Efficient Calculation of the Self-Inductance, Self-Force, and Internal Magnetic Field for Thin Electromagnetic Coils

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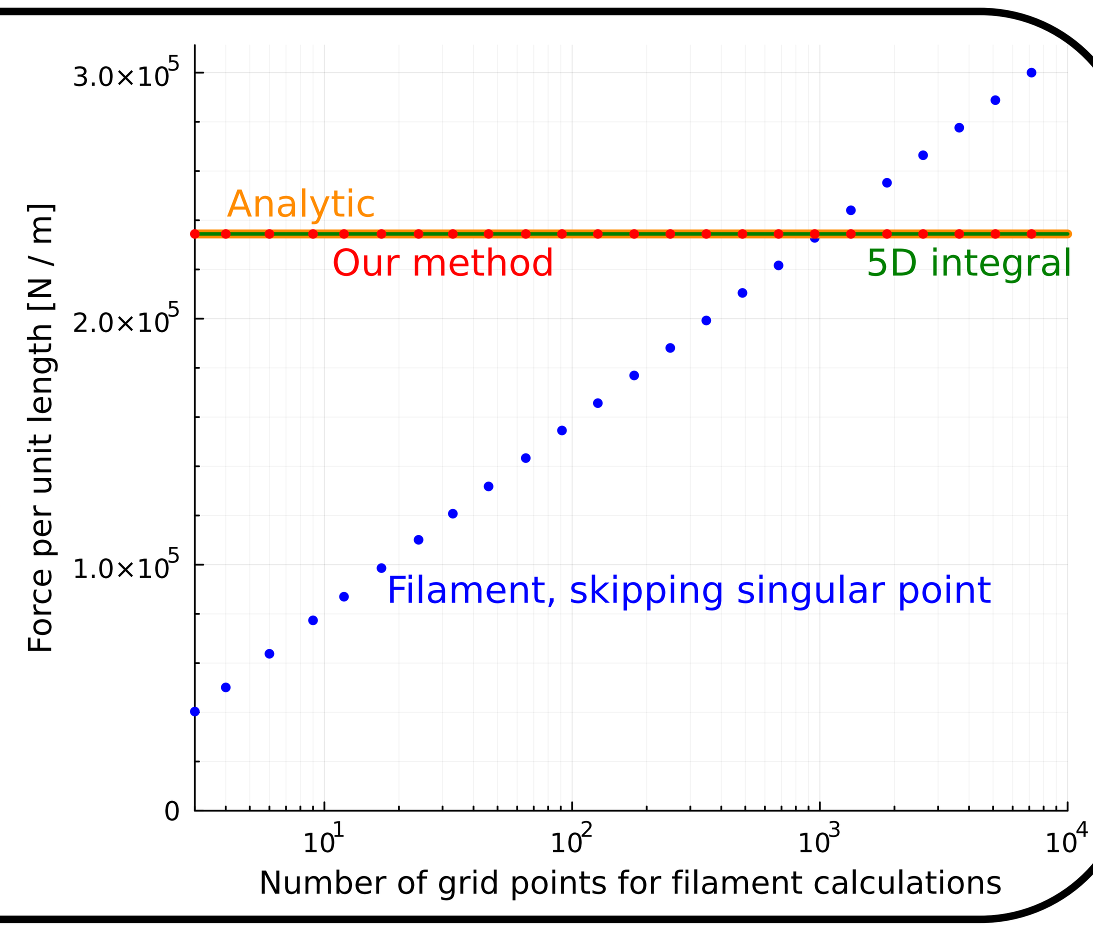
Motivations

- Quantities of high-field coils such as the self-inductance, self-force, and internal magnetic field may be important in coil design, and potential applications include optimization of stellarator magnets for low stresses and high critical current.
- The finite-thickness formulae are inefficient to compute due to high dimensionality of the integrals and a singularity at $\tilde{\mathbf{r}} = \mathbf{r}$, e.g.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\tilde{r}\,\frac{\tilde{\mathbf{J}}\times\Delta\mathbf{r}}{|\Delta\mathbf{r}|^3}. \qquad \Delta\mathbf{r} \equiv \mathbf{r} - \tilde{\mathbf{r}}_c$$

