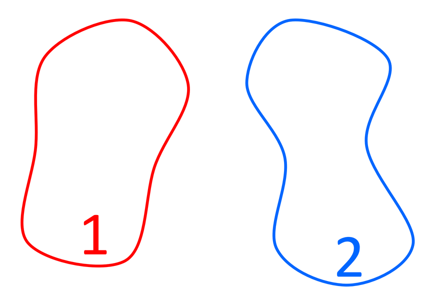
Reduced models of self-force, stored energy, & critical current for design optimization of electromagnets

Matt Landreman, Siena Hurwitz, Tom Antonsen

Motivation

- Forces ∝ B^2. High B limited by support structure.
- Superconductor quench limits depend on local **B**.
- Need to be able to dissipate stored energy $W = ½LI^2$.
- Coil shapes can probably be optimized for these quantities.

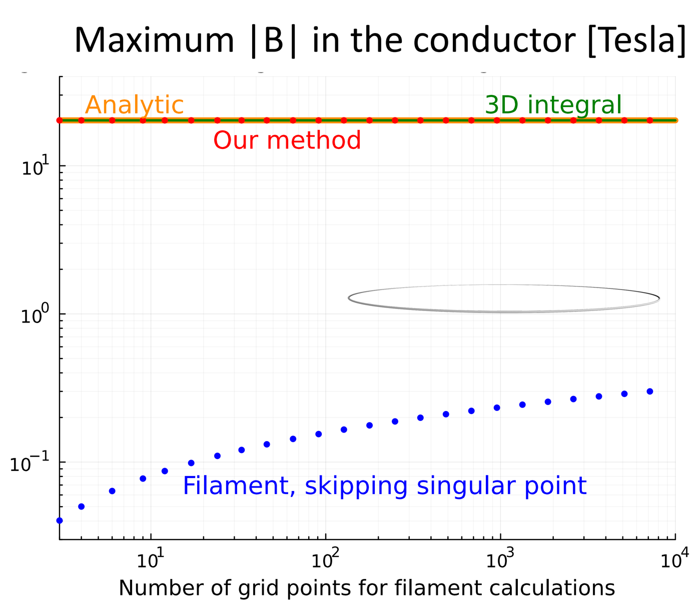
Field and force on coil 2 due to current in coil 1 can be computed quickly: 1D filament models are ok.

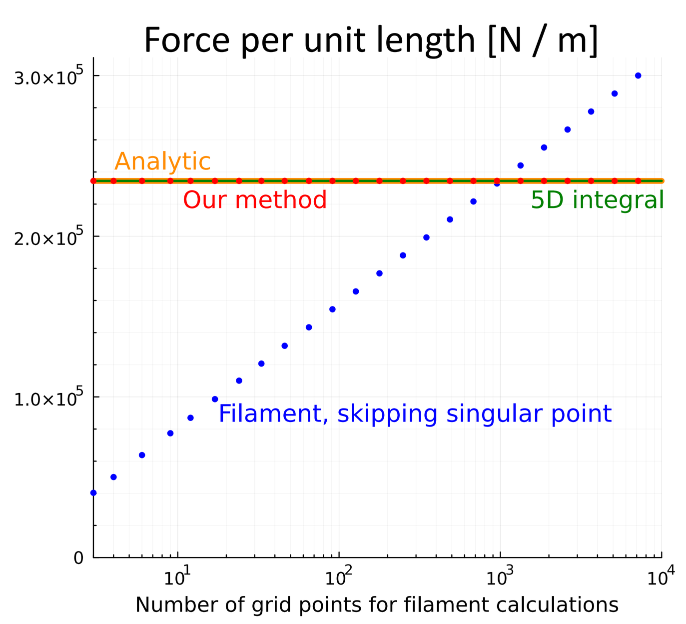


Tricky part is the self-field: singularity in Biot-Savart Law:

