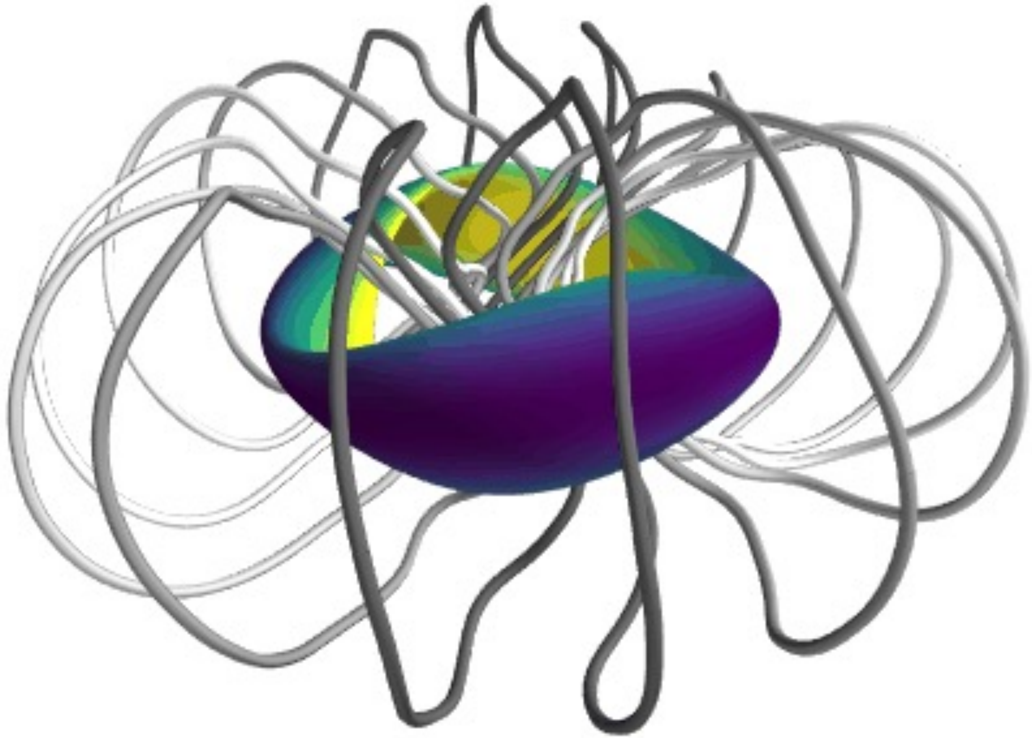
Stellarator optimization

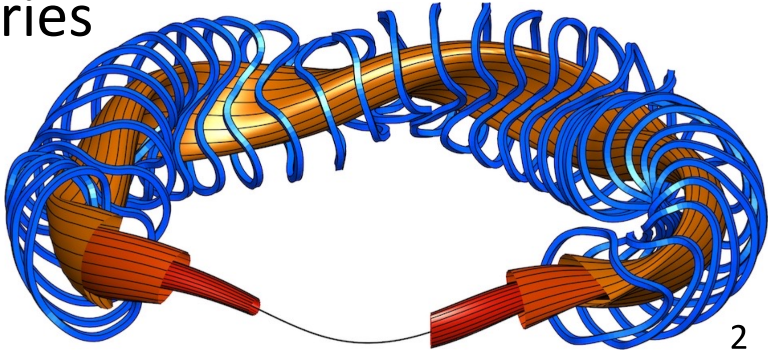
Where do these shapes come from?



Matt Landreman – mattland@umd.edu, University of Maryland

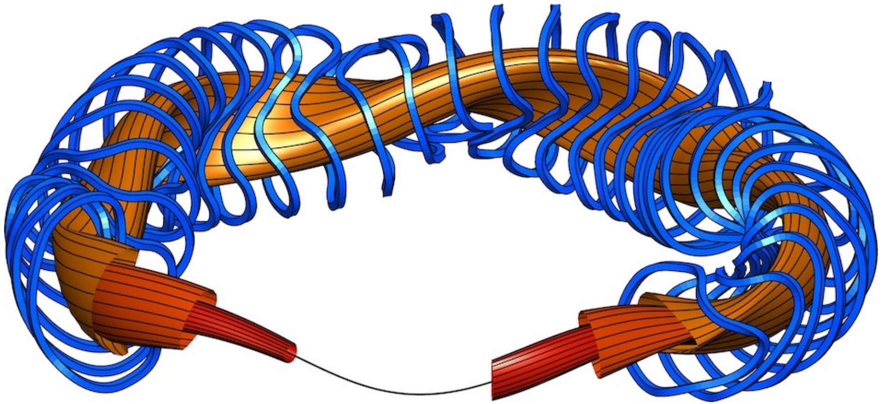
There is lots of freedom in the shape of a stellarator plasma and coils, which can be used to achieve many objectives

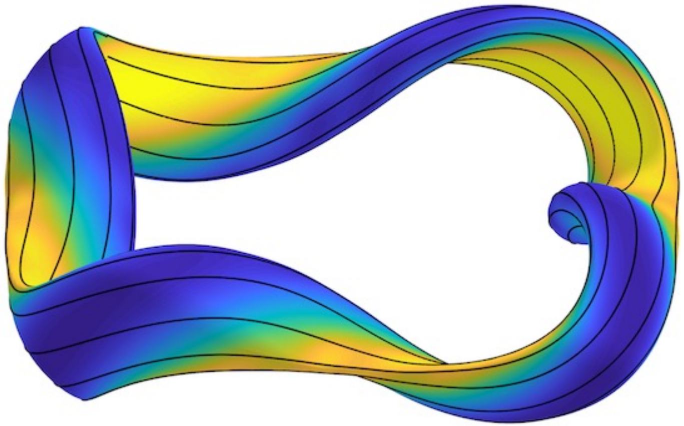
- Large volume of good magnetic surfaces (not islands & chaos)
- Enough rotational transform
- Plasma pressure & current doesn't modify **B** too much, i.e. maximum plasma pressure is not too low.
- Buildable coil shapes: low curvature, large clearances
- Magnetohydrodynamic (MHD) stability
- Good confinement of particle trajectories
- Low neoclassical transport
- Low turbulent transport



Outline

- Optimization in general
- Optimization for good flux surfaces
- Quasi-axisymmetry, quasi-helical symmetry, and quasi-isodynamic
- Optimization for stability & turbulence
- Coil optimization: current potential and filament methods

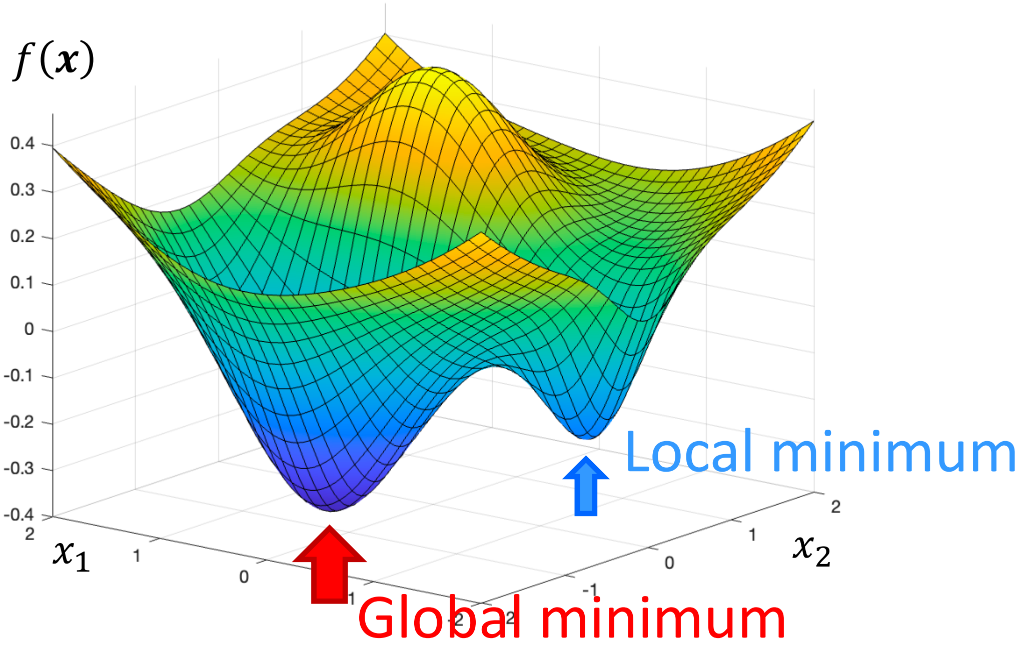




Optimization is a general technique with many applications

Given a "cost function" $f: \mathbb{R}^n \to \mathbb{R}$, minimize $f(\boldsymbol{x})$

(a.k.a. "objective function", "loss function")



Optimization is a general technique with many applications

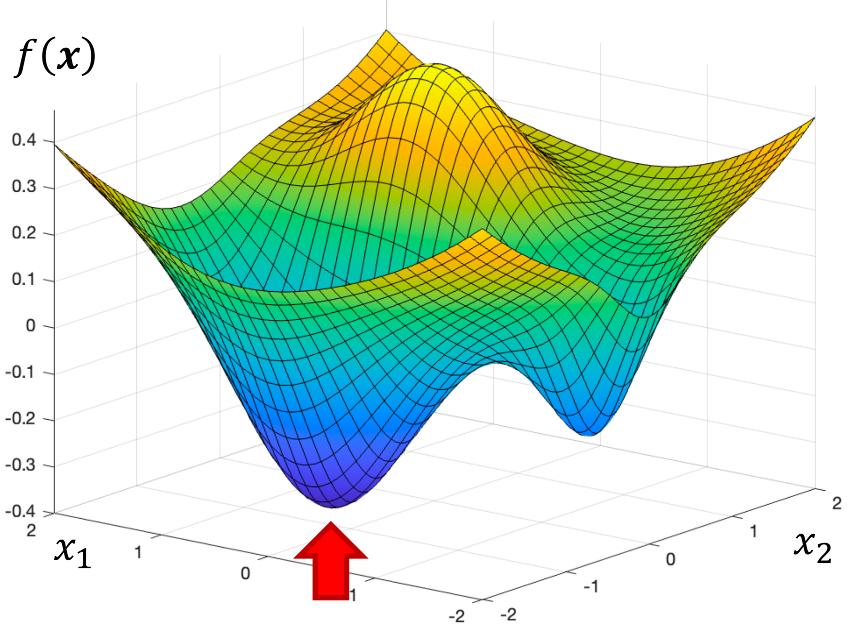
Given a "cost function" $f : \mathbb{R}^n \to \mathbb{R}$,

(a.k.a. "objective function", "loss function")

minimize $f(\boldsymbol{x})$

(Optional) Can add constraints: subject to $g_i(\boldsymbol{x}) = 0$,

$h_j(\boldsymbol{x}) < 0$



Suppose we want to minimize > 1 quantity, e.g. f_1, f_2, and f_3.

"Scalarization":

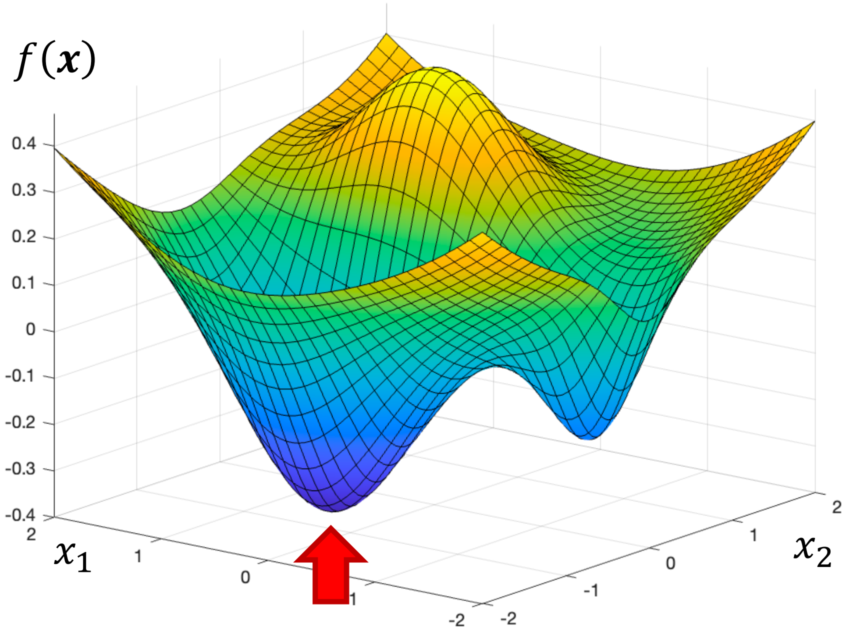
minimize $f = w_1 f_1 + w_2 f_2 + w_3 f_3$

where w_j are weights.

Optimization is natural to apply to stellarators

Given a "cost function" $f : \mathbb{R}^n \rightarrow \mathbb{R}$, minimize $f(\boldsymbol{x})$

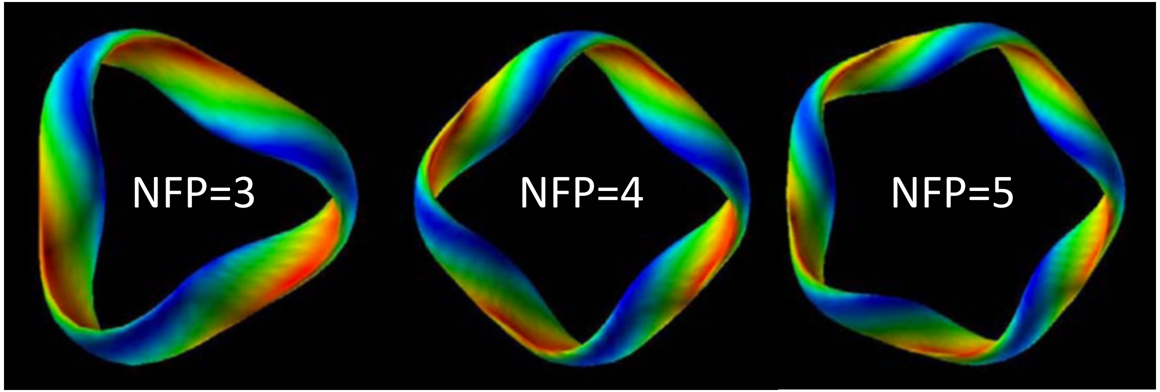
(a.k.a. "objective function", "loss function")

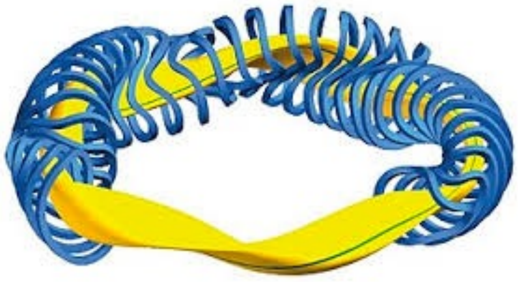


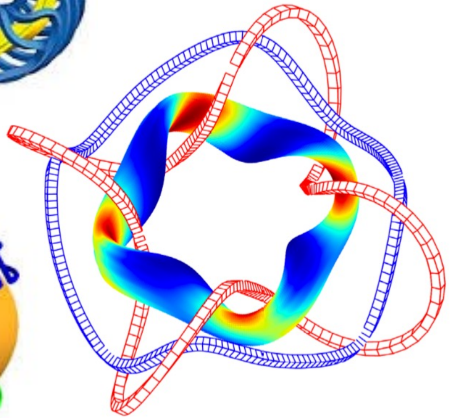
- Fewer lost particles are better, but may not be able to confine every trajectory.
- Lower coil curvature is better, but probably can't make it 0.
- Might not be able to achieve $\boldsymbol{J}\times\boldsymbol{B} = \nabla p$ with good flux surfaces exactly, but we can minimize the residual $(\boldsymbol{J}\times\boldsymbol{B} - \nabla p)^2$.

