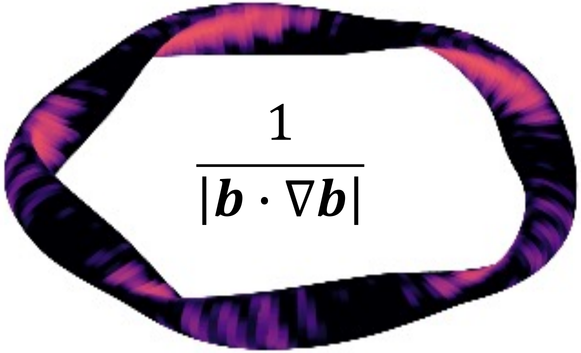
$$\frac{\sqrt{2}B}{\|\nabla \boldsymbol{B}\|}$$



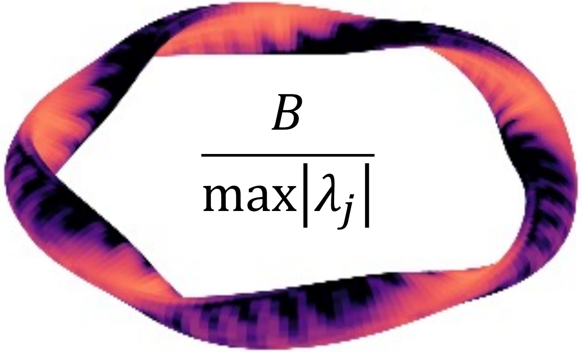
$$\frac{B}{|\nabla B|}$$



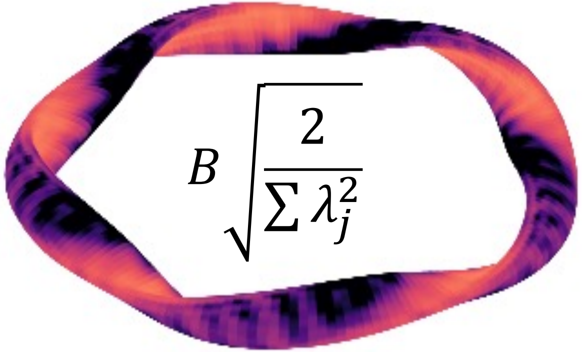
$$\frac{B}{\max(\boldsymbol{n}\cdot\nabla B, 0)}$$