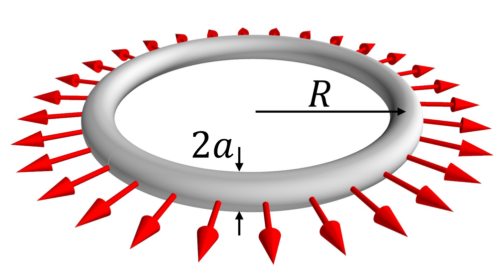
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int d\tilde{\phi}\, \frac{\frac{d\tilde{\mathbf{r}}}{d\tilde{\phi}} \times (\mathbf{r}-\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

Simply skipping the singular point in a 1D filament calculation gives a non-converging result with significant error:





Also apparent from analytic formulas for a circular coil:



$$\frac{d\boldsymbol{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4}\right]\boldsymbol{e}_R$$

Diverges if minor radius $a \to 0$

Can we avoid high-dimensional integrals or PDE solve?

Field: 3D integral

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\tilde{r}\, \frac{\mathbf{J}(\tilde{\mathbf{r}}) \times (\mathbf{r}-\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

Force per unit length: 5D integral

$$\frac{d\mathbf{F}}{d\ell} = \frac{\mu_0}{4\pi}\frac{d\phi}{d\ell}\int dx \int dy \int d^3\tilde{r}\sqrt{g}\, \frac{\mathbf{J}(\mathbf{r}) \times [\mathbf{J}(\tilde{\mathbf{r}}) \times (\mathbf{r}-\tilde{\mathbf{r}})]}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

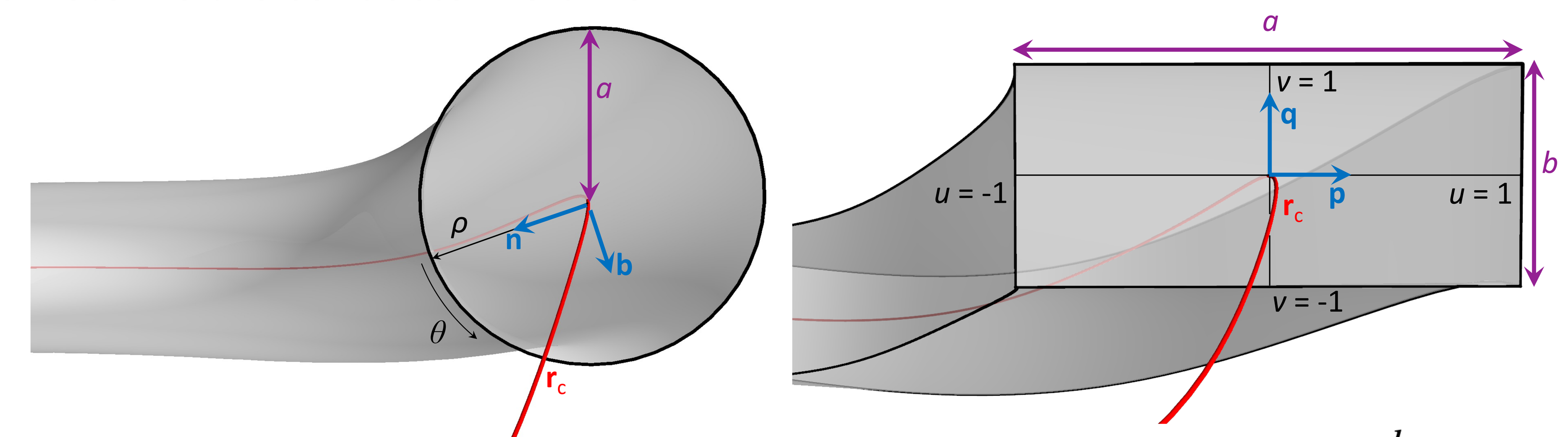
Self-inductance & stored energy: 6D integral

$$L = \frac{\mu_0}{4\pi I^2}\int d^3 r \int d^3\tilde{r}\, \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|}$$

