Optimized stellarators without optimization

Direct construction of stellarator shapes with good confinement

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Overview

In quasisymmetric designs to date (HSX, NCSX, etc), optimization has been done using "textbook" optimization algorithms to minimize symmetry-breaking Fourier modes in *B*. It works, but:

- Many local minima, so result depends on initial guess.
- Never sure you've found all the interesting regions of parameter space.
- Little insight as to the amount of freedom in the solution.

For a complementary approach without these shortcomings, here we extend work of Garren & Boozer [1, 2].

- Usually cited as a proof that quasisymmetry cannot be achieved to $(a/R)^3$.
- Less well known that it contains a useful constructive procedure.
- Provides insight & initial conditions for stellopt.
- Based on expansion in distance from the axis; valid in the core of low-*A* stellarators.

With this approach we can now generate quasisymmetric stellarator shapes in < 1 ms on a laptop!

Garren-Boozer Expansion [1, 2]

<u>Setup:</u> Frenet frame $(\mathbf{t},\mathbf{n},\mathbf{b})$ for the magnetic axis:

$$\frac{\partial \mathbf{r}_0}{\partial \ell}=\mathbf{t},\quad \frac{d\mathbf{t}}{d\ell}=\kappa\mathbf{n},\quad \frac{d\mathbf{n}}{d\ell}=-\kappa\mathbf{t}+\tau\mathbf{b},\quad \frac{d\mathbf{b}}{d\ell}=-\tau\mathbf{n}$$

$\mathbf{r}_0$ = magnetic axis, κ = curvature, τ = torsion

$\mathbf{t}$ = tangent, $\mathbf{n}$ = normal, $\mathbf{b}$ = binormal, ℓ = arclength

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)$$

$$\mathbf{B}=\beta\nabla\psi+I(r)\nabla\theta+G(r)\nabla\zeta=\nabla\psi\times\nabla\theta+\iota\nabla\zeta\times\nabla\psi$$

Apply dual relations: $\nabla\psi=(\nabla\psi\cdot\nabla\theta\times\nabla\zeta)\dfrac{\partial\mathbf{r}}{\partial\theta}\times\dfrac{\partial\mathbf{r}}{\partial\zeta}$ & cyclic permutations.

Expand in $r\propto\sqrt{\psi}$: $\iota(r)=\iota_0+O(r^2)$, $G(r)=G_0+O(r^2)$, $I(r)=r^2 I_2+O(r^4)$

$X(r,\theta,\zeta)=r\left[X_{1s}(\zeta)\sin\theta+X_{1c}(\zeta)\cos\theta\right]+O(r^2)$. Same for Y, Z.

<u>Results, to O(r^1):</u>

$$\boxed{\frac{d\sigma}{d\zeta}+\iota_0\left[\frac{\bar\eta^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar\eta^2}{\kappa^2}\left[I_2-\tau\right]=0 \qquad (1)}$$

I_2 = current density, ι_0 = rotational transform, $\bar\eta$ = some constant

Flux surface shape:

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)$$

$$X_{1c}(\zeta)=\bar\eta/\kappa(\zeta) \qquad Y_{1c}(\zeta)=\sigma(\zeta)\kappa(\zeta)/\bar\eta$$

$$Y_{1s}(\zeta)=\kappa(\zeta)/\bar\eta$$

Cylindrical Coordinates

We repeated the analysis without the Frenet frame, for cases in which the axis curvature vanishes, e.g. omnigenity with poloidally closed B contours.

Results in cylindrical coordinates (R,ϕ,z):

$$R(r,\theta,\phi)=R_0(\phi)+r\left[R_{1s}(\phi)\sin\theta+R_{1c}(\phi)\cos\theta\right]+O(r^2)$$

$$z(r,\theta,\phi)=z_0(\phi)+r\left[z_{1s}(\phi)\sin\theta+z_{1c}(\phi)\cos\theta\right]+O(r^2)$$

$$R_{1s}z_{1c}-R_{1c}z_{1s}=\frac{\bar B\ell'}{R_0 B_0} \qquad {}'=d/d\phi$$

$$\frac{B_{1s,c}}{B_0}=\left(\kappa\mathbf{n}\cdot\mathbf{e}_R+\frac{R_0'B_0'}{\ell'^2}\right)R_{1s,c}+\left(\kappa\mathbf{n}\cdot\mathbf{e}_z+\frac{z_0'B_0'}{\ell'^2}\right)z_{1s,c}$$