- One cannot take the zero-thickness limit as the results diverge logarithmically. For a circular wire loop with radii R and a,

$$\frac{d\mathbf{F}_\circ}{dl} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\frac{8R}{a} - \frac{3}{4}\right]\hat{\mathbf{r}}$$



Coils, Coordinate System, and Exact Model

We examine closed coils with a circular cross-section of minor radius, a, and uniform current, I. The coils are described by a center-line, $\mathbf{r}_c(\phi)$, where $\phi \in [0, 2\pi)$. The Frenet-Serret unit vectors can then be found for $\mathbf{r}_c$:

$$\mathbf{e}_1 = \mathbf{r}_c'/|\mathbf{r}_c'|,\ \mathbf{e}_2 = \mathbf{e}_1'/|\mathbf{e}_1'|,\ \mathbf{e}_3 = \mathbf{e}_1\times\mathbf{e}_2$$

We form the coordinate system,

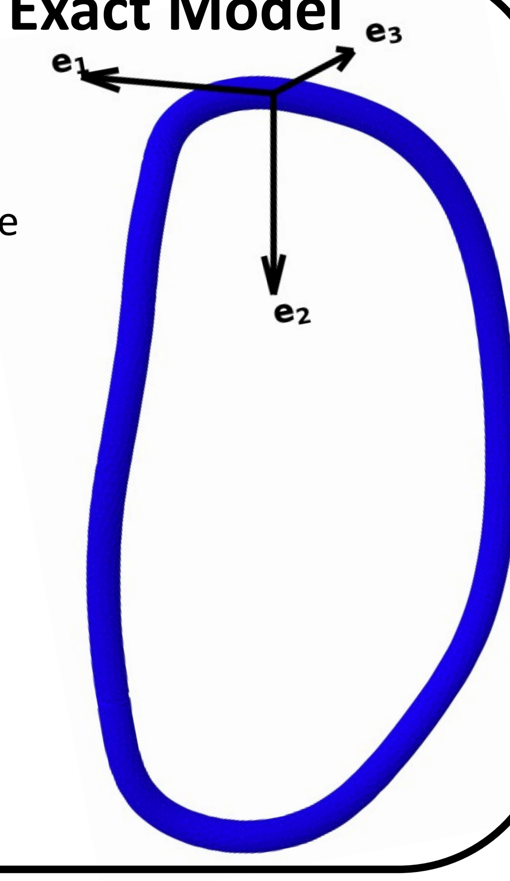
$\mathbf{r}(s,\theta,\phi) = \mathbf{r}_c + s\cos\theta\,\mathbf{e}_2 + s\sin\theta\,\mathbf{e}_3,$

where $\theta \in [0, 2\pi)$ and $s \geq 0$. This gives:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a^2}\int_0^{2\pi} d\tilde{\phi}\int_0^{2\pi} d\tilde{\theta}\int_0^{a} d\tilde{s}\,\tilde{s}(1-\tilde{\kappa}\tilde{s})\frac{\tilde{\mathbf{r}}_c'}{|\Delta\mathbf{r}|}$$

$$L = \frac{1}{\pi a^2 I}\int_0^{2\pi} d\phi\int_0^{2\pi} d\theta\int_0^{a} ds\, s(1-\kappa s)|\mathbf{r}_c'|(\mathbf{e}_1\cdot\mathbf{A})$$

$$\frac{d\mathbf{F}}{dl}(\phi) = \frac{I}{\pi a^2}\int_0^{2\pi} d\theta\int_0^{a} ds\, s(1-\kappa s)(\mathbf{e}_1\times\mathbf{B})$$