High fidelity model

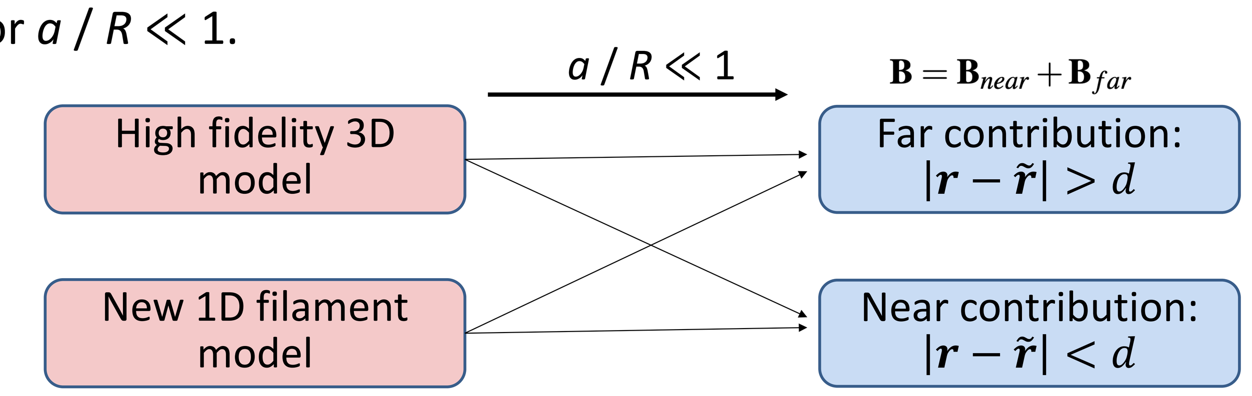
Derivation of reduced models

- Parameterize the conductor volume:



$$\mathbf{r}(\rho,\theta,\phi) = \mathbf{r}_c(\phi) + \rho a\cos\theta\,\mathbf{n}(\phi) + \rho a \sin\theta\,\mathbf{b}(\phi) \qquad \mathbf{r}(u,v,\phi) = \mathbf{r}_c(\phi) + \frac{au}{2}\mathbf{p}(\phi) + \frac{bv}{2}\mathbf{q}(\phi)$$

- Expansion parameter: $a / R \ll 1$, where $R \sim$ scales of curve centerline, and $b \sim a$.
- Introduce intermediate scale d, with $a \ll d \ll R$. Split integrals into "near part" + "far part".
- Far part defined by $|\boldsymbol{r} - \tilde{\boldsymbol{r}}| > d$. Finite cross-section can be neglected.
- Near part defined by $|\boldsymbol{r} - \tilde{\boldsymbol{r}}| < d$. Coil centerline can be Taylor-expanded, so integrals can be done explicitly.
- Identify a 1D integral that has the same near part and far part as the above "high fidelity" calculation for $a / R \ll 1$.



Self-field

$$\mathbf{B} = \mathbf{B}_{reg} + \mathbf{B}_0 + \mathbf{B}_\kappa + \mathbf{B}_b,$$

$$\mathbf{B}_{reg}(\phi) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi}\, \frac{\tilde{\mathbf{r}}_c' \times (\mathbf{r}_c - \tilde{\mathbf{r}}_c)}{\left(|\mathbf{r}_c - \tilde{\mathbf{r}}_c|^2 + \delta a b\right)^{3/2}}$$

$$\delta = \exp\left(-\frac{25}{6} + k\right)$$

$$k = \frac{4b}{3a}\tan^{-1}\frac{a}{b} + \frac{4a}{3b}\tan^{-1}\frac{b}{a} + \frac{b^2}{6a^2}\ln\frac{b}{a} + \frac{a^2}{6b^2}\ln\frac{a}{b} - \frac{a^4 - 6a^2b^2 + b^4}{6a^2b^2}\ln\left(\frac{a}{b} + \frac{b}{a}\right)$$

