Reduced models of self-force, stored energy, & critical current for design optimization of electromagnets

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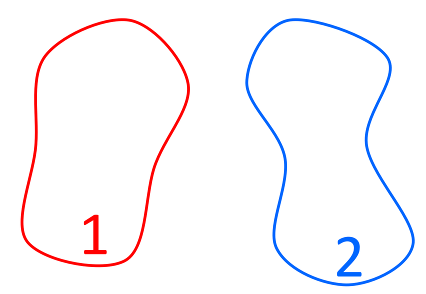
Motivation

Want to rapidly assess & optimize Lorentz forces, charge/discharge times, stored energy, quench limits, etc.

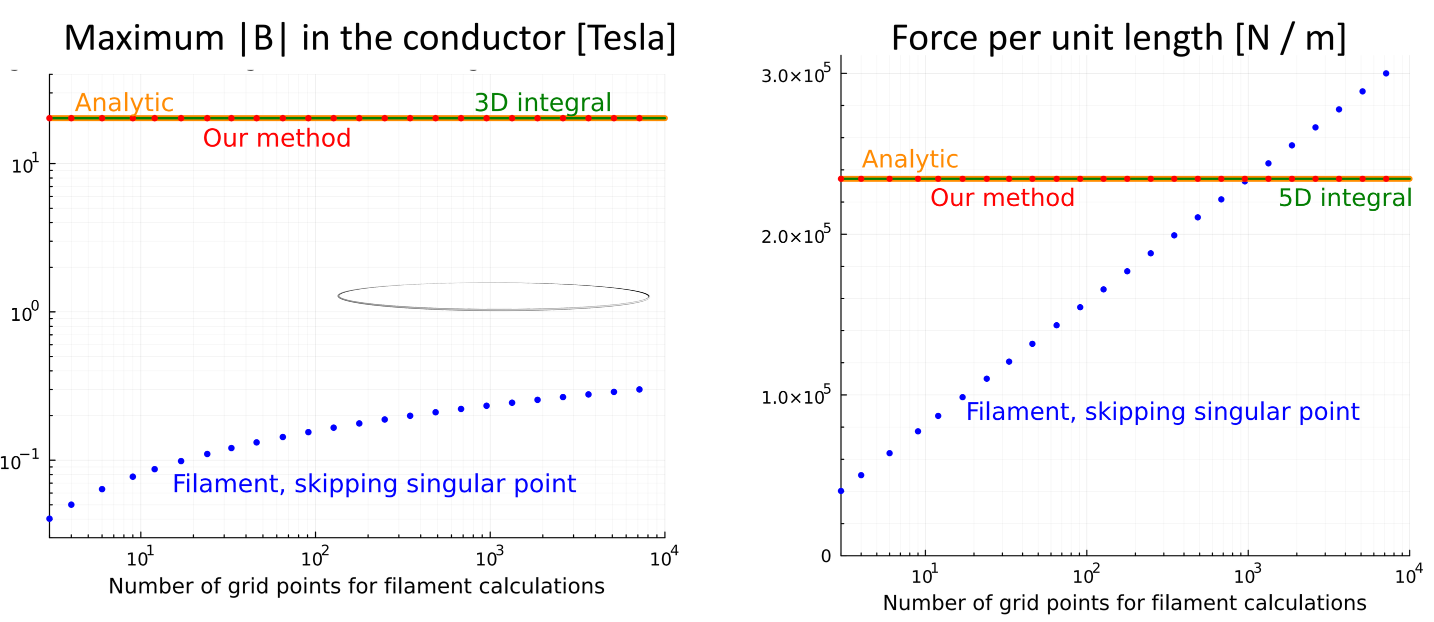
Field and force on coil 2 due to current in coil 1 can be computed quickly: 1D filament models are ok.