Some stellarator parameters are integers. These are mostly optimized by hand.

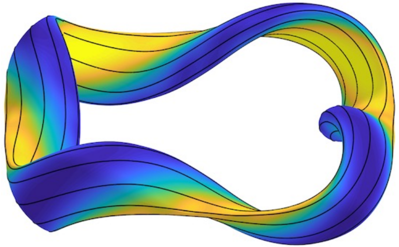
- Number of “field periods”.
- Number of coils.







- Do coils link the plasma poloidally, helically, or not at all?
- Do *B* contours link the torus toroidally (QA), helically (QH), or poloidally (QI)?

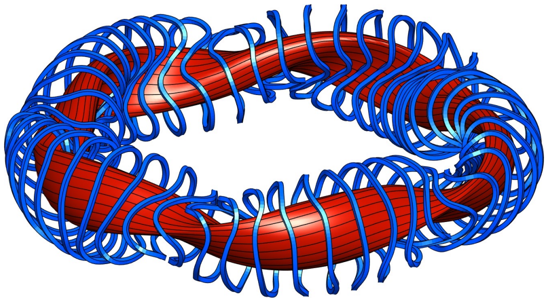


There are several possible choices of parameter space to optimize in

Minimize $f(\boldsymbol{x})$ where $\boldsymbol{x} \in \mathbb{R}^n$. What is $\boldsymbol{x}$?

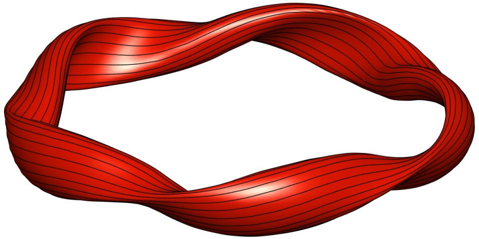
$\boldsymbol{x}$ = shapes of coils:

- Much of this space is very bad: not good flux surfaces, can't evaluate physics objectives
- "Free boundary equilibrium" calculations can be fragile



$\boldsymbol{x}$ = shape of plasma:

- Can use "fixed boundary equilibrium", which is very reliable.
- Some of parameter space corresponds to unphysical self-intersecting shapes.
- Otherwise, most of this parameter space is good.
- But the plasma shape is not what you build – you build coils.

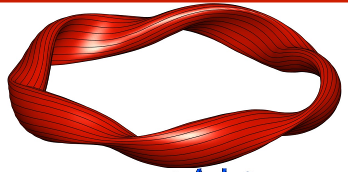


Other options:

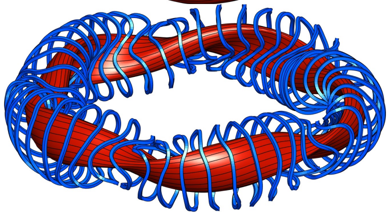
$\boldsymbol{x}$ = shape of plasma *and* coils, $\boldsymbol{x}$ = shape of magnetic axis, …

Most transport-optimized stellarators have used 2 sequential optimization stages

1. Parameters = shape of boundary toroidal surface. Objective = physics (confinement, stability, etc.)



2. Parameters = coil shapes.
   Objective = error in **B** on boundary shape from stage 1.



Shape of a toroidal boundary surface (+ pressure & current vs *r* inside, & total **B** flux) determines **B** everywhere inside:

$$\text{Consider a low-pressure plasma so } 0 \approx \mathbf{J} = \nabla \times \mathbf{B} \quad \Rightarrow \quad \mathbf{B} = \nabla \Phi.$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \nabla^2 \Phi = 0.$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \text{ on boundary} \quad \Rightarrow \quad \mathbf{n} \cdot \nabla \Phi = 0.$$

$$\Rightarrow \quad \text{Laplace's eq with Neuman condition.}$$

$$\Rightarrow \quad \text{Unique solution up to scale factor + constant.}$$

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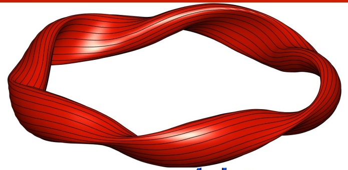

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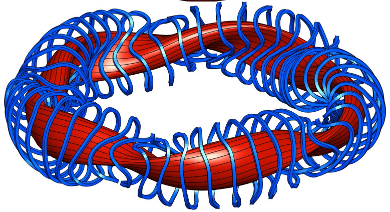
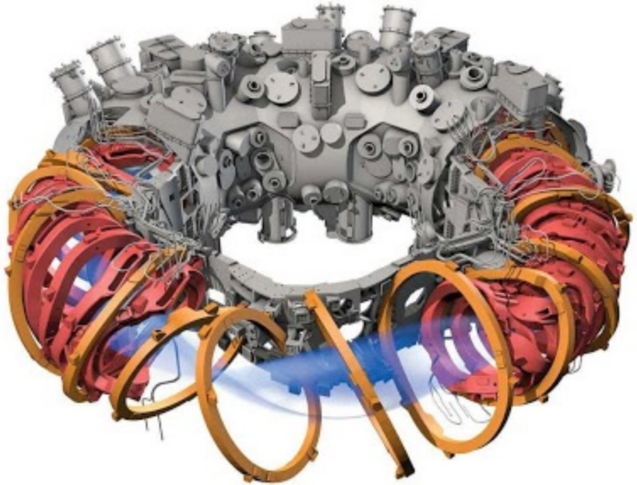
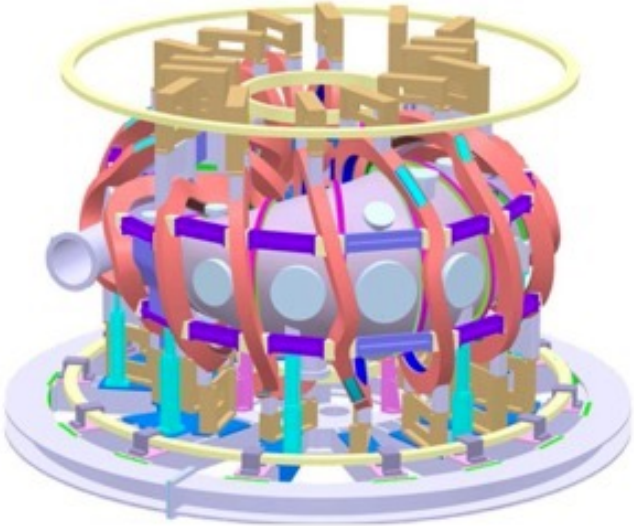


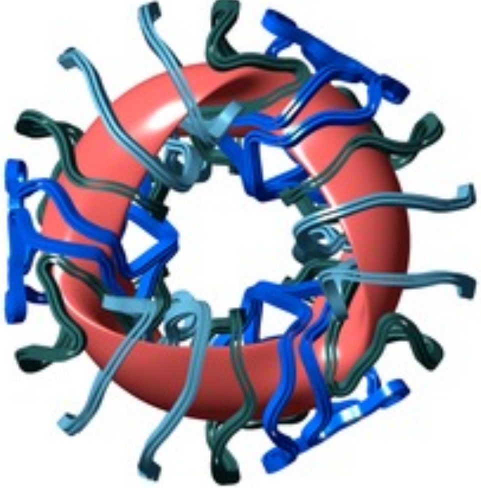
W7-X (Germany)



CFQS (China), under construction



NCSX (Princeton)



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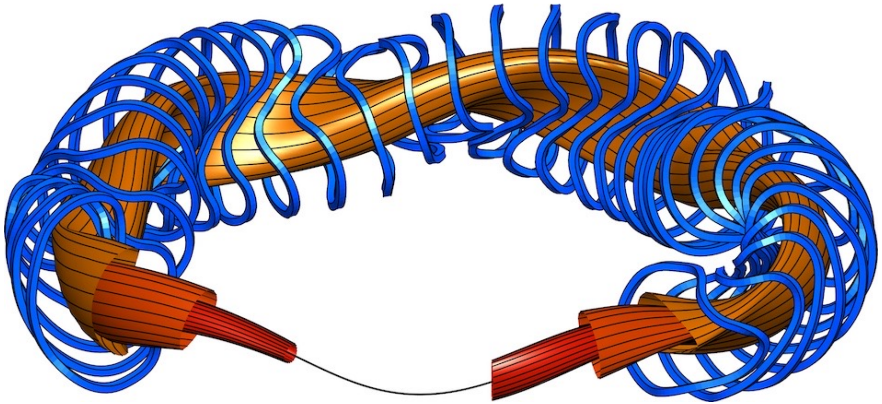
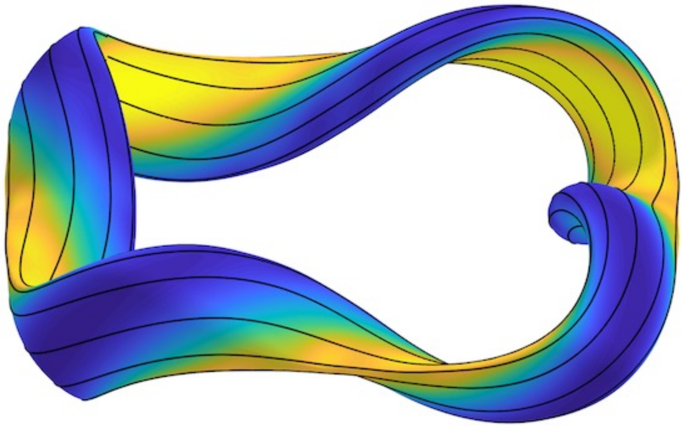
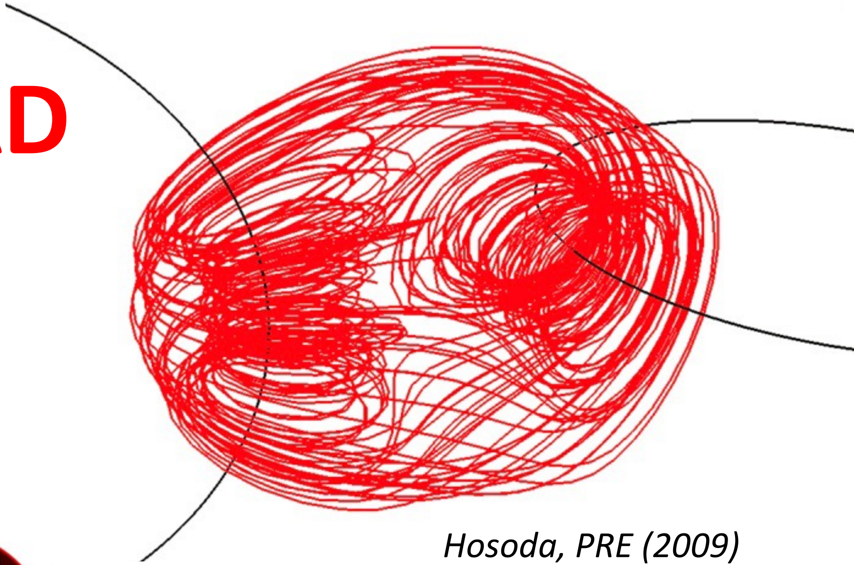




One goal of stellarator optimization is having field lines lie on surfaces.

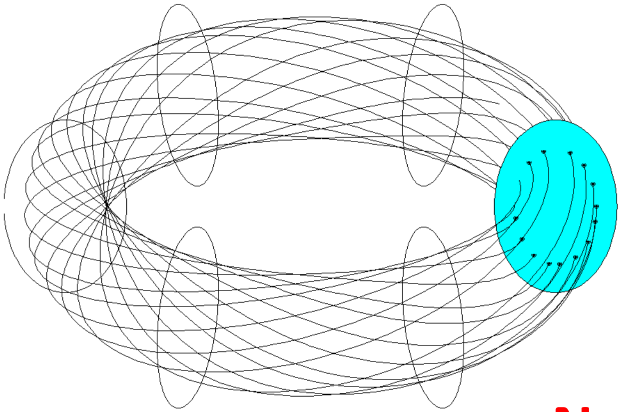
Chaotic (volume-filling) **B** field lines would allow inside & outside to mix even without cross-**B** drift.



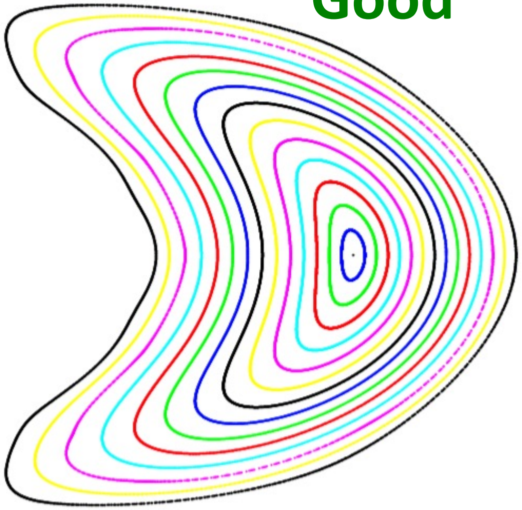
Hosoda, PRE (2009)

One goal of stellarator optimization is having field lines lie on surfaces.