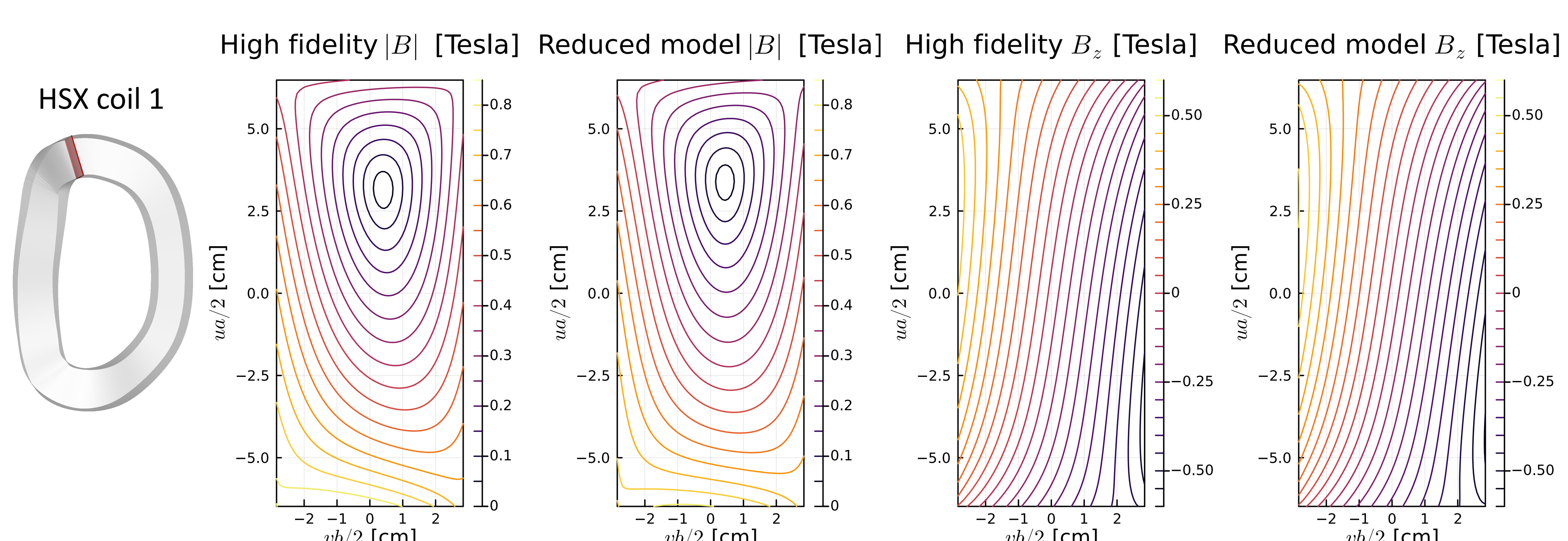
$$\mathbf{B}_b = \frac{\mu_0 I \kappa \mathbf{b}}{8\pi}\left(4 + 2\ln 2 + \ln\delta\right)$$

$$\mathbf{B}_0 = \frac{\mu_0 I}{4\pi a b}\sum_{s_u, s_v} s_u s_v \left[G\left(b(v - s_v),\, a(u - s_u)\right)\mathbf{q} - G\left(a(u - s_u),\, b(v - s_v)\right)\mathbf{p}\right]$$

$$G(x,y) = y\tan^{-1}\frac{x}{y} + \frac{x}{2}\ln\left(1 + \frac{y^2}{x^2}\right)$$

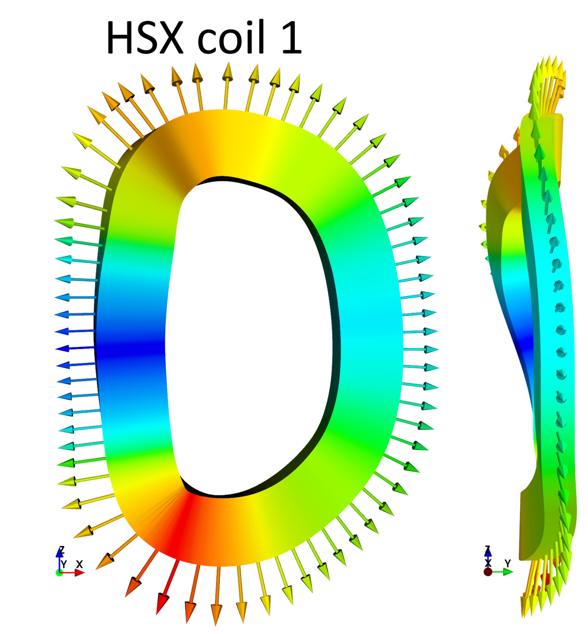
$$\mathbf{B}_\kappa = \frac{\mu_0 I}{64\pi}\sum_{s_u, s_v} s_u s_v \mathbf{K}\left(u - s_u,\, v - s_v\right),$$