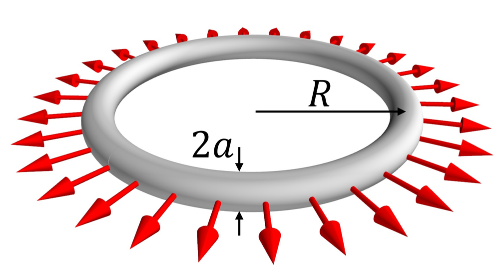
Tricky part is the self-field: singularity in Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int d\tilde{\phi}\,\frac{\frac{d\tilde{\mathbf{r}}}{d\tilde{\phi}}\times(\mathbf{r}-\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

Simply skipping the singular point in a 1D filament calculation gives a non-converging result with significant error:



Also apparent from analytic formulas for a circular coil:



$$\frac{d\boldsymbol{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\left(\frac{8R}{a}\right)-\frac{3}{4}\right]\boldsymbol{e}_R$$

Diverges if minor radius $a \to 0$

Can we avoid high-dimensional integrals or PDE solve?

Field: 3D integral

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\tilde{r}\,\frac{\mathbf{J}(\tilde{\mathbf{r}})\times(\mathbf{r}-\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

Self-inductance & stored energy: 6D integral

$$L = \frac{\mu_0}{4\pi I^2}\int d^3r\int d^3\tilde{r}\,\frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|}$$

High fidelity models

References

Supported by DE-FG02-93ER54197 & the Simons Foundation (560651, TA).

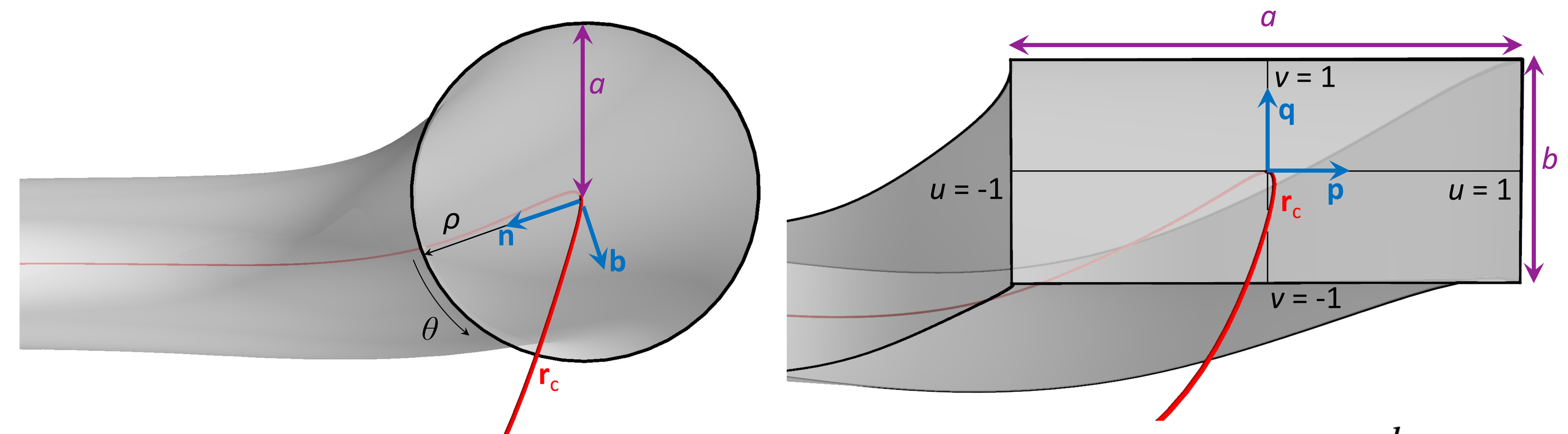
S Hurwitz, M Landreman, T Antonsen, *IEEE Trans. Magnetics* **60**, 1 (2024)

M Landreman, S Hurwitz, T Antonsen, *Nucl. Fusion* **65**, 036008 (2025)

S Hurwitz, M Landreman, et al, *Nucl. Fusion* **65**, 056044 (2025)