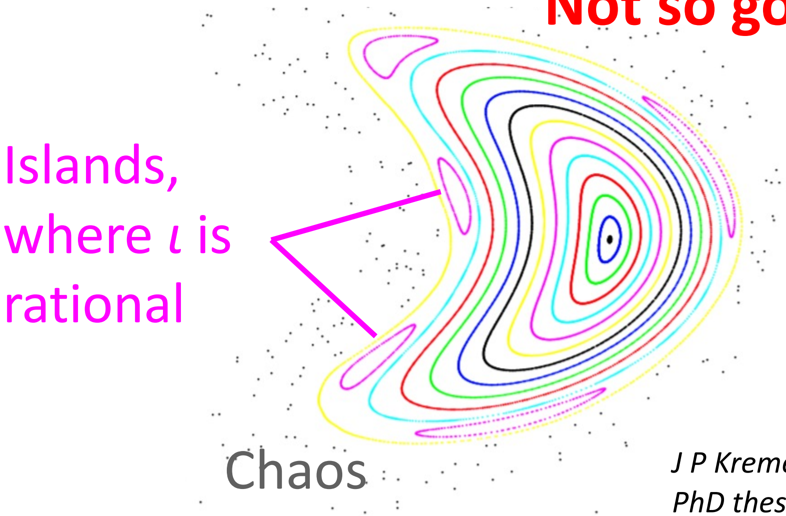
Magnetic surfaces (a.k.a flux surfaces) can be visualized with a “Poincare plot”:



Good



Not so good



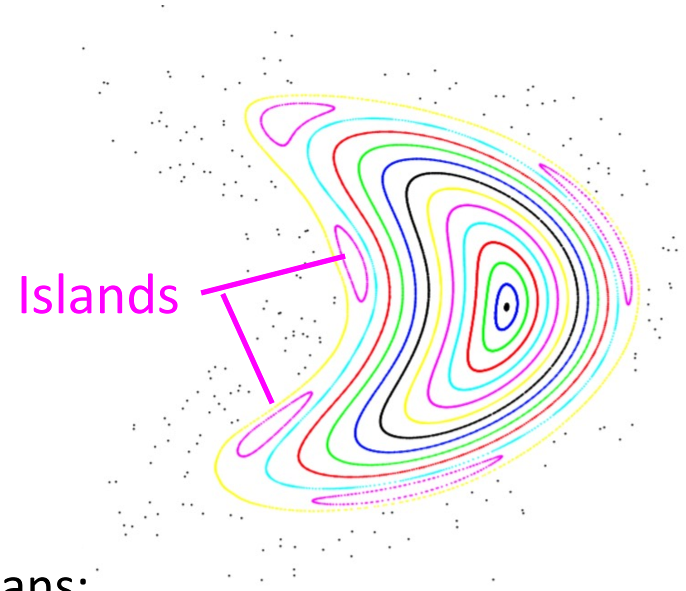
J P Kremer, PhD thesis, Columbia

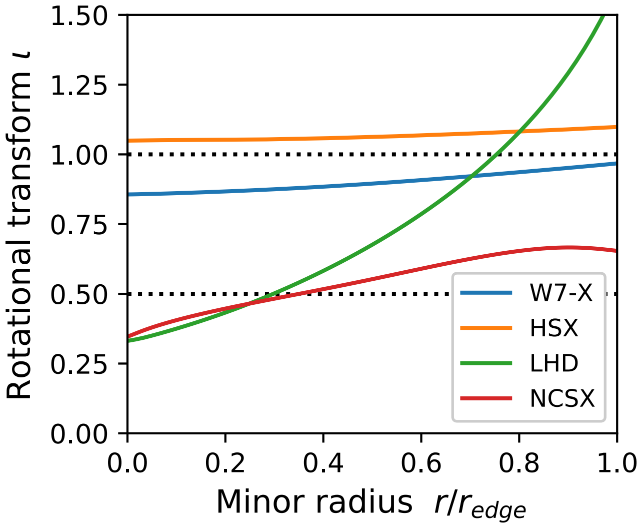
How much rotational transform do you want?

Avoid rationals like ι = 1 or ½: islands form there.

So, maybe want low "magnetic shear" = $|\nabla\iota|$.

Or, maybe want *high* magnetic shear since it makes islands thin. (width $\propto |\nabla\iota|^{-1/2}$)





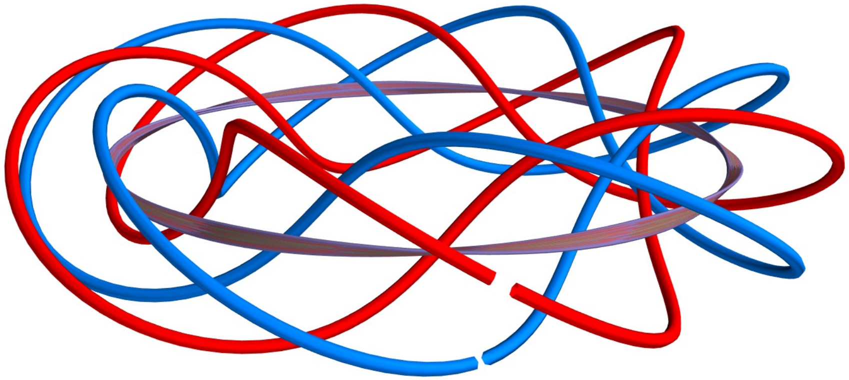
Larger ι means:

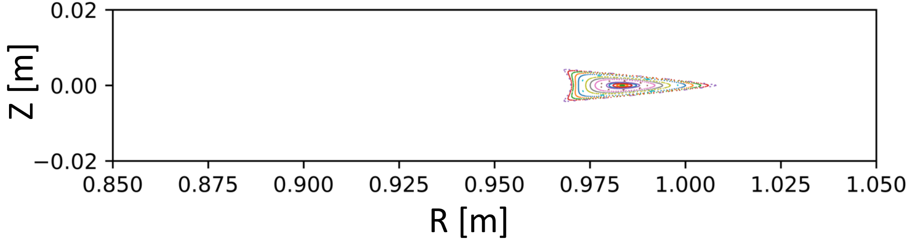
- Thinner orbits, so better confinement.
- **B** changes less due to plasma current. (Higher "equilibrium β limit".)
- But, more complicated coils.

Perhaps the first type of stellarator optimization was to achieve good flux surfaces

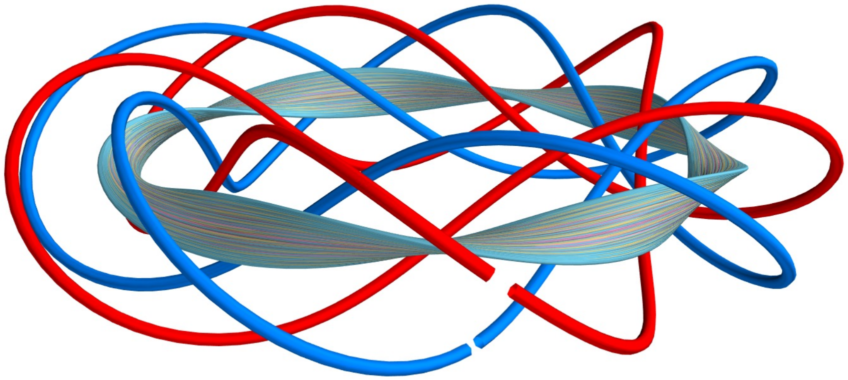
Reproduction of Cary & Hanson (1986) by Rogerio Jorge

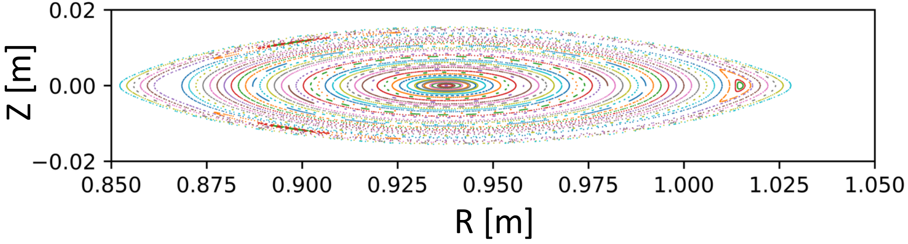
Unoptimized





Optimized



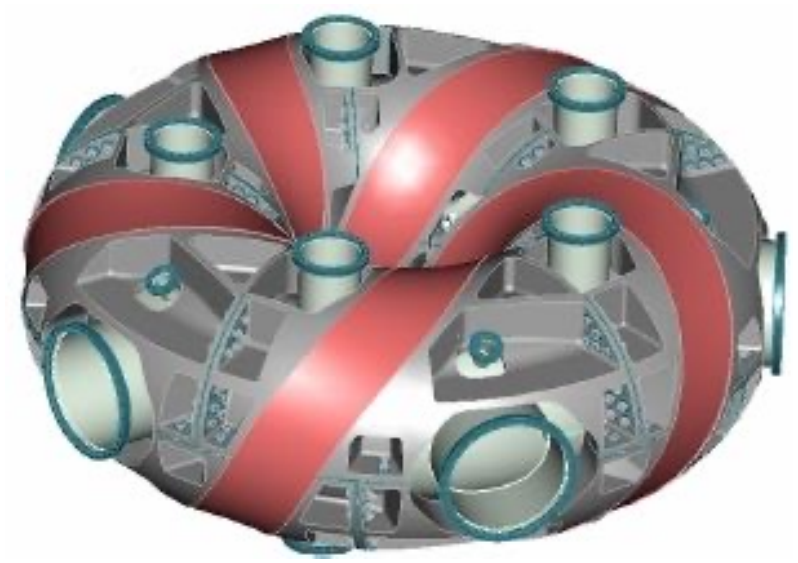


Parameter space x = Fourier modes of coil shapes.

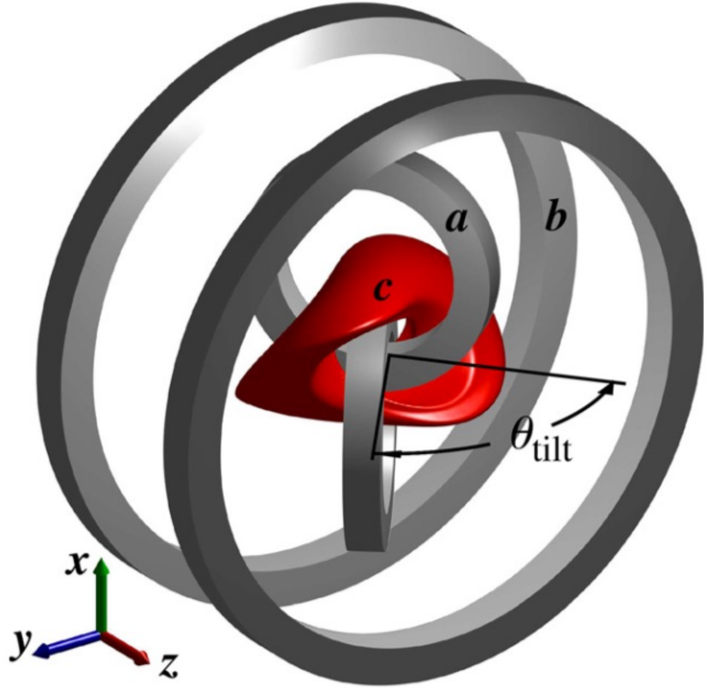
Cost function f = square of "Greene's residue"

Optimization for good flux surfaces continues to be a principle behind recent stellarators

CTH:
(Compact Toroidal Hybrid, at Auburn)



CNT (Columbia Non-neutral Torus):
Optimize *expected* volume over possible coil position errors



Pedersen (2004), Hammond (2016)

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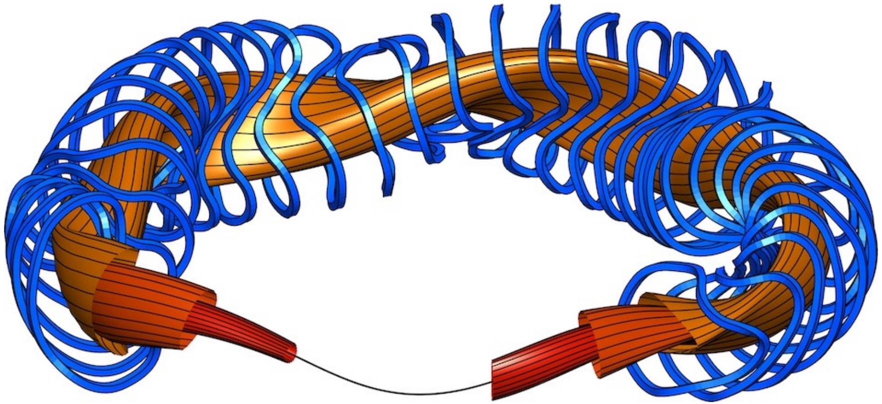
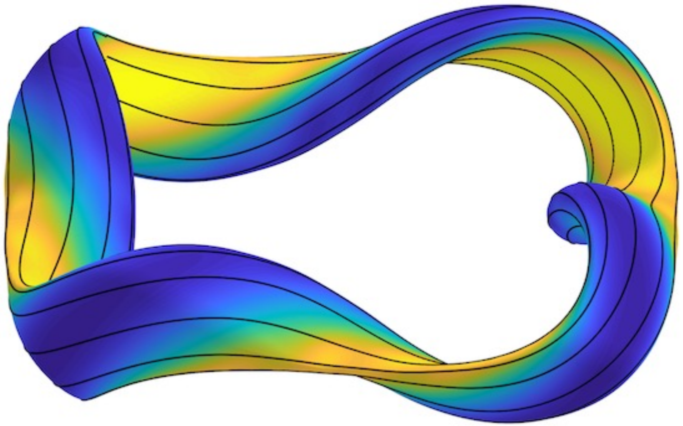
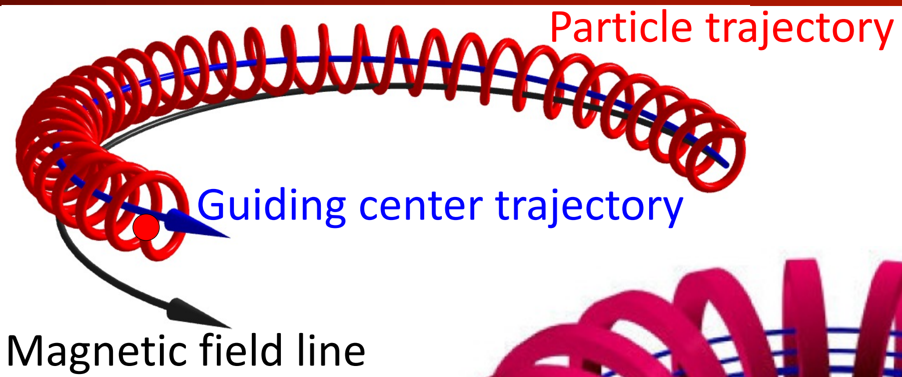
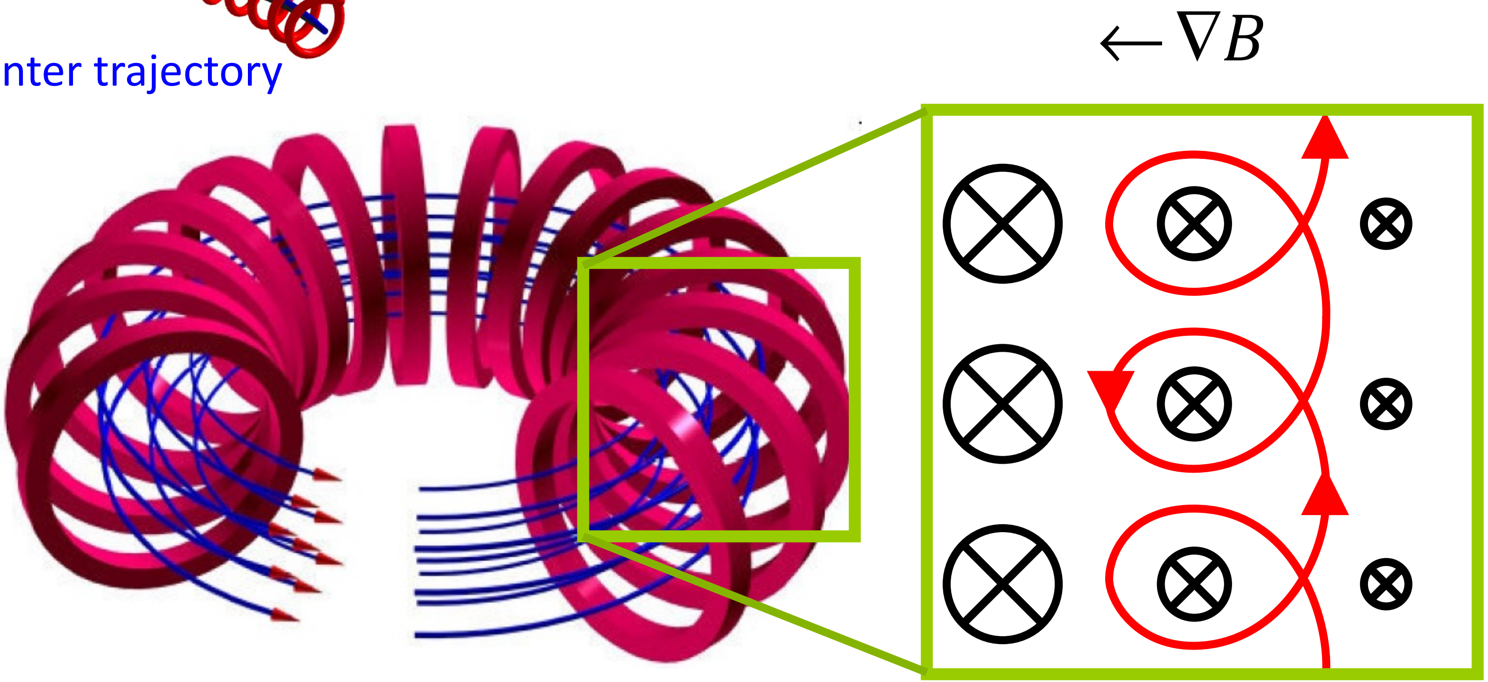