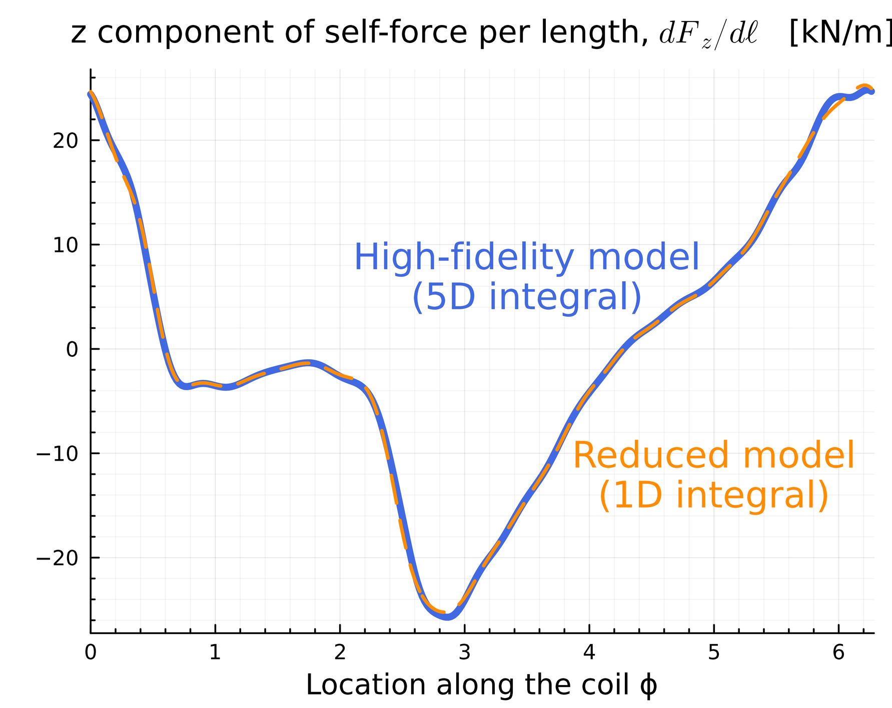
$$\mathbf{K}(U, V) = -2UV\left(\kappa_1\mathbf{q} - \kappa_2\mathbf{p}\right)\ln\left(\frac{aU^2}{b} + \frac{bV^2}{a}\right) + \left(\kappa_2\mathbf{q} - \kappa_1\mathbf{p}\right)\left(\frac{aU^2}{b} + \frac{bV^2}{a}\right)\ln\left(\frac{aU^2}{b} + \frac{bV^2}{a}\right) + \frac{4aU^2\kappa_2\mathbf{p}}{b}\tan^{-1}\frac{bV}{aU} - \frac{4bV^2\kappa_1\mathbf{q}}{a}\tan^{-1}\frac{aU}{bV},$$