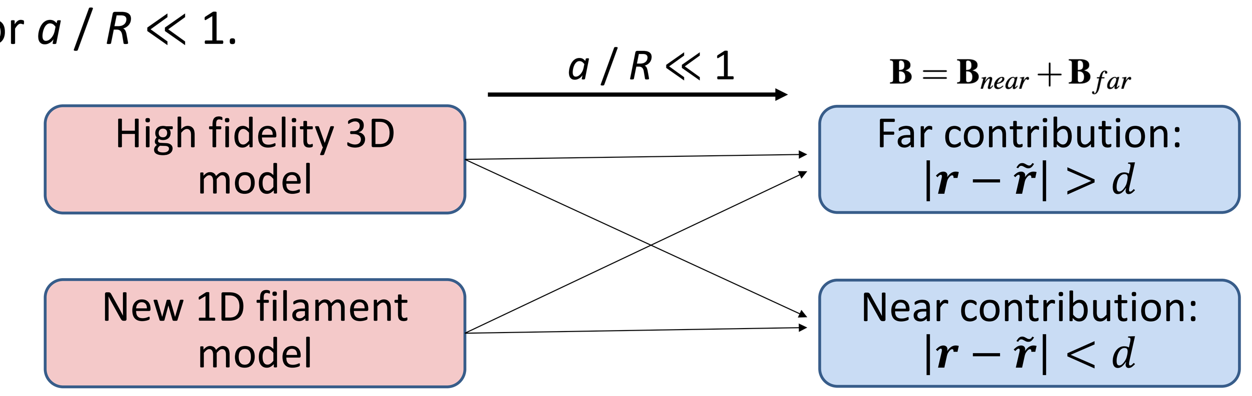
Derivation of reduced models

- Parameterize the conductor volume:



$$\mathbf{r}(\rho,\theta,\phi)=\mathbf{r}_c(\phi)+\rho a\cos\theta\,\mathbf{n}(\phi)+\rho a\sin\theta\,\mathbf{b}(\phi)\qquad \mathbf{r}(u,v,\phi)=\mathbf{r}_c(\phi)+\frac{au}{2}\mathbf{p}(\phi)+\frac{bv}{2}\mathbf{q}(\phi)$$

- Expansion parameter: $a / R \ll 1$, where $R \sim$ scales of curve centerline, and $b \sim a$.
- Introduce intermediate scale d, with $a \ll d \ll R$. Split integrals into "near part" + "far part".
- Far part defined by $|\boldsymbol{r}-\tilde{\boldsymbol{r}}| > d$. Finite cross-section can be neglected.
- Near part defined by $|\boldsymbol{r}-\tilde{\boldsymbol{r}}| < d$. Coil centerline can be Taylor-expanded, so integrals can be done explicitly.
- Identify a 1D integral that has the same near part and far part as the above "high fidelity" calculation for $a / R \ll 1$.



Self-field

$$\mathbf{B} \approx \mathbf{B}_{reg} + \mathbf{B}_0$$

Field of a filament, regularized

$$\mathbf{B}_{reg}(\phi) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi}\,\frac{\tilde{\mathbf{r}}_c'\times(\mathbf{r}_c-\tilde{\mathbf{r}}_c)}{\left(|\mathbf{r}_c-\tilde{\mathbf{r}}_c|^2+\delta ab\right)^{3/2}}$$