Reminder: the ∇B and curvature drifts make confinement challenging

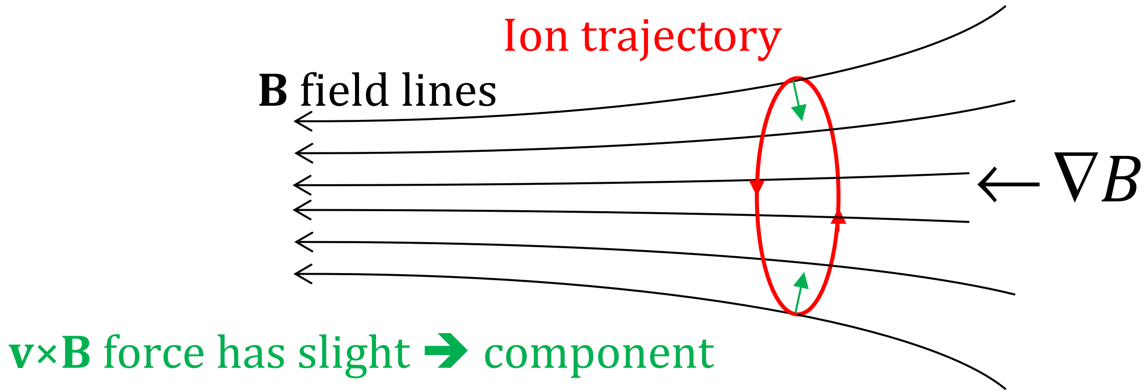




Ions drift up: they are not confined!

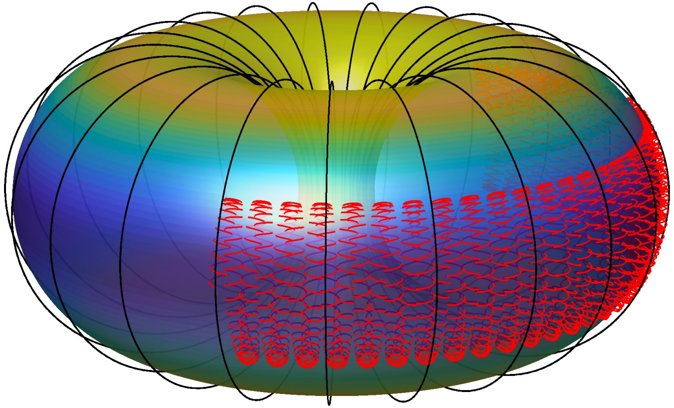
Particles drift in the $q\mathbf{B}\times\nabla B$ direction

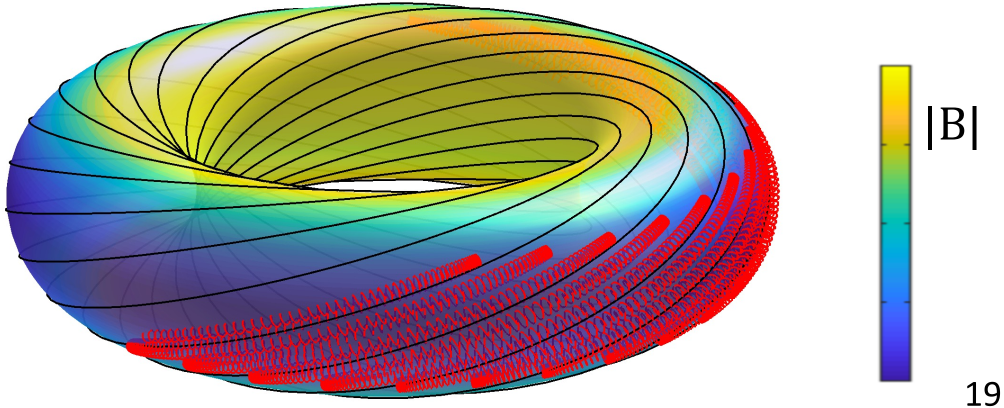
Mirror force: particles are pushed away from regions of high |B|



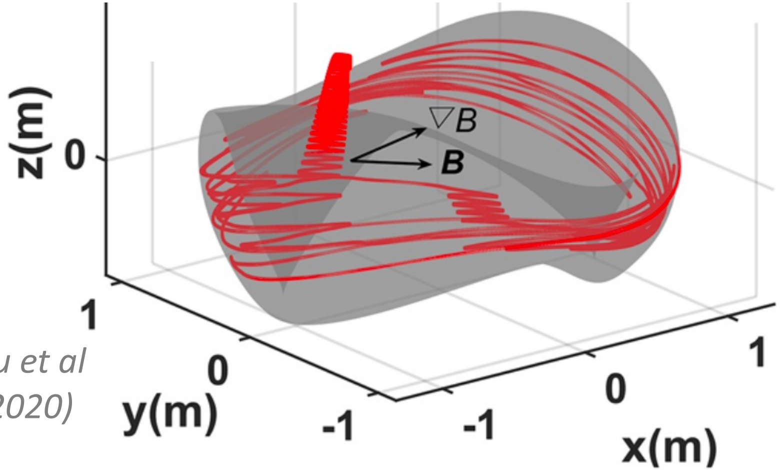
$$\frac{dv_{||}}{dt} = -\frac{v_\perp^2}{2B}\mathbf{b}\cdot\nabla B$$

A few particles with very small $v_{||}$ = **v**·**B** “bounce” and are “trapped” in low-|B| regions.





Flux surfaces are not enough: *Trapped* particles are not confined without a further condition like "quasisymmetry" or "omnigenity"

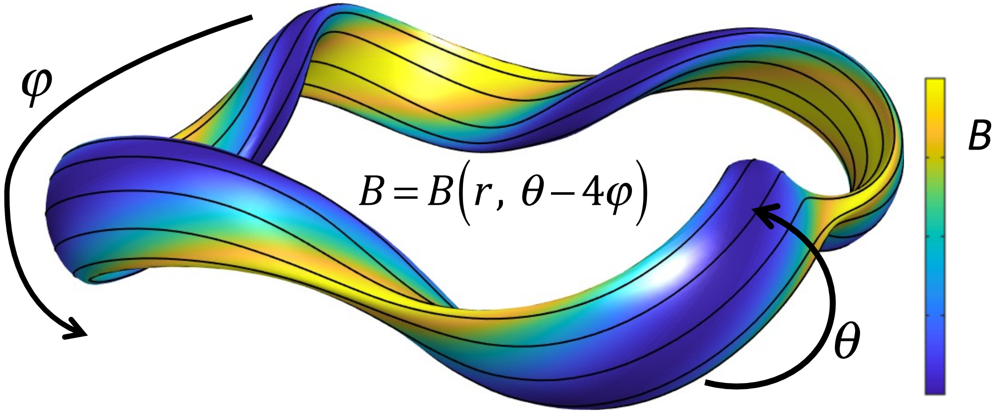


Su et al (2020)

In general: trapped particles do not sample the whole surface, so cross-field drift does not average to 0.

$\Longrightarrow$ Large neoclassical transport.

One solution is quasisymmetry: make $B(r, \theta, \varphi) = B(r, M\theta - N\varphi)$ for special angles θ, φ.

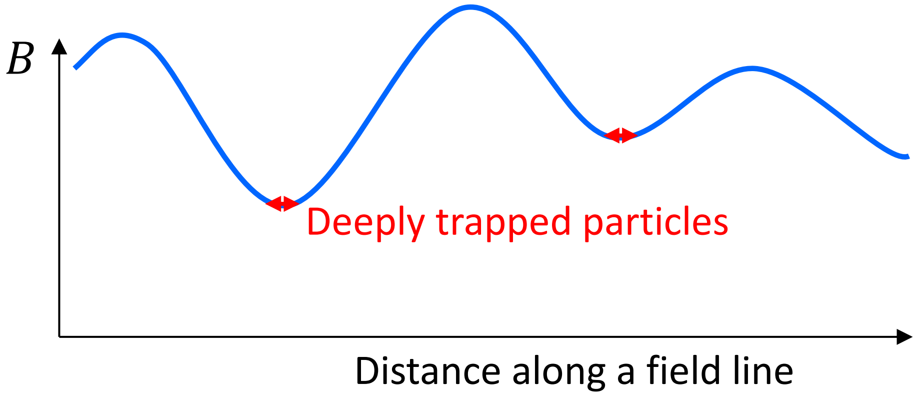


Symmetry direction

$\Longrightarrow$ Conserved quantity.

$\Longrightarrow$ Drift averages to 0.

Lemma: deeply trapped particles move so $|B|$ is constant



$$\frac{dB}{dt} = \boldsymbol{v} \cdot \nabla B$$

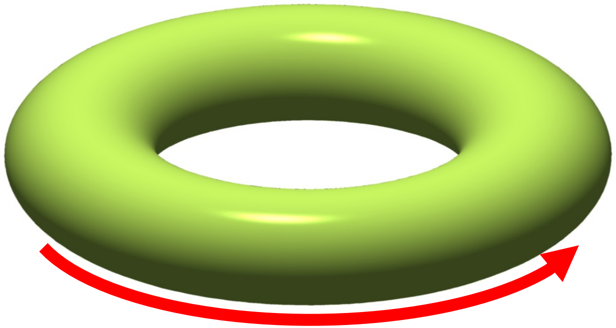
$$v_{||} \approx 0 \;\text{ so } \boldsymbol{v} \approx \text{the } \nabla B \text{ drift} \quad \Longrightarrow \quad \boldsymbol{v} \,||\, \boldsymbol{B} \times \nabla B$$

$$\frac{dB}{dt} \propto (\boldsymbol{B} \times \nabla B) \cdot \nabla B = 0.$$

For low neoclassical transport, recent stellarators have come in 3 flavors

- Trapped particles should drift toroidally, helically, or poloidally on a surface.
- *B* contours on a surface have the same topology as these drifts.

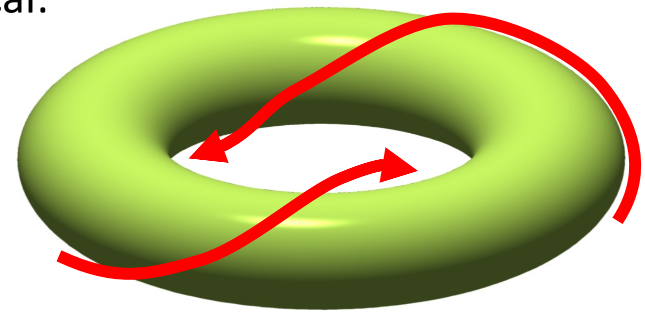
Toroidal:



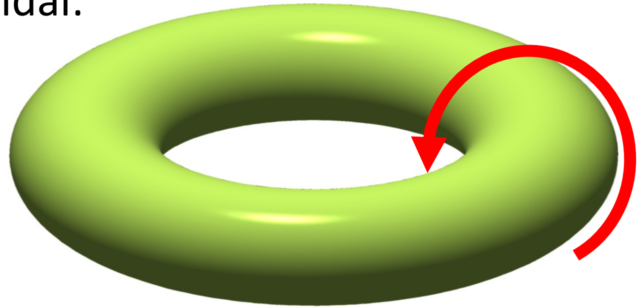
E.g., particles with $v_{||}$=0 move along a constant-*B* contour:

$$\left(\nabla B \text{ drift}\right)\cdot\nabla B \propto \mathbf{B}\times\nabla B\cdot\nabla B = 0$$

Helical:



Poloidal:



For low neoclassical transport, recent stellarators have come in 3 flavors

- Trapped particles should drift toroidally, helically, or poloidally on a surface.
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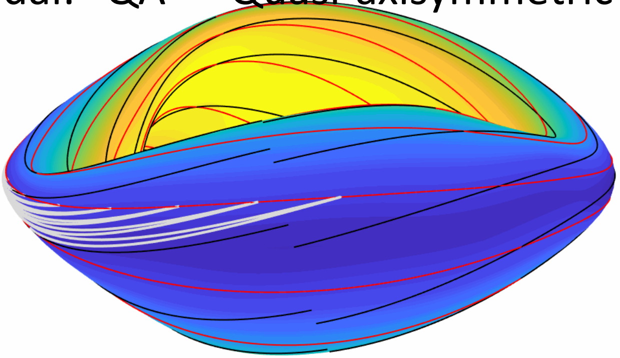
Toroidal: “QA” = Quasi-axisymmetric

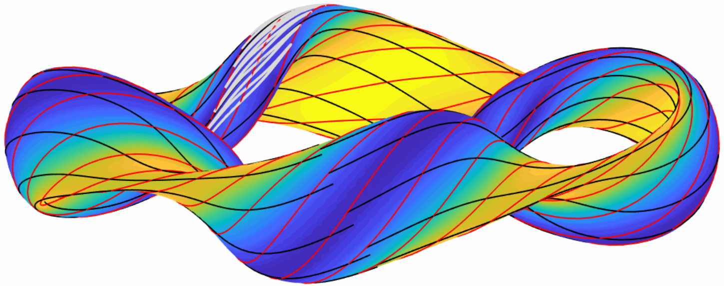


Field lines

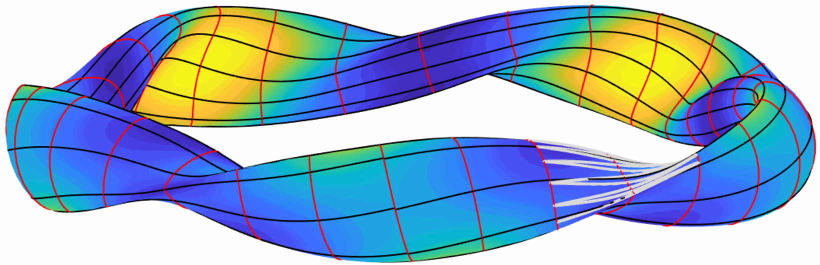
|B| contours (slightly idealized)