Self-force

$$\frac{d\mathbf{F}}{d\ell} = I\mathbf{t} \times \mathbf{B}_{reg},$$





~ 18,000x speed-up for same precision

Self-inductance & stored energy

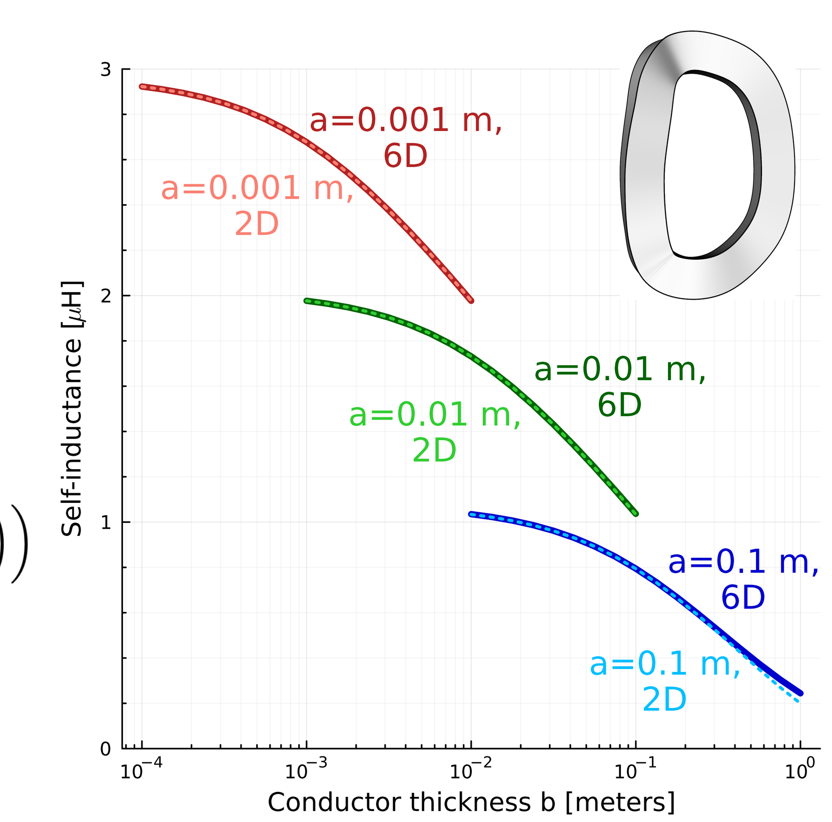
$$L = \frac{\mu_0}{4\pi}\int d\phi \int d\tilde{\phi}\, \frac{1}{\sqrt{|\mathbf{r}_c - \tilde{\mathbf{r}}_c|^2 + \Delta}}\frac{d\mathbf{r}_c}{d\phi} \cdot \frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}}$$

$\Delta = a^2/\sqrt{e}$ for circular x-section,

$$\Delta = ab\exp\left(-\frac{25}{6} + \frac{4b}{3a}\tan^{-1}\frac{a}{b} + \frac{4a}{3b}\tan^{-1}\frac{b}{a} + \frac{b^2}{6a^2}\ln\frac{b}{a} + \frac{a^2}{6b^2}\ln\frac{a}{b} - \frac{a^4 - 6a^2b^2 + b^4}{6a^2b^2}\ln\left(\frac{a}{b} + \frac{b}{a}\right)\right)$$

for rectangular x-section,

$$W = \frac{1}{2}LI^2.$$

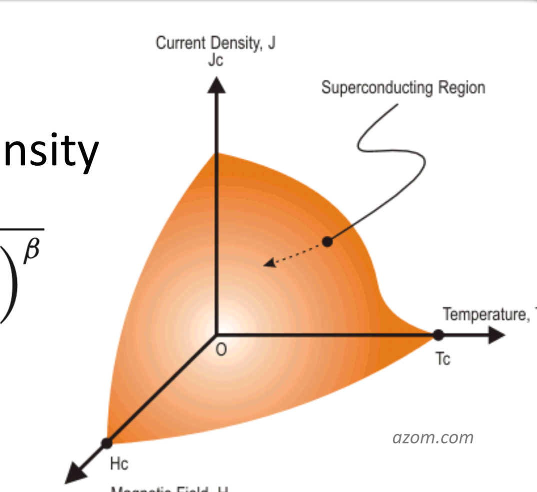


Critical current

Given a model for how the local critical current density depends on $\boldsymbol{B}$, e.g.

$$j_c(x,y) = \frac{j_{c0}}{\left(1 + \frac{\sqrt{k^2 B_x^2(x,y) + B_y^2(x,y)}}{B_0}\right)^\beta}$$

Gömöry and Klinčok (2006)



estimate the global critical current as

$$I_c = \min_\phi \int_{x-section} d^2a\; j_c\big(\boldsymbol{B}(u,v,\phi,I)\big)$$

Solve for I such that $I_c = I$.

Not self-consistent, but maybe good enough for optimization/estimates?

References

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S Hurwitz, Poster GP11.00098 (Tuesday morning)

S Hurwitz, M Landreman , & T Antonsen, arXiv:2310.09313

M Landreman, S Hurwitz, & T Antonsen, arXiv:2310.12087