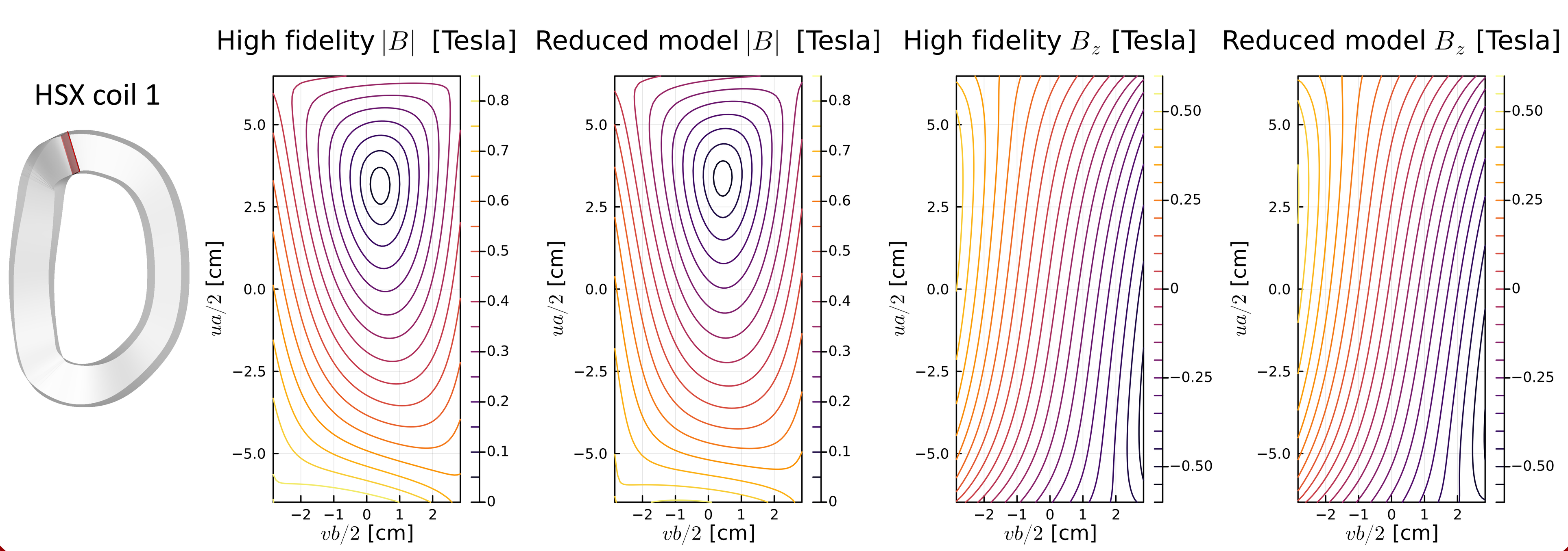
$$\delta = \exp\left(-\frac{25}{6}+k\right)$$

$$k = \frac{4b}{3a}\tan^{-1}\frac{a}{b}+\frac{4a}{3b}\tan^{-1}\frac{b}{a}+\frac{b^2}{6a^2}\ln\frac{b}{a}+\frac{a^2}{6b^2}\ln\frac{a}{b} - \frac{a^4-6a^2b^2+b^4}{6a^2b^2}\ln\left(\frac{a}{b}+\frac{b}{a}\right)$$