Trapped particle

Helical: “QH”= Quasi-helically symmetric



Poloidal: “QI”= Quasi-isodynamic



Pros and cons of the 3 classes

QA:

+ Lowest aspect ratio
+ Fewest coils, largest clearances
+ Large bootstrap current increases iota
- Wider orbits mean worse confinement
- Large current may contribute to MHD instability

QH:

+ Extremely good confinement
+ Can build on experience with HSX
- Seems to require high aspect ratio and many coils

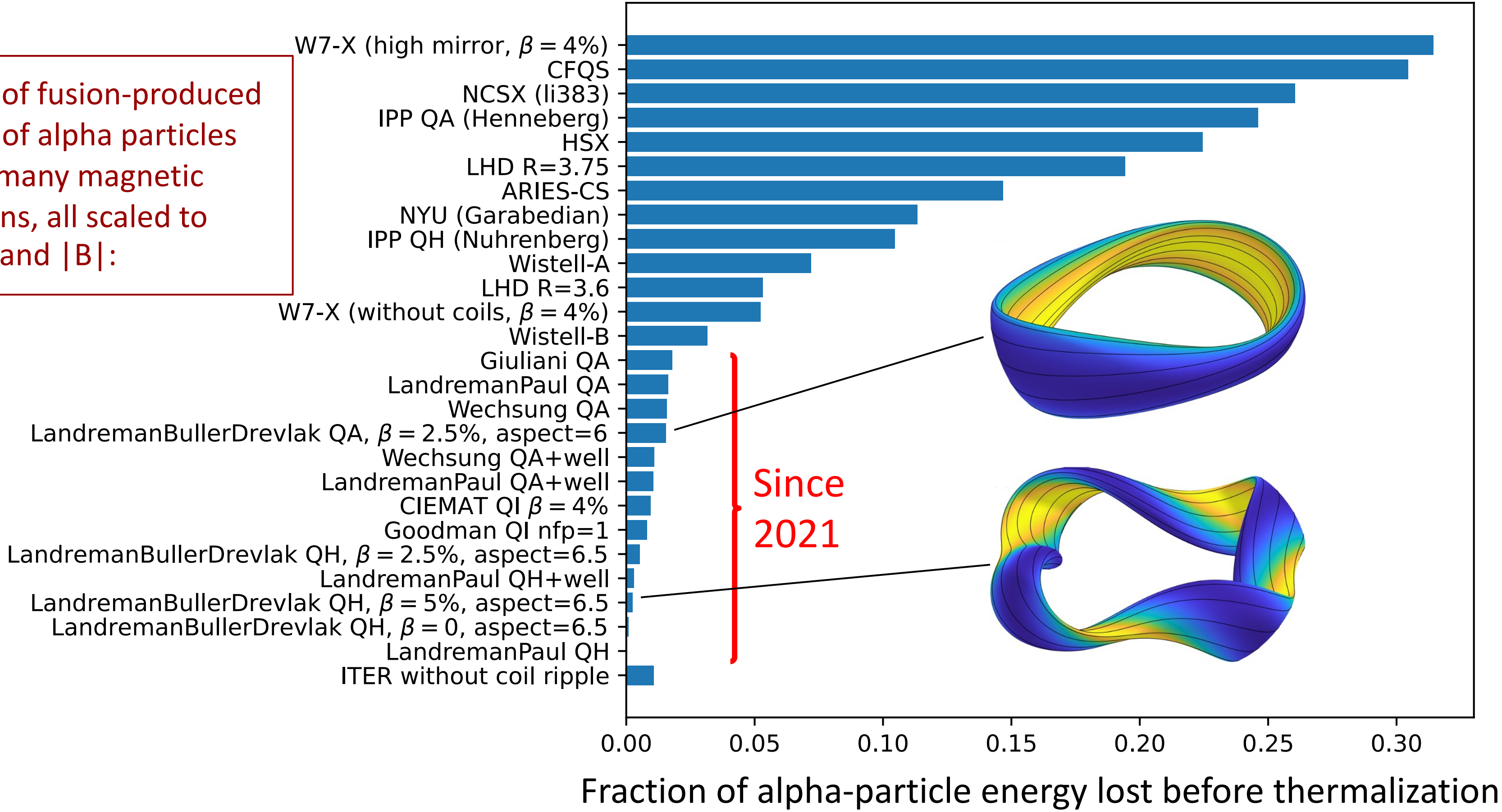
? Intermediate bootstrap current between QA and QI

QI:

+ Low bootstrap current means high robustness to different pressure profiles
+ Can use island divertor
+ Can build on experience from W7-X
- Seems to require high aspect ratio and many coils
- Optimization is generally trickier

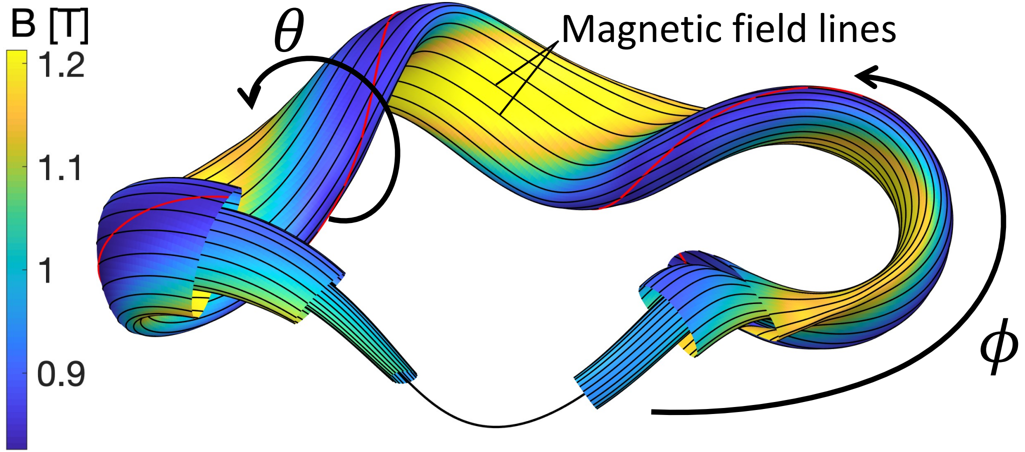
There has been great progress recently in optimizing stellarator neoclassical confinement

Trajectories of fusion-produced distribution of alpha particles followed in many magnetic configurations, all scaled to reactor size and |B|:



Landreman et al, Phys Plasmas (2022).

The parameter space for stage-1 optimization is typically a set of Fourier amplitudes for the boundary surface in cylindrical coordinates



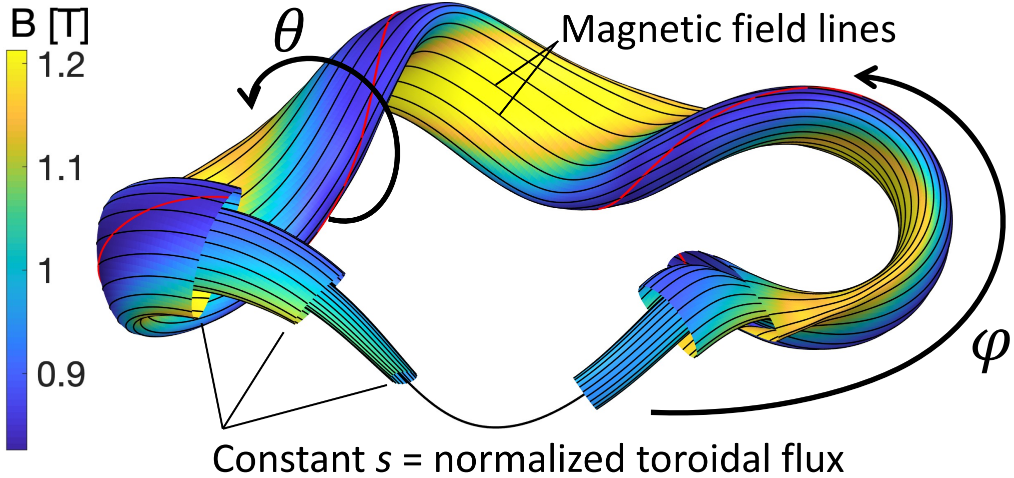
Parameterization of boundary surface:

$$R(\theta,\phi) = \sum_{m,n=-5}^{5} R_{m,n}\cos\left(m\theta - n_{fp} n\phi\right) \qquad Z(\theta,\phi) = \sum_{m,n=-5}^{5} Z_{m,n}\sin\left(m\theta - n_{fp} n\phi\right)$$

ϕ = standard cylindrical angle, n_{fp} = number of field periods

Parameter space for optimization: $\boldsymbol{x} = \left[R_{m,n}, Z_{m,n}\right]$

Quasisymmetry is a sufficient (though not necessary) condition for confinement, & a useful surrogate



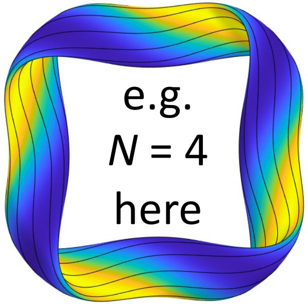
$$B = B(s,\, \theta - N\,\varphi)$$

"Boozer angles"

Objective: $f_{QS} = \int d^3x \left(\frac{1}{B^3} \left[(N - \iota) \boldsymbol{B} \times \nabla B \cdot \nabla \psi - G \boldsymbol{B} \cdot \nabla B \right] \right)^2$

For quasi-axisymmetry, $N = 0$.

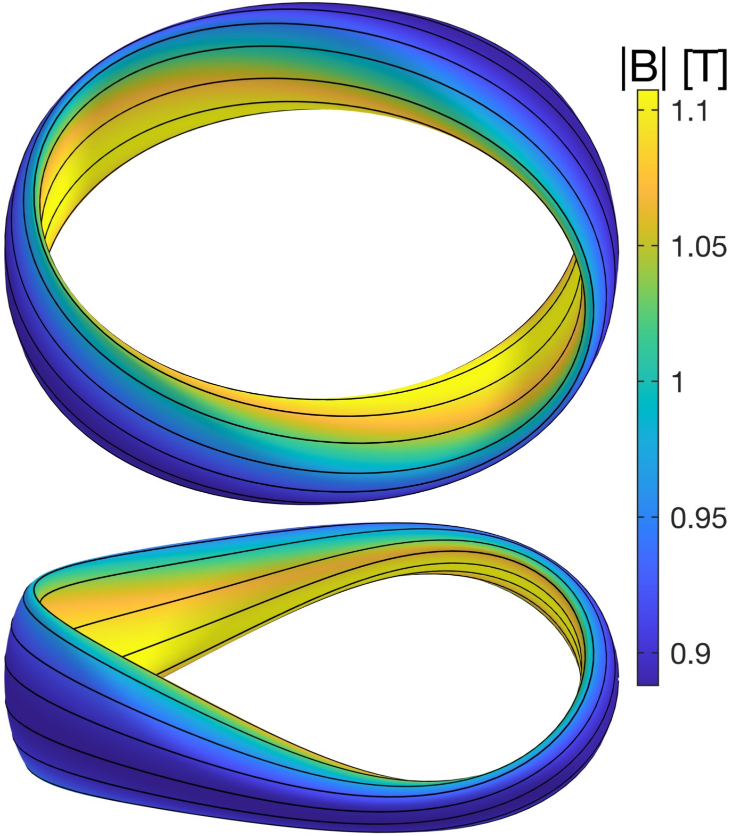
For quasi-helical symmetry, N is the number of field periods,



$$f_{QH} = \left(A - A_* \right)^2 + f_{QS}$$

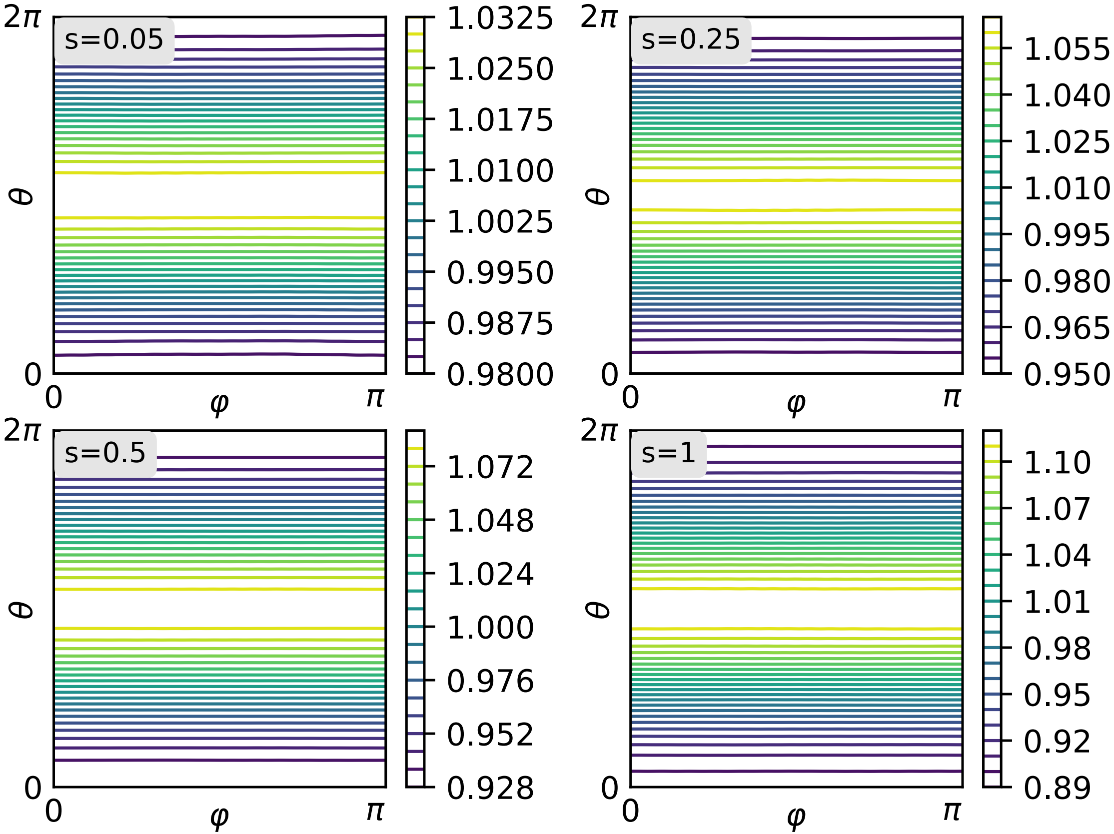
Boundary aspect ratio

Example of quasi-axisymmetry optimization

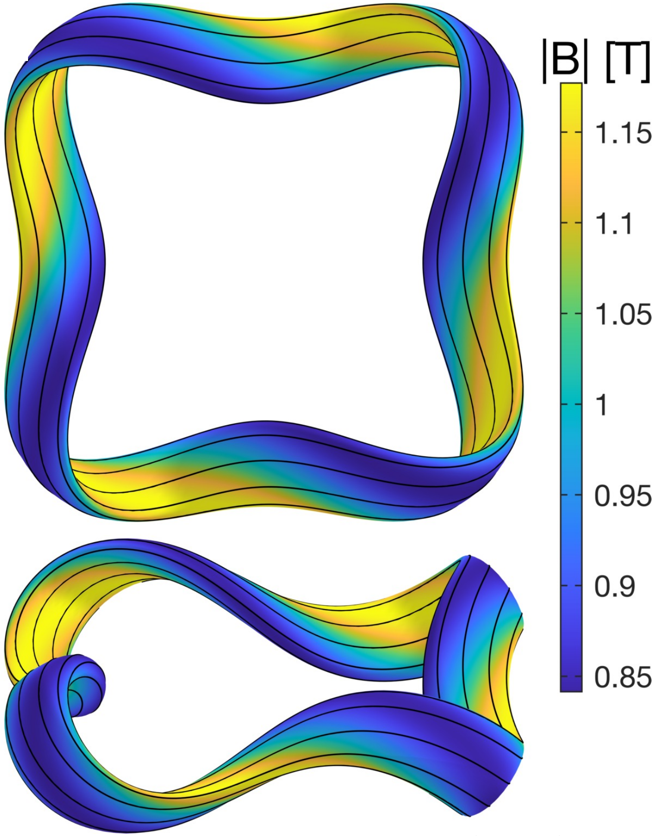


ML & Paul, Phys Rev Lett (2022)