$$\iota_0=T/V \qquad (2)$$

$$\begin{aligned}T=\frac{|G_0|}{\ell'^3 B_0}\Big[&R_0^2\left(R_{1c}R_{1s}'-R_{1s}'R_{1c}'+z_{1c}z_{1s}'-z_{1s}z_{1c}'\right)\\&+\left(R_{1c}z_{1s}-R_{1s}z_{1c}\right)\left(R_0'z_0''+2R_0z_0'-z_0'R_0''\right)\\&+\left(z_{1c}z_{1s}'-z_{1s}z_{1c}'\right)R_0'^2+\left(R_{1c}R_{1s}'-R_{1s}R_{1c}'\right)z_0'^2\\&+\left(R_{1s}z_{1c}'-z_{1c}R_{1s}'+z_{1s}R_{1c}'-R_{1c}z_{1s}'\right)R_0'z_0'\Big]+\frac{2G_0I_2}{B_0^2}\end{aligned}$$

$$\begin{aligned}V=\frac{1}{\ell'^2}\Big[&R_0^2\left(R_{1c}^2+R_{1s}^2+z_{1c}^2+z_{1s}^2\right)+R_0'^2\left(z_{1s}^2+z_{1c}^2\right)\\&-2R_0'z_0'\left(R_{1c}z_{1c}-R_{1s}z_{1s}\right)+z_0'^2\left(R_{1c}^2+R_{1s}^2\right)\Big]\end{aligned}$$

Equivalent to [1,2]. Eq (2) is equivalent to Mercier's result [5] that ι comes from toroidal current, axis torsion, & rotating elongation, but (2) is valid even if the axis curvature vanishes.

Number of Solutions

To precisely count & parameterize quasisymmetric shapes, we have proved an existence & uniqueness theorem [4]:

Given $P(\zeta)>0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$ 2π-periodic, bounded, and integrable, a solution to

$$\frac{d\sigma}{d\zeta}+\iota\left(P+\sigma^2\right)+Q=0 \qquad (3)$$

is a pair $\{\iota,\ \sigma(\zeta)\}$ solving (3) where $\sigma(\zeta)$ is 2π-periodic.

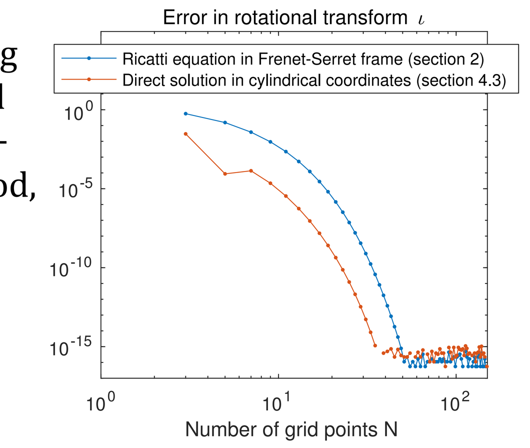
<u>Theorem:</u> A solution exists and it is unique.

We can now precisely state the amount of freedom in 1st-order-in-r quasisymmetry:

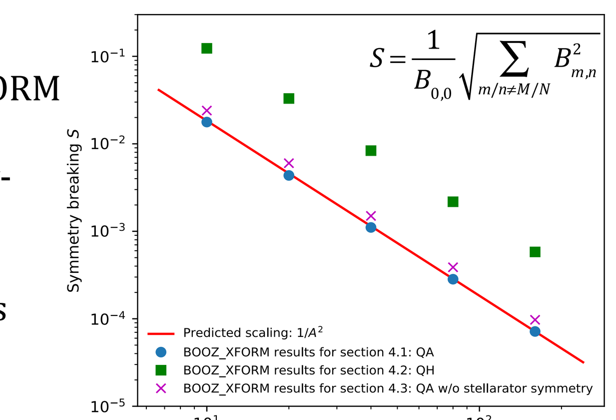
- For every magnetic axis shape (2 functions of ϕ) with nonvanishing curvature, and 3 real numbers ($\bar\eta$, $\sigma(0)$, and I_2), there is precisely 1 way to shape the near-axis surfaces consistent with quasisymmetry. Also,
  - For stellarator symmetry, $\sigma(0)=0$.
  - In the usual case of no current density on axis, then $I_2=0$.
- This solution is quasi-axisymmetric or quasi-helically symmetric depending on whether $\mathbf{n}$ loops around the axis poloidally when you follow the axis toroidally.
- However many of these solutions have absurdly high elongation.