$$\frac{1}{|\boldsymbol{b}\cdot\nabla \boldsymbol{b}|}$$



$$\frac{B}{\max|\lambda_j|}$$



$$B\sqrt{\frac{2}{\sum \lambda_j^2}}$$

λ_j = eigenvalues of $\nabla \boldsymbol{B}$

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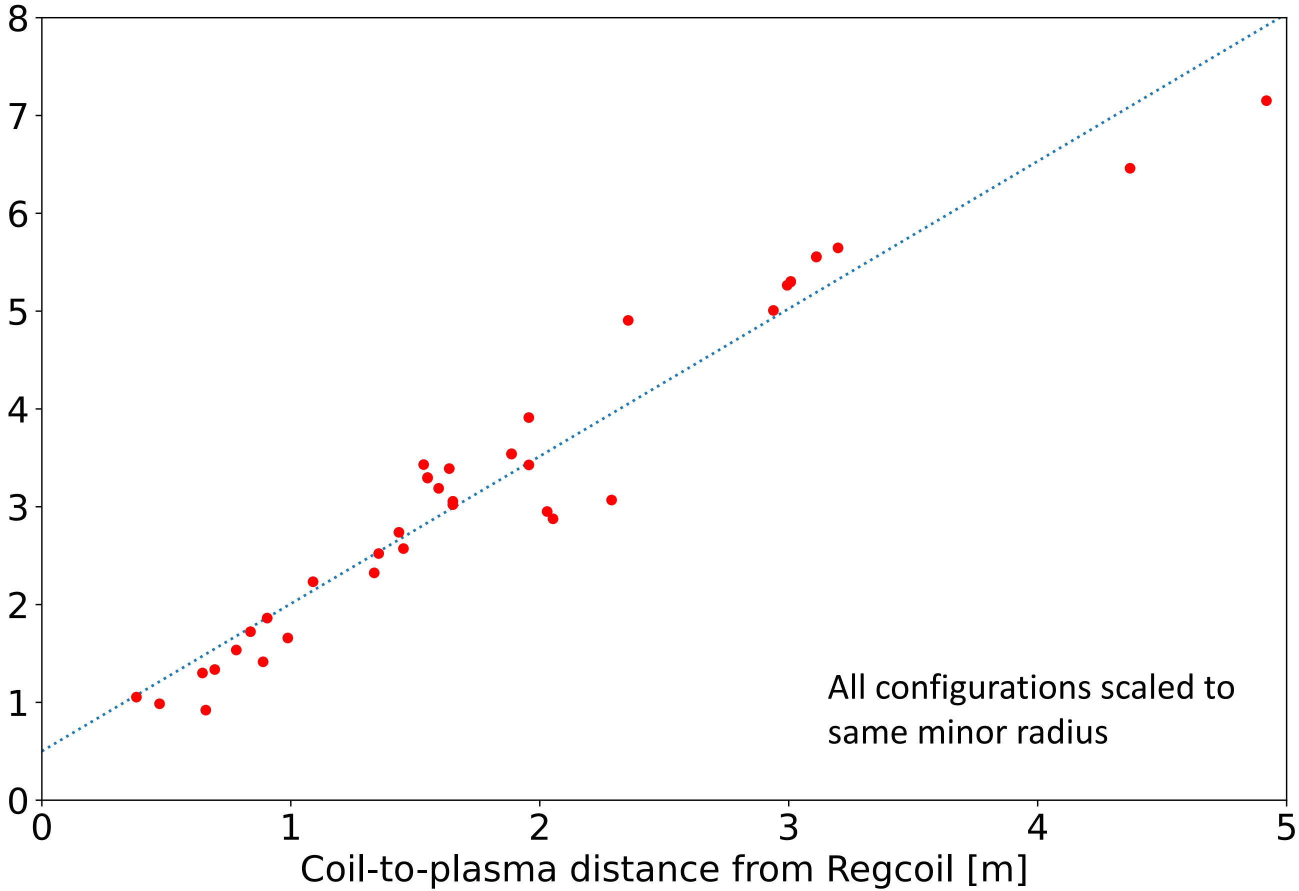
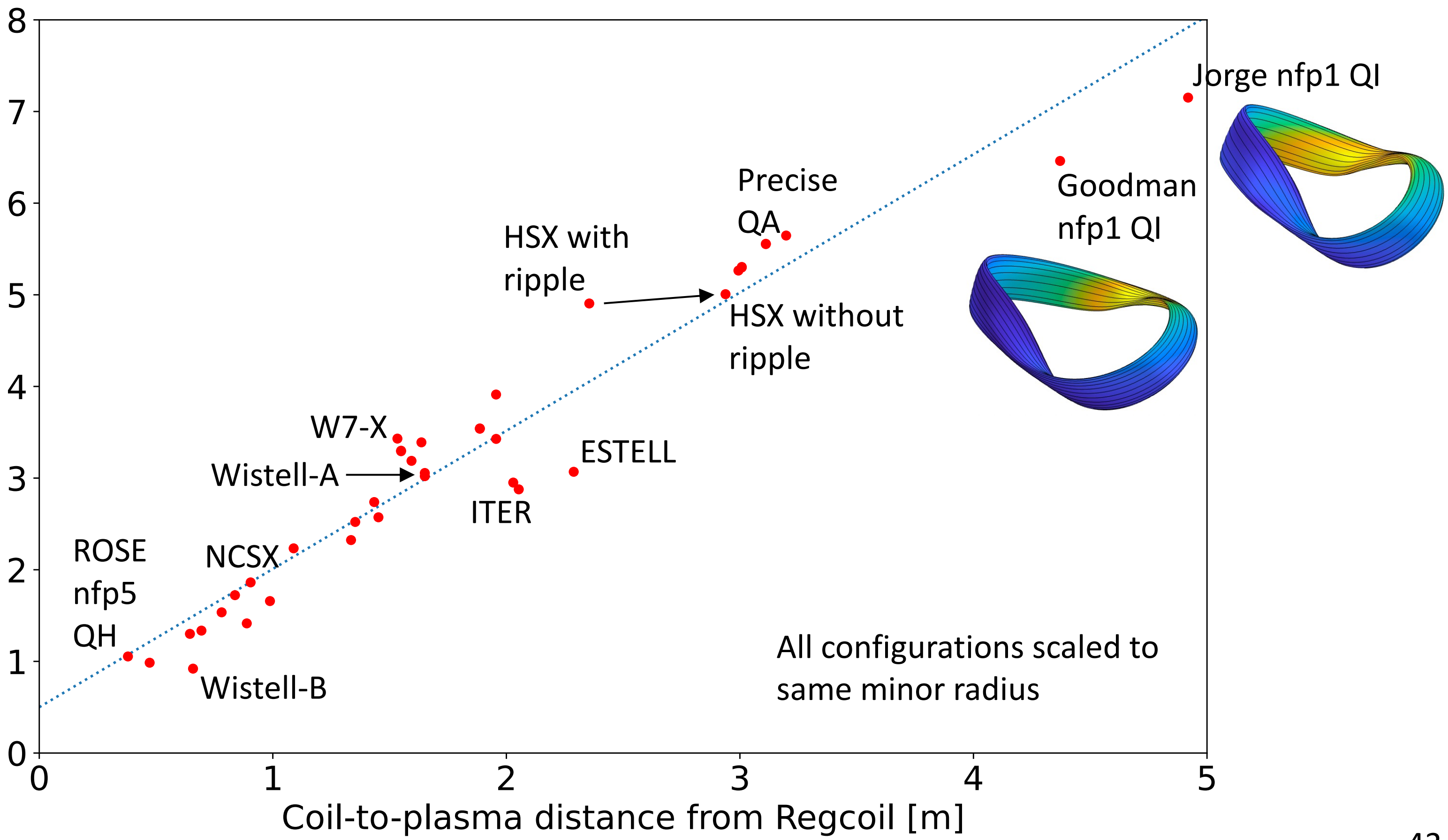
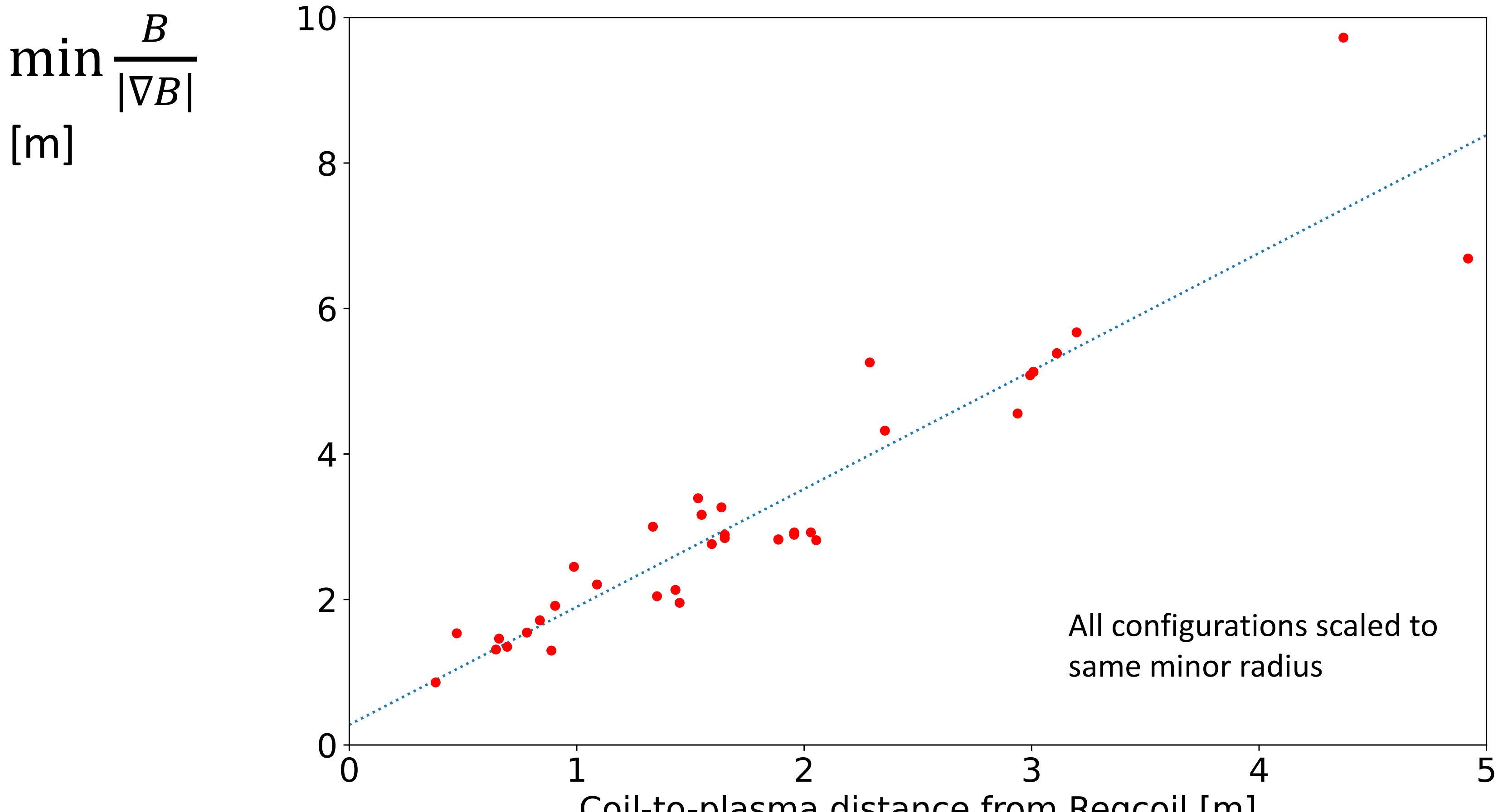


Main result: ∇B length is well correlated with real coil designs

$$\min \frac{\sqrt{2}B}{||\nabla \boldsymbol{B}||} = L_{\nabla \boldsymbol{B}}\ [\mathrm{m}]$$



Regcoil coil-to-plasma distance is actually well correlated with other definitions of B length scale too



Regcoil coil-to-plasma distance is actually well correlated with other definitions of B length scale too

$$\min \frac{1}{|\boldsymbol{b}\cdot\nabla\boldsymbol{b}|}$$

[m]