|B| on flux surfaces of the quasi-axisymmetric field

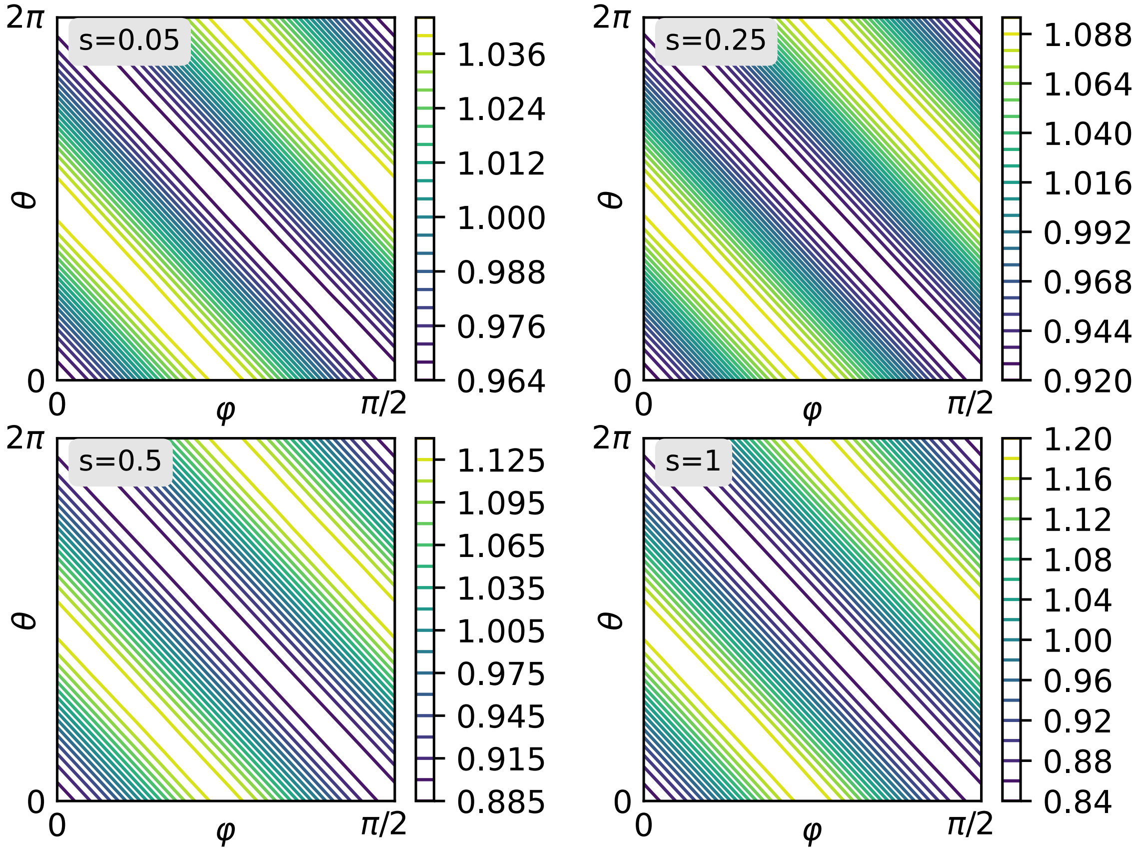


Example of quasi-helical symmetry optimization

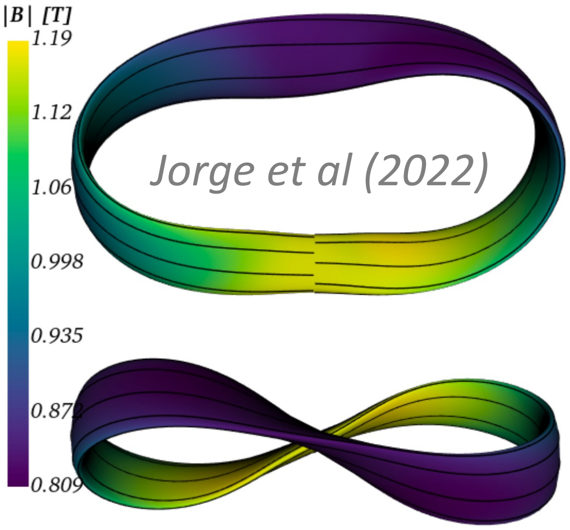


ML & Paul, Phys Rev Lett (2022)

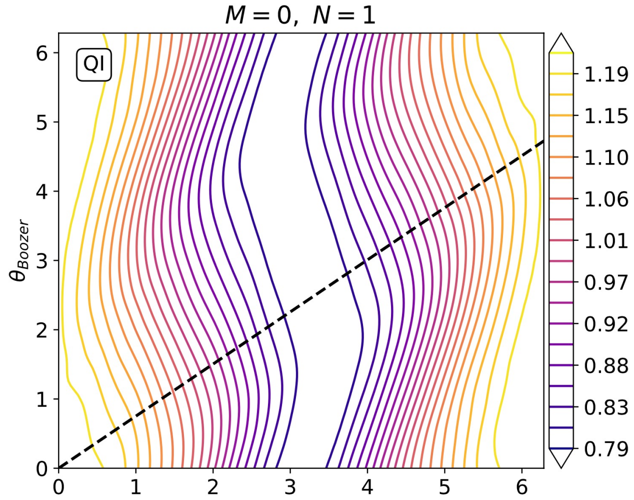
|B| on flux surfaces of the quasi-helically symmetric field



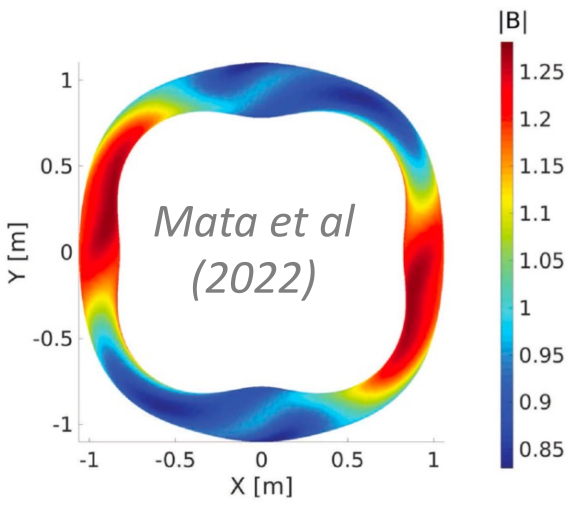
There has been similar recent progress in finding quasi-isodynamic configurations



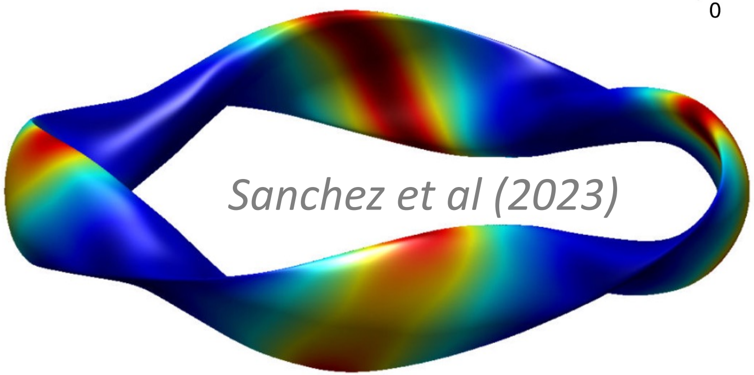
Jorge et al (2022)



Dudt et al (2023)

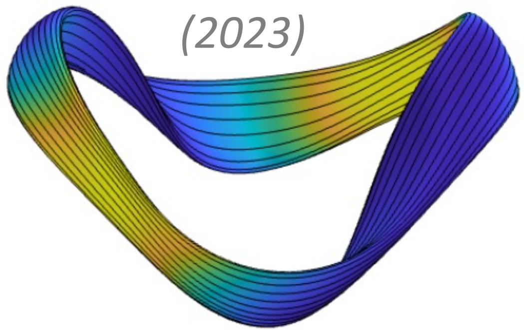


Mata et al (2022)



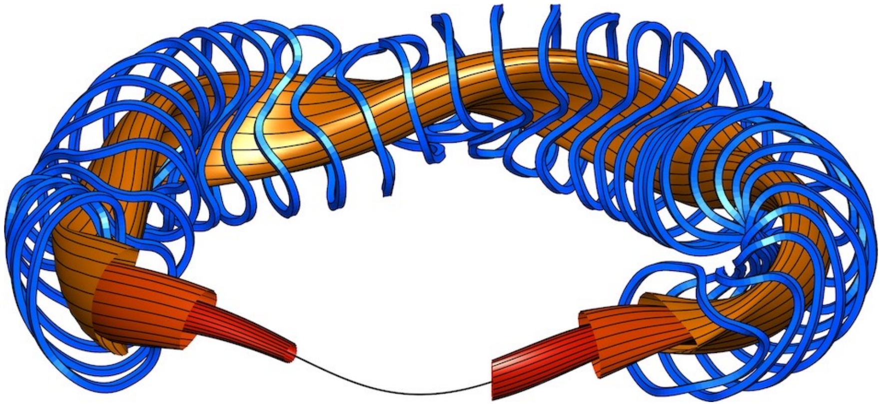
Sanchez et al (2023)

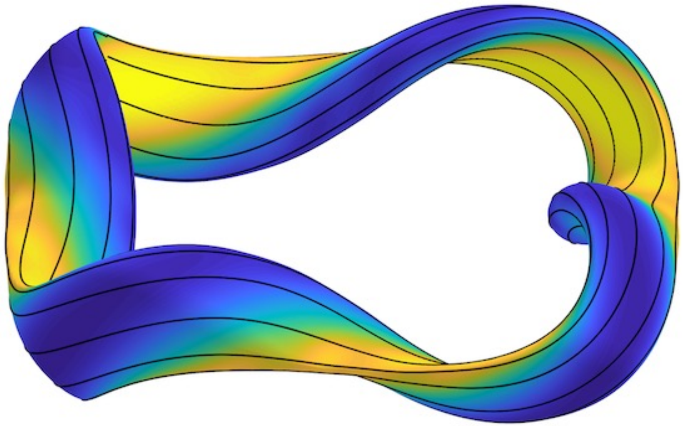
Goodman et al (2023)



Outline

- Optimization in general
- Which parameter space to optimize in?
- Quasi-axisymmetry, quasi-helical symmetry, and quasi-isodynamic
- Optimization for stability & turbulence
- Coil optimization: current potential and filament methods

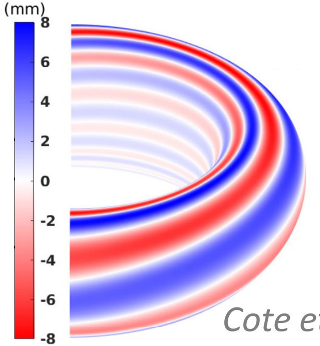




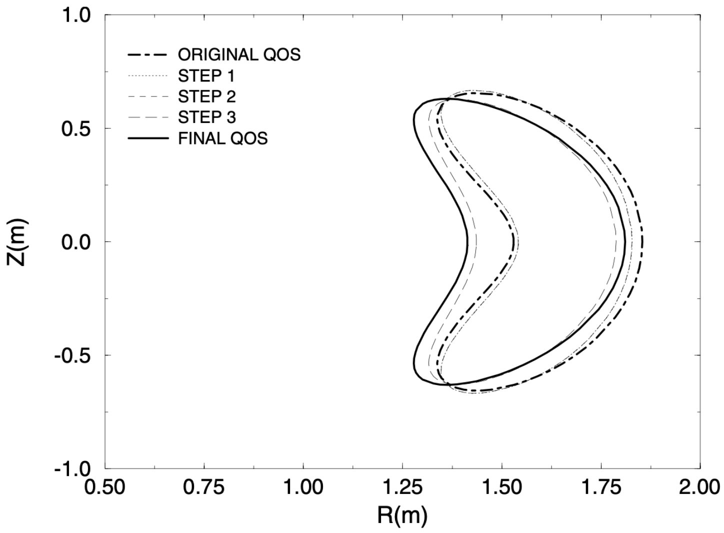
Stellarator geometry can be optimized for MHD stability

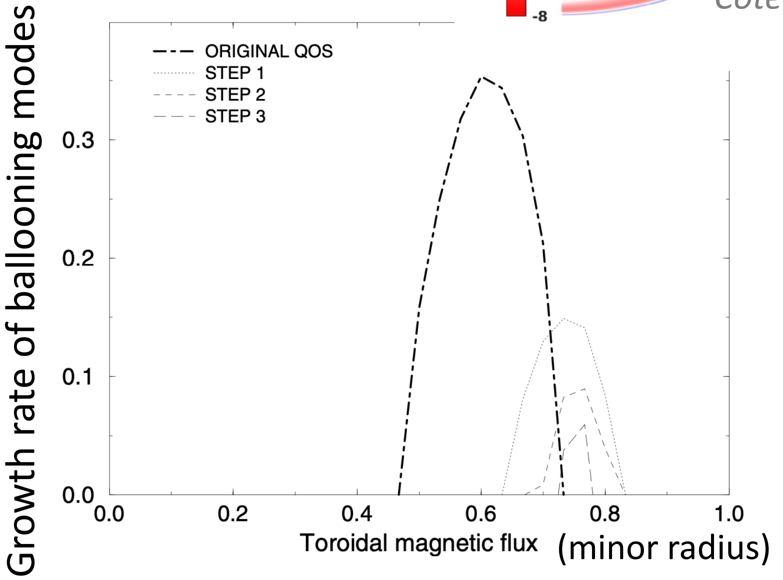
Types of MHD stability calculations, in increasing complexity:

- Magnetic well & Mercier's criterion (interchange)
- Ballooning modes (short wavelength ⊥ to **B**)
- Finite wavelength (everything)



Cote et al, (2019)



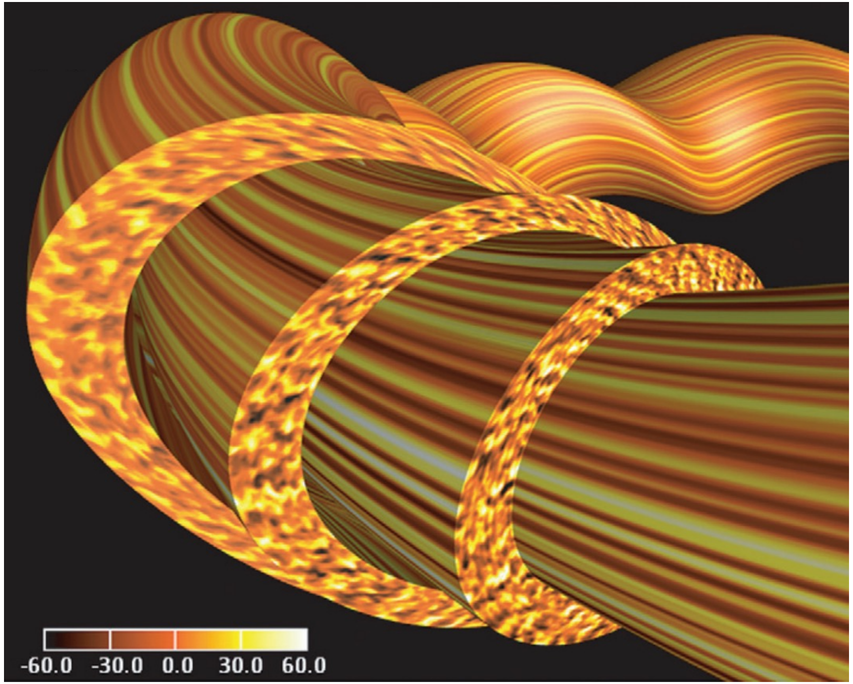


Sanchez et al, Plasma Phys. Control. Fusion (2000)

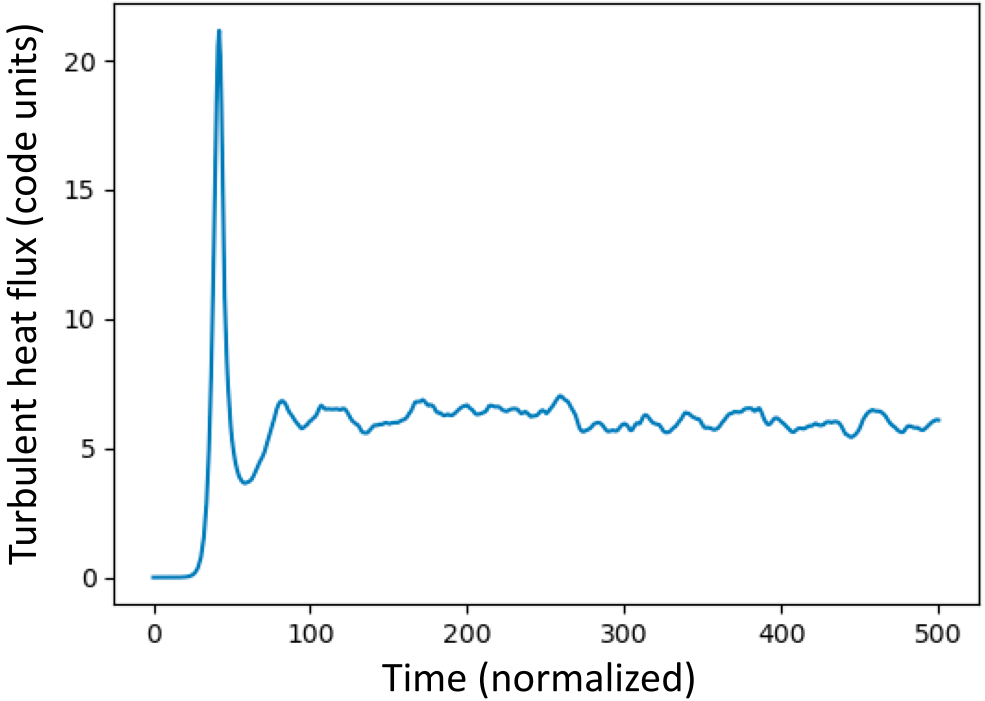
Optimization for reduced turbulence has mostly used simplified proxies in the cost function

Turbulent heat flux can be simulated, but it is computationally expensive and noisy

⇒ not good for an objective function



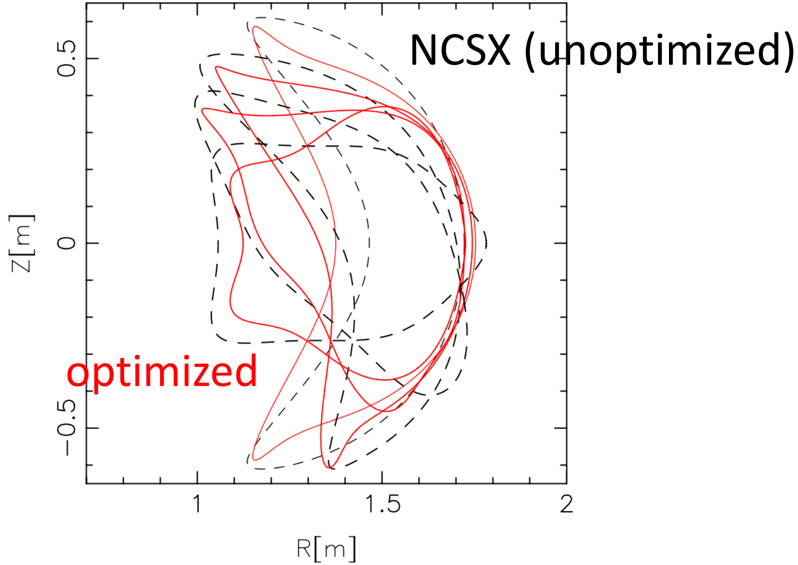
Nunami 2017

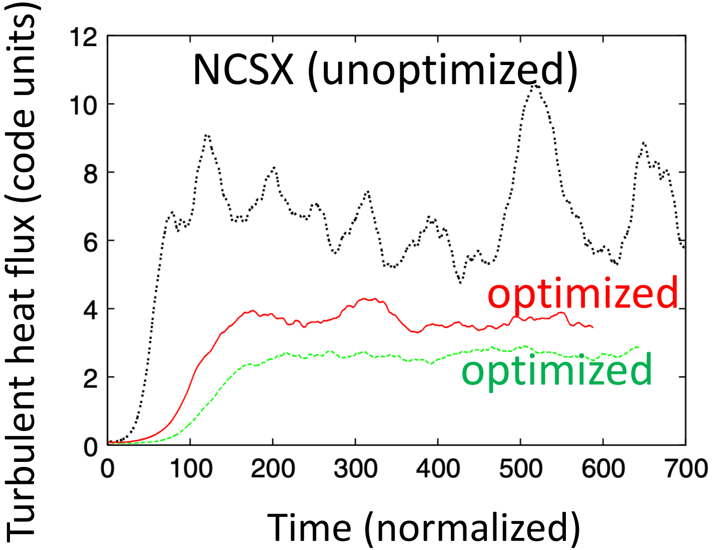


Optimization for reduced turbulence has mostly used simplified proxies in the cost function

$$f = -0.959 n g^{xx} \frac{dT}{dx} \sum_k \frac{\gamma_k}{k_x^2} \qquad \gamma_k = -\frac{c k_y T_i}{eB} \sqrt{\kappa_1 (\kappa_p - \kappa_{cr})} H(\kappa_p - \kappa_{cr}) H(-\kappa_1) \qquad \kappa_{cr} = 0.053$$

$$\frac{1}{k_x^2} = \rho_i^2 + \frac{\rho_i L_p}{1 + \left\langle (1.12 s_l)^2 \right\rangle_\theta} \qquad s_l = \frac{\partial (g^{xy}/g^{xx})}{\partial \theta} \qquad \kappa_1 = (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \frac{\partial \mathbf{r}}{\partial x} \qquad \kappa_p = \frac{1}{L_p} = -\frac{1}{p}\frac{dp}{dx}$$

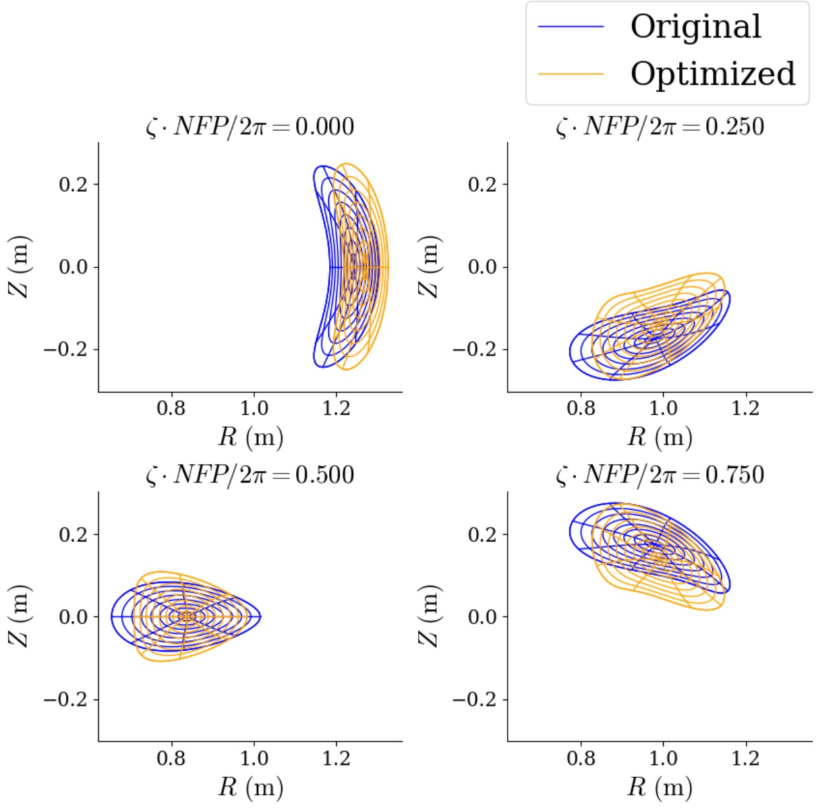


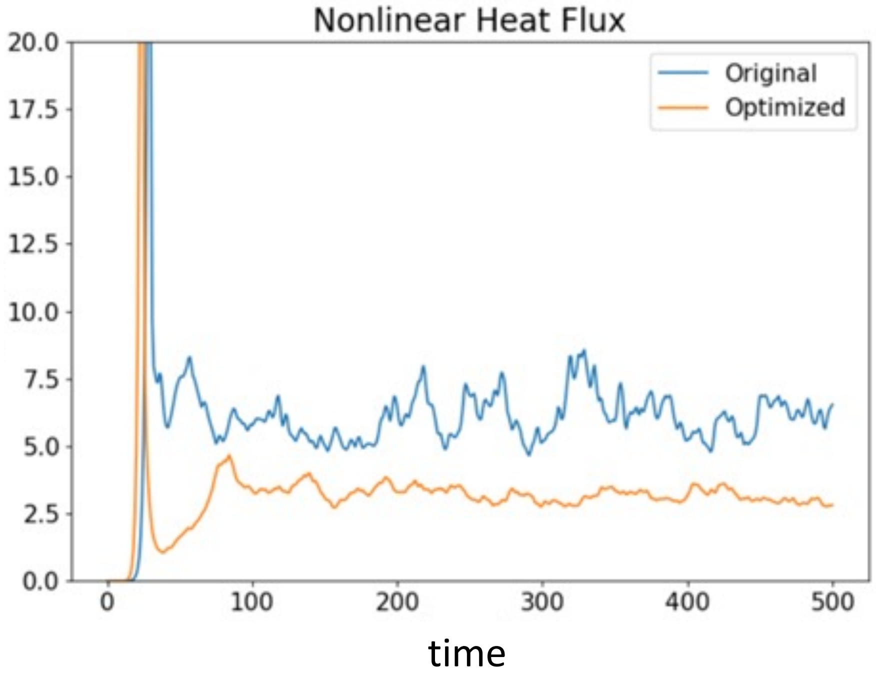


Mynick et al, Physical Review Letters (2010)

The first optimizations with nonlinear turbulence calculations in the objective are becoming possible

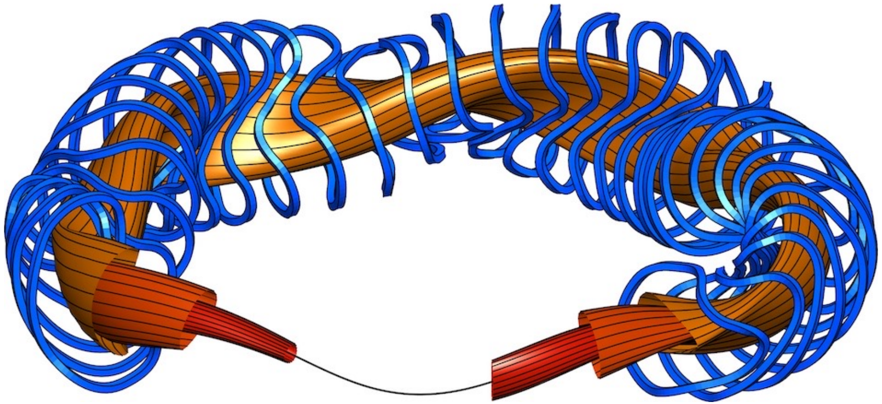
By Patrick Kim (Maryland undergraduate!)