Field of infinite straight conductor

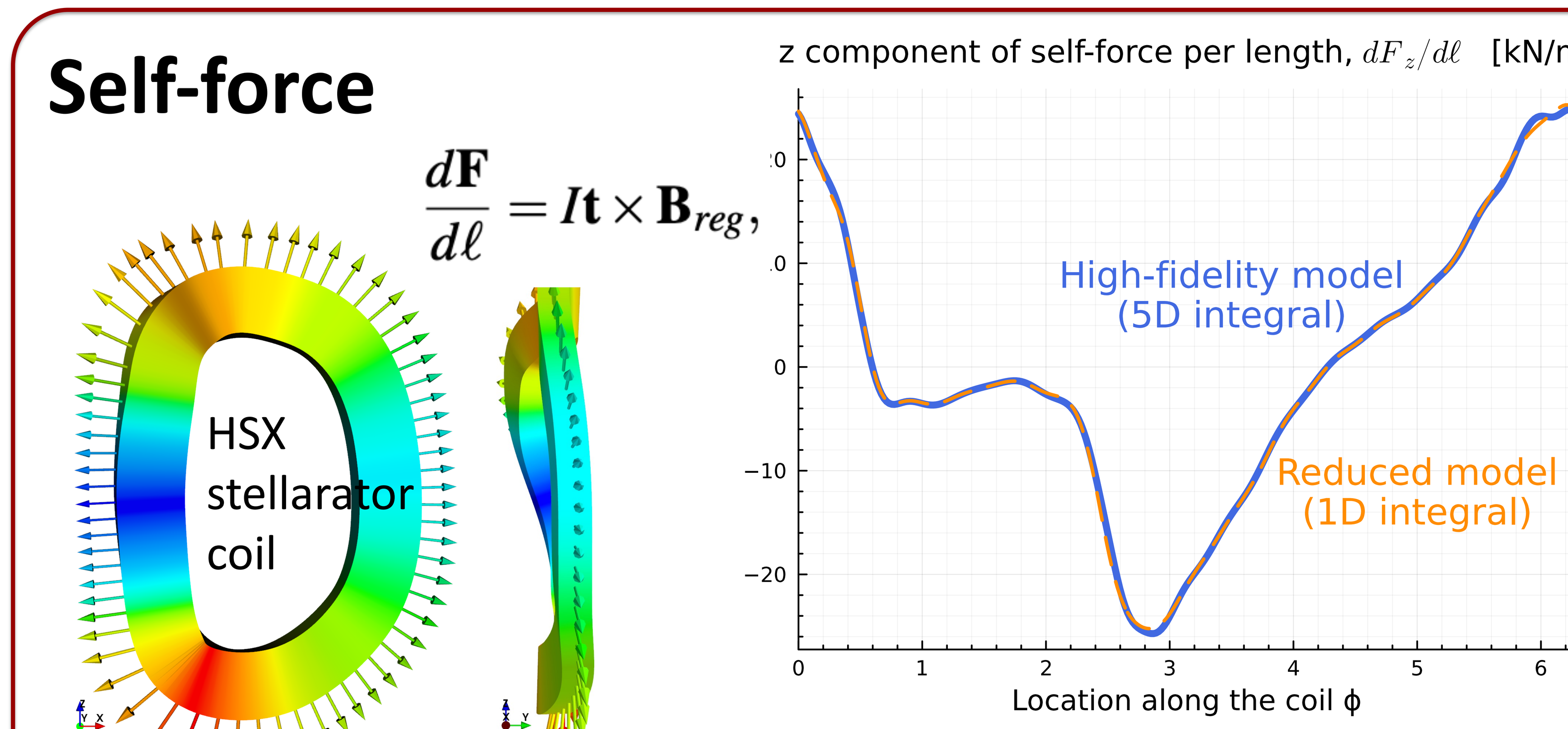
$$\mathbf{B}_0 = \frac{\mu_0 I}{4\pi ab}\sum_{s_u,s_v} s_u s_v\left[G\left(b(v-s_v),\,a(u-s_u)\right)\mathbf{q} - G\left(a(u-s_u),\,b(v-s_v)\right)\mathbf{p}\right]$$