Convergence Tests

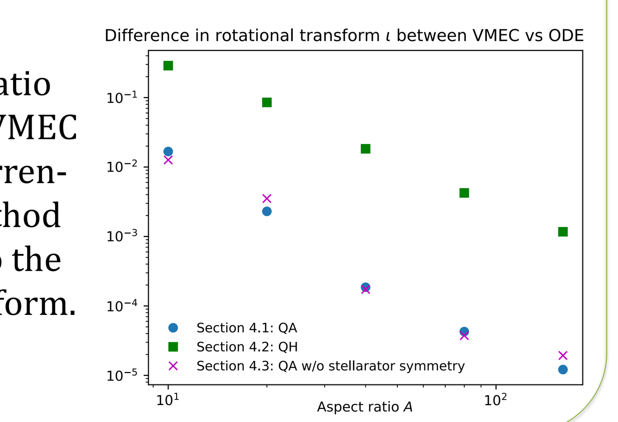
Solving (1) using pseudo-spectral differentiation + Newton's method, we achieve spectral convergence.



VMEC + BOOZ_XFORM shows the symmetry-breaking Boozer harmonics scale as expected.



As aspect ratio increases, VMEC and the Garren-Boozer method converge to the same transform.



Numerical Solutions

<u>Quasi-axisymmetry</u>

Inputs:

axis shape:

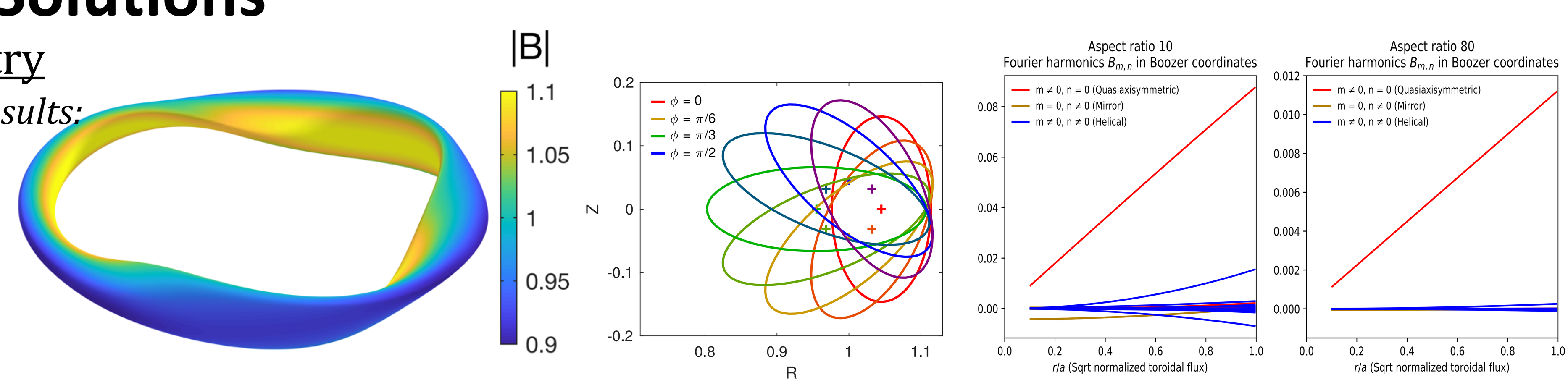
$R_0(\phi)=1+0.045\cos(3\phi)$,

$Z_0(\phi)=-0.045\sin(3\phi)$,

$\bar\eta=-0.9$, $\sigma(0)=0$

$R/a=10$

Results:



<u>Quasi-helical symmetry</u>

Inputs:

axis shape:

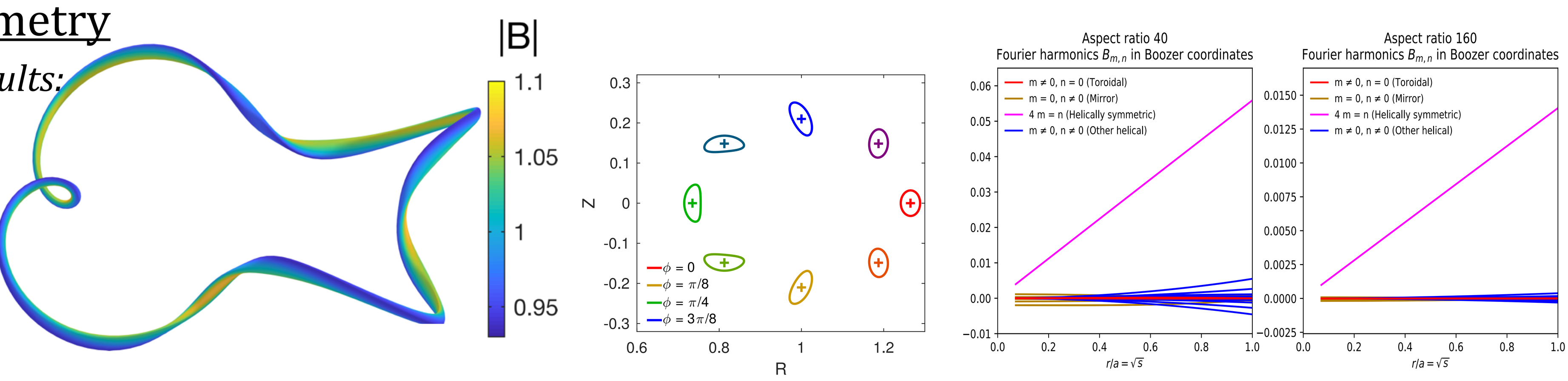
$R_0(\phi)=1+0.265\cos(4\phi)$,

$Z_0(\phi)=-0.21\sin(4\phi)$,

$\bar\eta=-2.25$, $\sigma(0)=0$

$R/a=40$

Results:



<u>Quasi-axisymmetry without stellarator symmetry</u>

May have large intrinsic rotation?

Inputs:

axis shape:

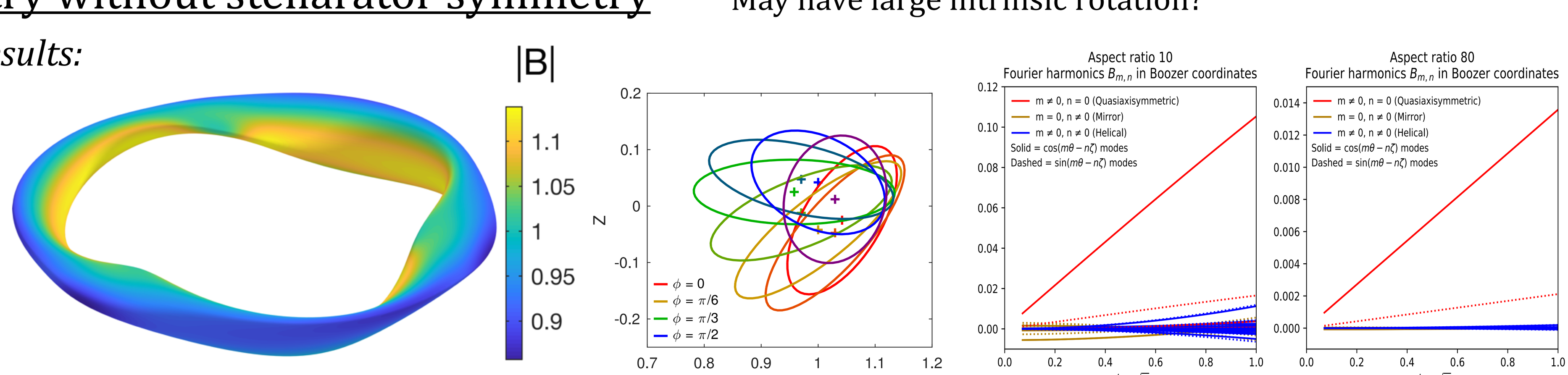
$R_0(\phi)=1+0.042\cos(3\phi)$,

$Z_0(\phi)=-0.042\sin(3\phi)$
$\quad -0.025\cos(3\phi)$,

$\bar\eta=-1.1$, $\sigma(0)=-0.6$

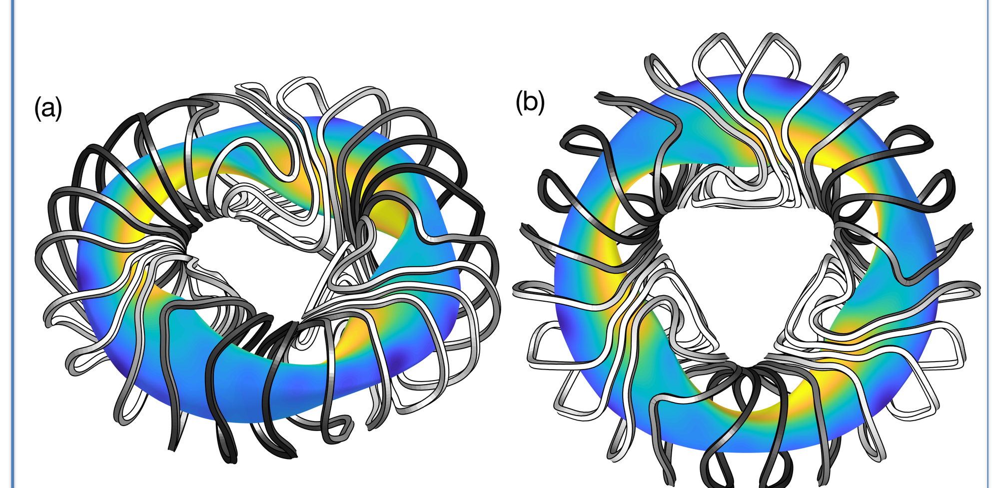
$R/a=10$

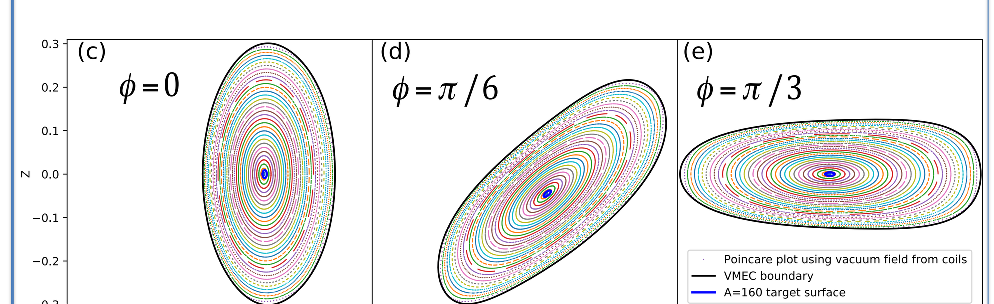
Results:

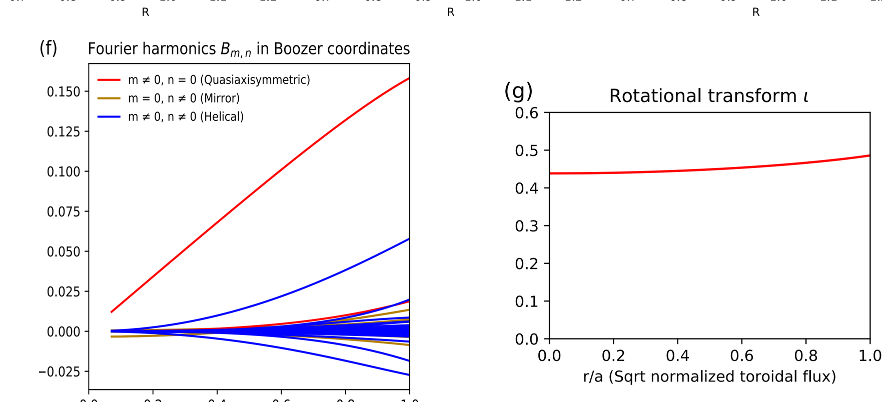


<u>Outward extrapolation using coils</u>

- Using REGCOIL, compute coil shapes that produce the R/a=160 Garren-Boozer QA solution.
- Good flux surfaces exist out to R/a=5