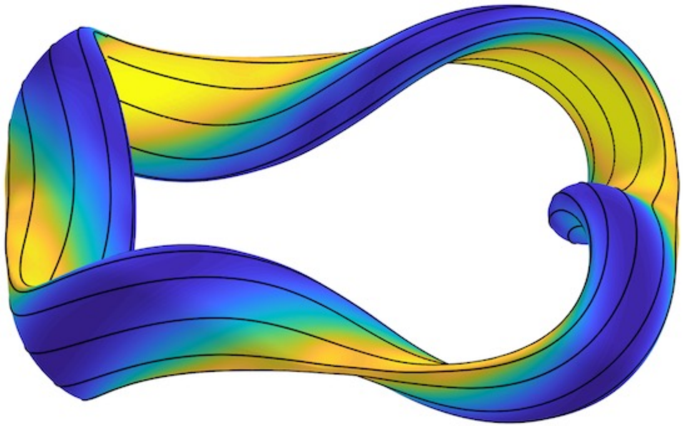




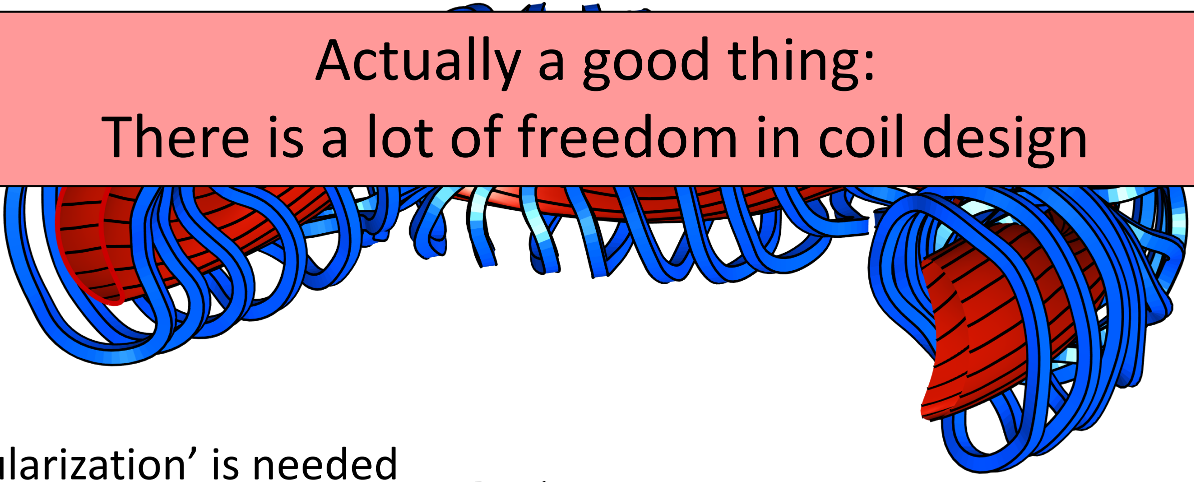
Outline

- Optimization in general
- Optimization for good flux surfaces
- Quasi-axisymmetry, quasi-helical symmetry, and quasi-isodynamic
- Optimization for stability & turbulence
- Coil optimization: current potential and filament methods





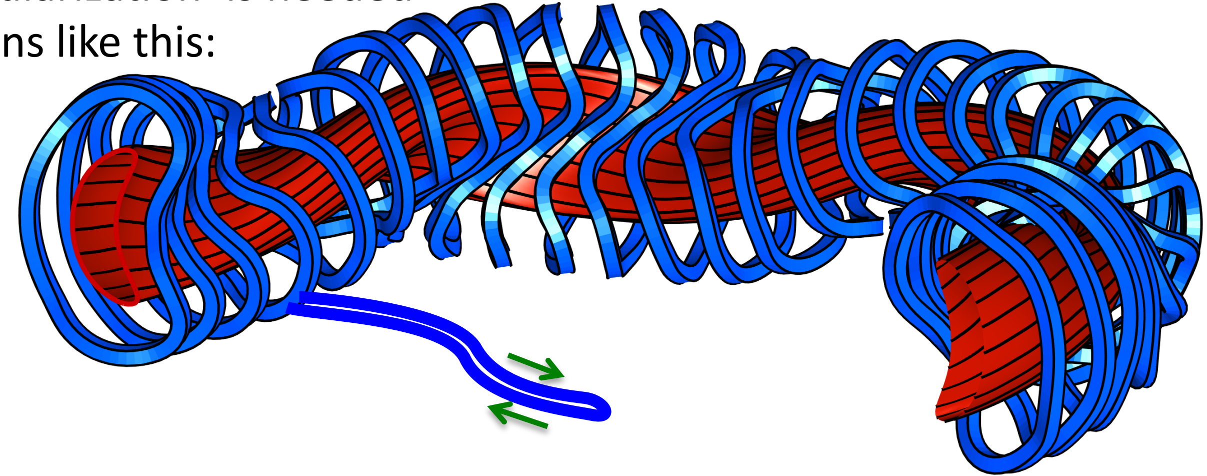
Calculating the currents that produce a given B is an “ill-posed inverse problem”: solution is not unique.



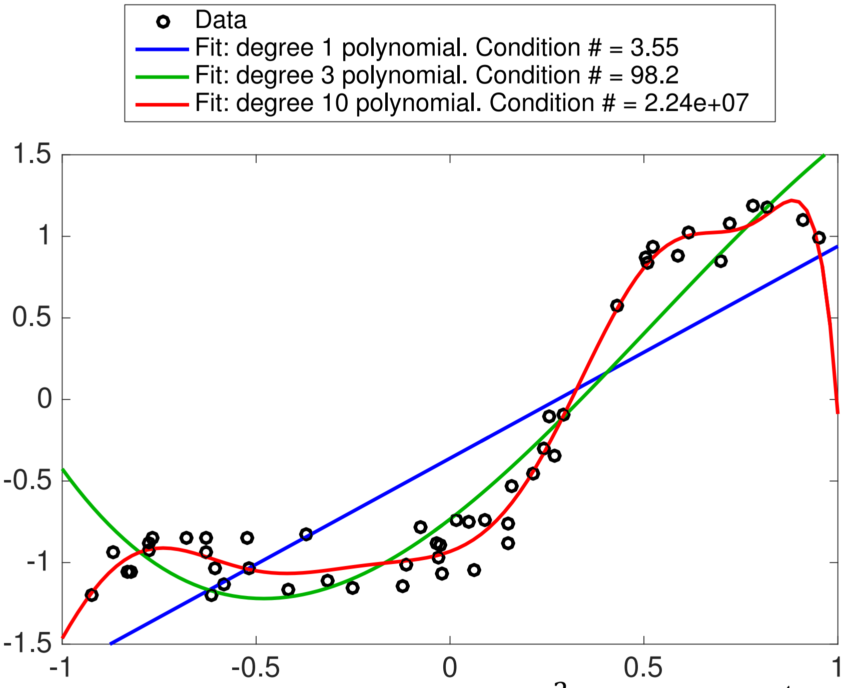
Actually a good thing:

There is a lot of freedom in coil design

Some kind of ‘regularization’ is needed to exclude solutions like this:



Finding coils that produce a given B is analogous to fitting data with a polynomial



Linear least-squares: $\min_{\vec{\mathbf{x}}} \left| \overleftrightarrow{\mathbf{A}}\vec{\mathbf{x}} - \vec{\mathbf{b}} \right|^2 \quad \Rightarrow \quad \vec{\mathbf{x}} = \left(\overleftrightarrow{\mathbf{A}}^T \overleftrightarrow{\mathbf{A}} \right)^{-1} \left(\overleftrightarrow{\mathbf{A}}^T \vec{\mathbf{b}} \right)$

As polynomial degree increases, fit is closer to data but less ‘regular’.
Polynomial degree is the ‘regularization parameter’.

Current potential methods: NESCOIL & REGCOIL

Regcoil: Consider sheet current on a "coil winding surface"

ML, Nuclear Fusion (2017).

$$\boldsymbol{K} = \boldsymbol{n} \times \nabla \phi$$

Surface current

"current potential"

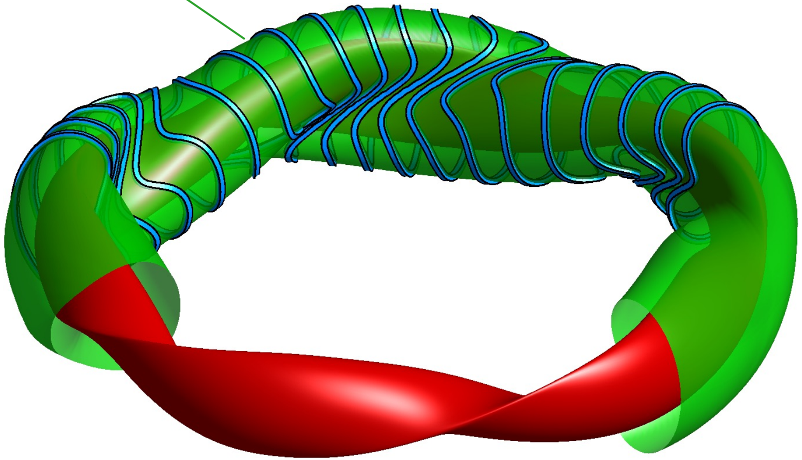
Normal to winding surface

$$\min_{\phi} \left(\int_{\text{Plasma surface}} d^2x \left[\left(\boldsymbol{B} - \boldsymbol{B}_{\text{target}} \right) \cdot \boldsymbol{n} \right]^2 + \lambda \int_{\text{Coil surface}} d^2x \left| \boldsymbol{K} \right|^2 \right)$$

B field error

Coil complexity

Regularization parameter



ϕ contours = coils

$K = |\boldsymbol{K}| \propto 1/\text{distance between coils}$

Pros:

- *Linear* least-squares: no local optima besides the global one.
- Only 2 parameters to vary: coil-to-plasma distance and λ.

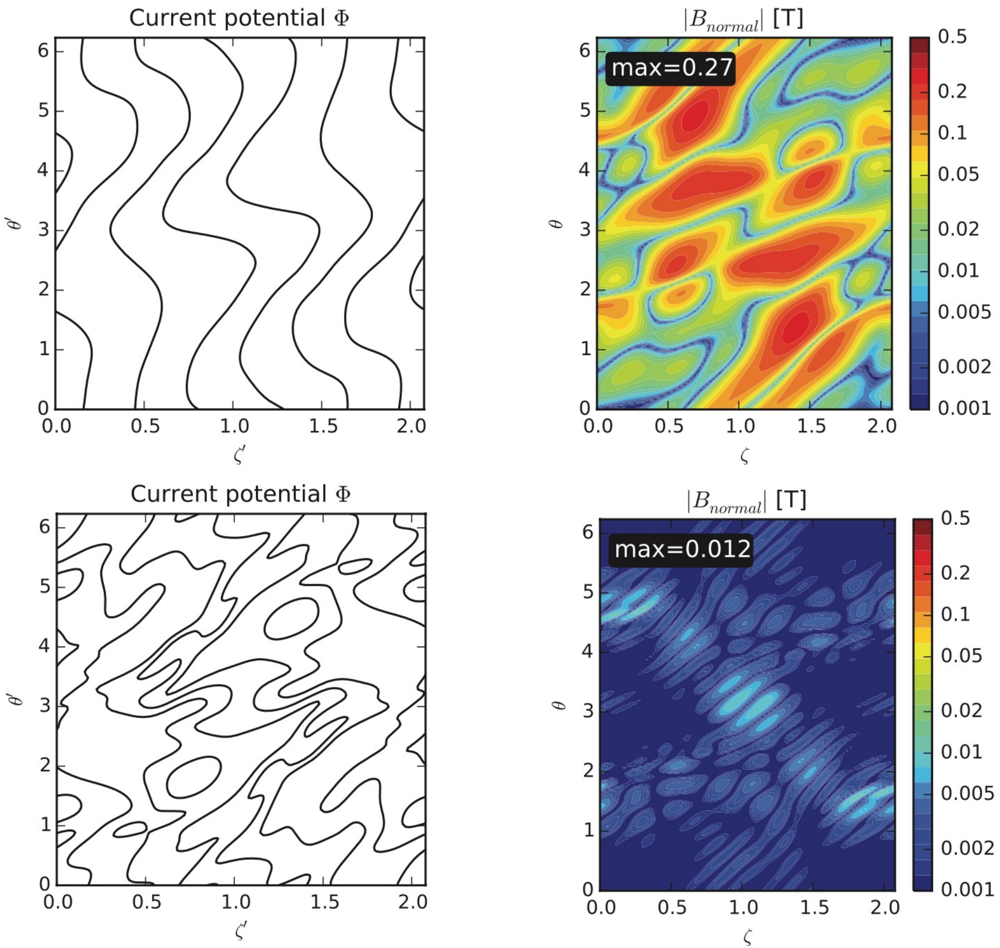
Cons:

- Neglects ripple from discrete coils.
- Coils can't move in 3rd dimension.

In stage-2 coil optimization, there is a trade-off between field accuracy and coil simplicity

High regularization λ:
Simpler coils
but large field error

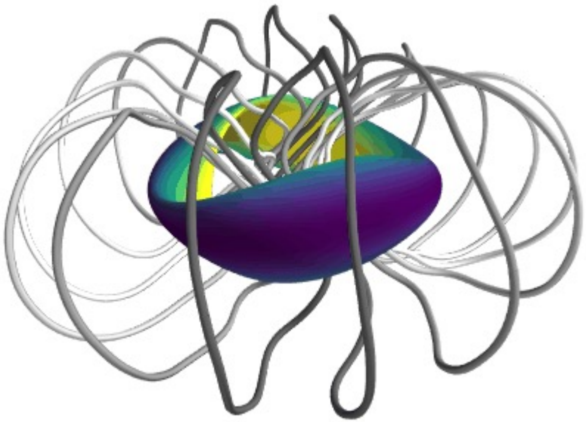
Low regularization λ:
Complicated coils
but small field error



Filament coil optimization

Zhu, Hudson, et al, Nuclear Fusion (2018).

Assume plasma shape has already been optimized, so target **B** field is known.



Coils represented as space curves.

Design variables: Fourier modes of Cartesian components.

$$x(t) = x_{c,0} + \sum_{n=1}^{N_F} [x_{c,n} \cos(nt) + x_{s,n} \sin(nt)]$$

Objective:

$$f = \underbrace{\int_{\substack{plasma \\ surf}} \left[(\boldsymbol{B} - \boldsymbol{B}_{target}) \cdot \boldsymbol{n} \right]^2}_{\text{Match target B}} + \underbrace{\lambda (length - target)^2 + \ldots}_{\text{Regularization}}$$

- Does account for *B* ripple from discreteness of coils.
- Non-convex, so there are multiple local minima. Need good initial guess.

Open questions for stellarator optimization

- How best to combine coil and plasma design?
- How to find designs that tolerate errors in coil shape/position?
- How to avoid getting stuck in little local minima? How to find global optima?
- How to optimize for expensive & noisy objectives (turbulence & fast-particle confinement)?
- How to balance multiple competing objectives?
- How to optimize coil topology?
- How to find configurations that are flexible?
  - Good confinement for different plasma pressures.
  - Ability to tune physics properties by changing coil currents.

More resources

Introductory papers:

Imbert-Gerard, Paul, & Wright, https://arxiv.org/abs/1908.05360
Helander, http://dx.doi.org/10.1088/0034-4885/77/8/087001

Summer schools:

https://hiddensymmetries.princeton.edu/summer-school/summer-school-2020/schedule
https://hiddensymmetries.princeton.edu/summer-school/summer-school-2019/schedule
https://gss.pppl.gov/2021/
https://suli.pppl.gov/2022/course/index.html
https://suli.pppl.gov/2021/course/index.html
https://suli.pppl.gov/2020/course/index.html
https://suli.pppl.gov/2019/course/index.html

Open-source software:

https://github.com/PrincetonUniversity/STELLOPT
https://desc-docs.readthedocs.io/
https://simsopt.readthedocs.io/
https://github.com/landreman/regcoil
https://gitlab.com/wistell/StellaratorOptimization.jl