$$G(x,y) = y\tan^{-1}\frac{x}{y}+\frac{x}{2}\ln\left(1+\frac{y^2}{x^2}\right)$$



Self-force

$$\frac{d\mathbf{F}}{d\ell} = I\mathbf{t}\times\mathbf{B}_{reg},$$



~ 18,000x speed-up for same precision

Self-inductance & stored energy

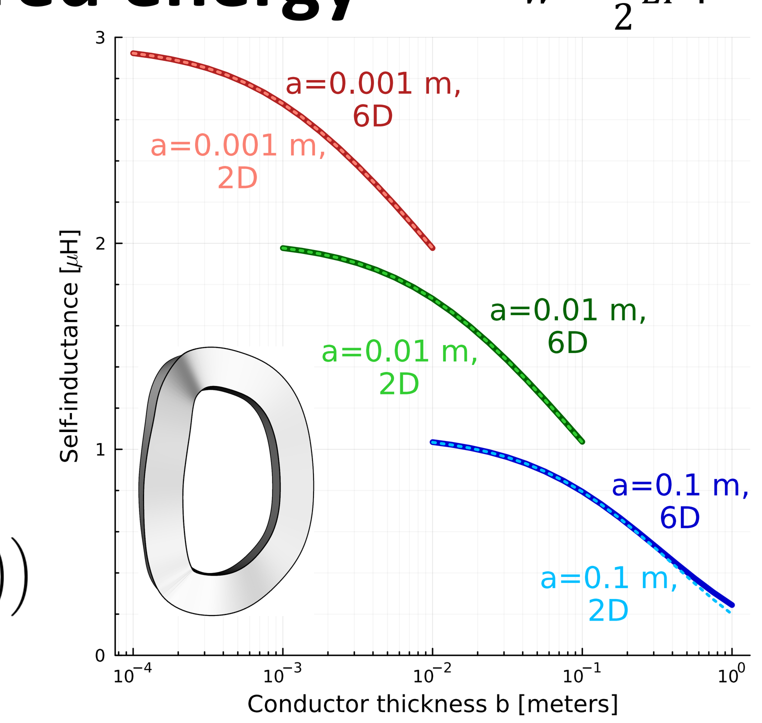
$$W = \frac{1}{2}LI^2.$$

$$L = \frac{\mu_0}{4\pi}\int d\phi\int d\tilde{\phi}\,\frac{1}{\sqrt{|\mathbf{r}_c-\tilde{\mathbf{r}}_c|^2+\Delta}}\,\frac{d\mathbf{r}_c}{d\phi}\cdot\frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}}$$

$\Delta = a^2/\sqrt{e}$ for circular x-section,

$$\Delta = ab\exp\left(-\frac{25}{6}+\frac{4b}{3a}\tan^{-1}\frac{a}{b}+\frac{4a}{3b}\tan^{-1}\frac{b}{a} + \frac{b^2}{6a^2}\ln\frac{b}{a}+\frac{a^2}{6b^2}\ln\frac{a}{b}-\frac{a^4-6a^2b^2+b^4}{6a^2b^2}\ln\left(\frac{a}{b}+\frac{b}{a}\right)\right)$$

for rectangular x-section,



Critical current

Given a model for how the local critical current density depends on $\boldsymbol{B}$, e.g.

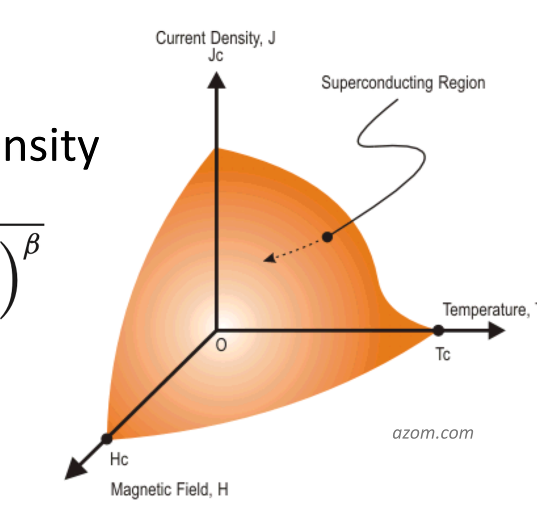
$$j_c(x,y) = \frac{j_{c0}}{\left(1+\frac{\sqrt{k^2B_x^2(x,y)+B_y^2(x,y)}}{B_0}\right)^{\beta}}$$

Gömöry and Klinčok (2006)

estimate the global critical current as

$$I_c = \min_{\phi}\int_{x-section} d^2a\; j_c(\boldsymbol{B}(u,v,\phi,I))$$

Solve for I such that $I_c = I$.



Optimization

Differentiate through these expressions using auto-differentiation for derivative-based optimization