Landscape of Solutions

E.g. Scan over $(R_{0c}, Z_{0s}, \bar\eta)$

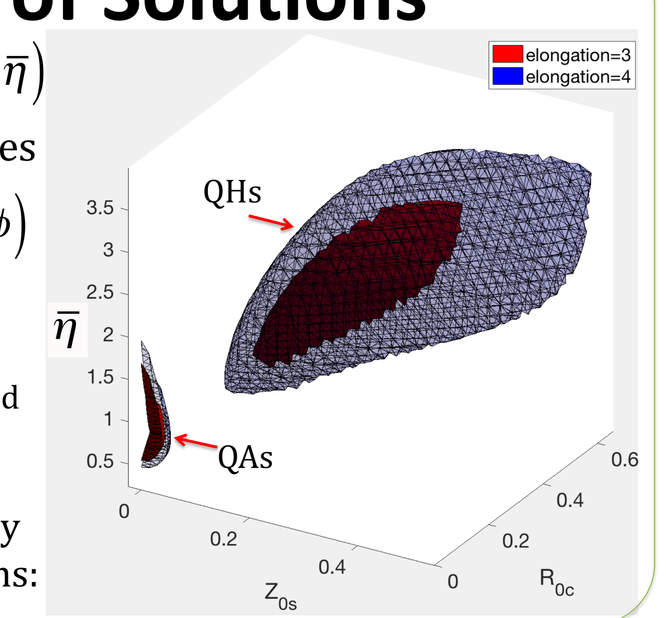
for magnetic axis shapes

$R_0(\phi)=1+R_{0c}\cos(4\phi)$

$Z_0(\phi)=Z_{1s}\sin(4\phi)$

274,560 solutions generated in <30s on a laptop.

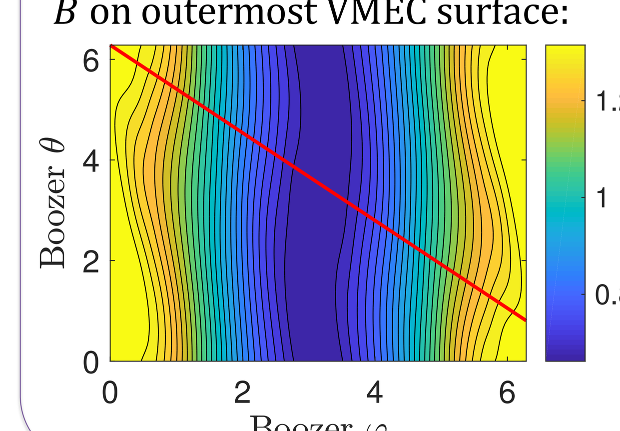
Elongation is reasonably small only in two regions:

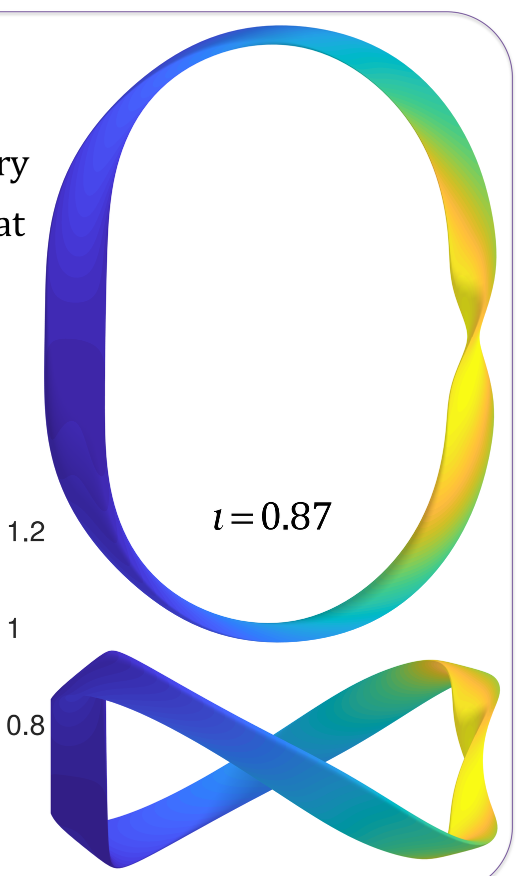


Omnigenity [6]

The same Garren-Boozer machinery can be applied to a $B(\theta,\zeta)$ such that $\oint_{bounce}\mathbf{v}_d\cdot\nabla\psi=0$ even without quasisymmetry.

B on outermost VMEC surface:





Future Directions

- Fully map the landscape of possible 1st-order quasisymmetric shapes by considering more Fourier modes in the axis shape. (How do I plot this?)
- Optimize in the space of axis shapes.
- Can we construct shapes with quasisymmetry imposed at a mid-radius surface?
- Expand about a nearly circular axis.
- 2nd order in distance from the axis.
- Connect to analysis of the difficulty of producing various plasma shapes.

References

[1] D A Garren & A H Boozer, *Phys Fluids B* **3**, 2805 (1991).

[2] D A Garren & A H Boozer, *Phys Fluids B* **3**, 2822 (1991).

[3] M Landreman & W Sengupta, arXiv:1809.10233 (2018).

[4] M Landreman, W Sengupta, & G G Plunk, arXiv:1809.10246 (2018).

[5] C Mercier, *Nuclear Fusion* **4**, 213 (1964).

[6] G G Plunk & M Landreman, *in preparation*.

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