Derivation Overview

1. Assume $a/\mathcal{L} \ll 1$, where $\mathcal{L}$ is any length scale of $\mathbf{r}_c$ (e.g., $|\mathbf{r}_c'|, \kappa^{-1}, \tau^{-1}$, etc).
2. Partition the integral over $\tilde{\phi}$ in $\mathbf{A}(\mathbf{r})$ into two sections about ϕ_0, where ϕ_0 satisfies $a/\mathcal{L} \ll \phi_0 \ll 1$.
   - The "near region" ($|\phi - \tilde{\phi}| < \phi_0$) satisfies $|\Delta\phi| \ll 1$.
   - The "far region" ($|\phi - \tilde{\phi}| > \phi_0$) satisfies $a/|\Delta\mathbf{r}| \ll 1$.
3. Solve for $\mathbf{A}(\mathbf{r})$ and simplify under these assumptions.
4. Using this new formula, obtain desire formulae by
   - positing a regularized form of the magnetic field, $\mathbf{B}_{reg}$
   - calculating $\mathbf{B} = \nabla\times\mathbf{A}$, comparing to $\mathbf{B}_{reg}$, and substituting in $\mathbf{B}_{reg}$ where appropriate
   - applying previous two methods to the formulae for L and $d\mathbf{F}/dl$

Self-Magnetic Field

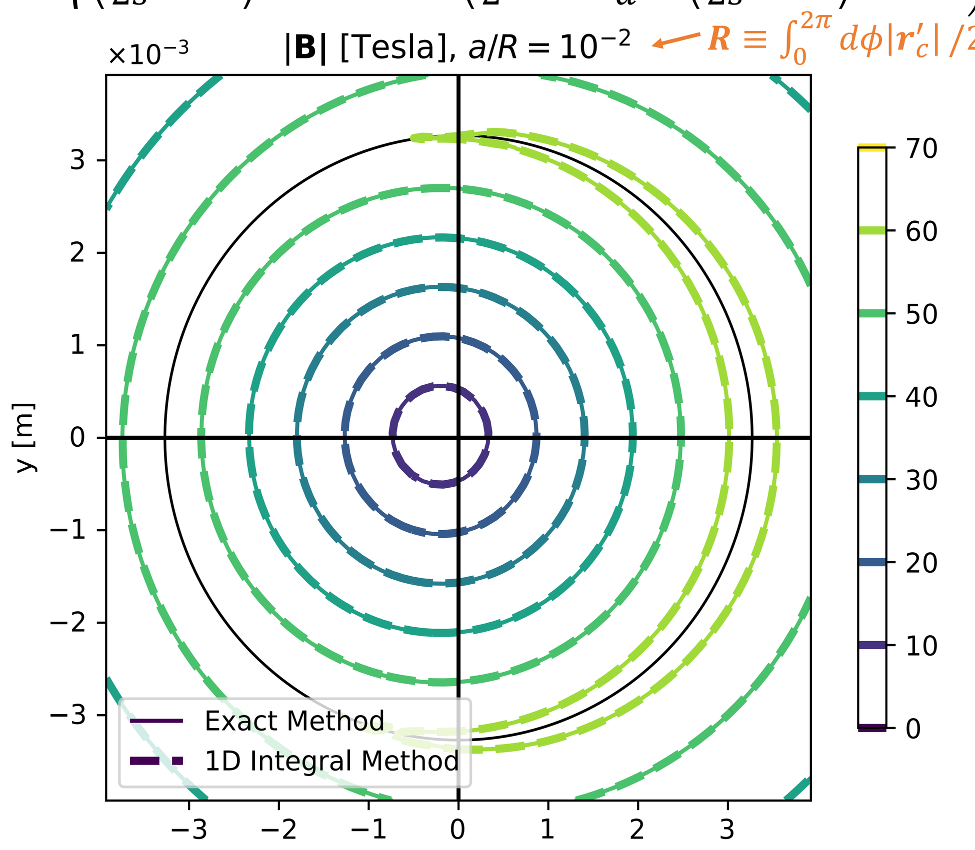
$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_{reg} + \mathbf{B}_{cyl} + \mathbf{B}_{hi},$$

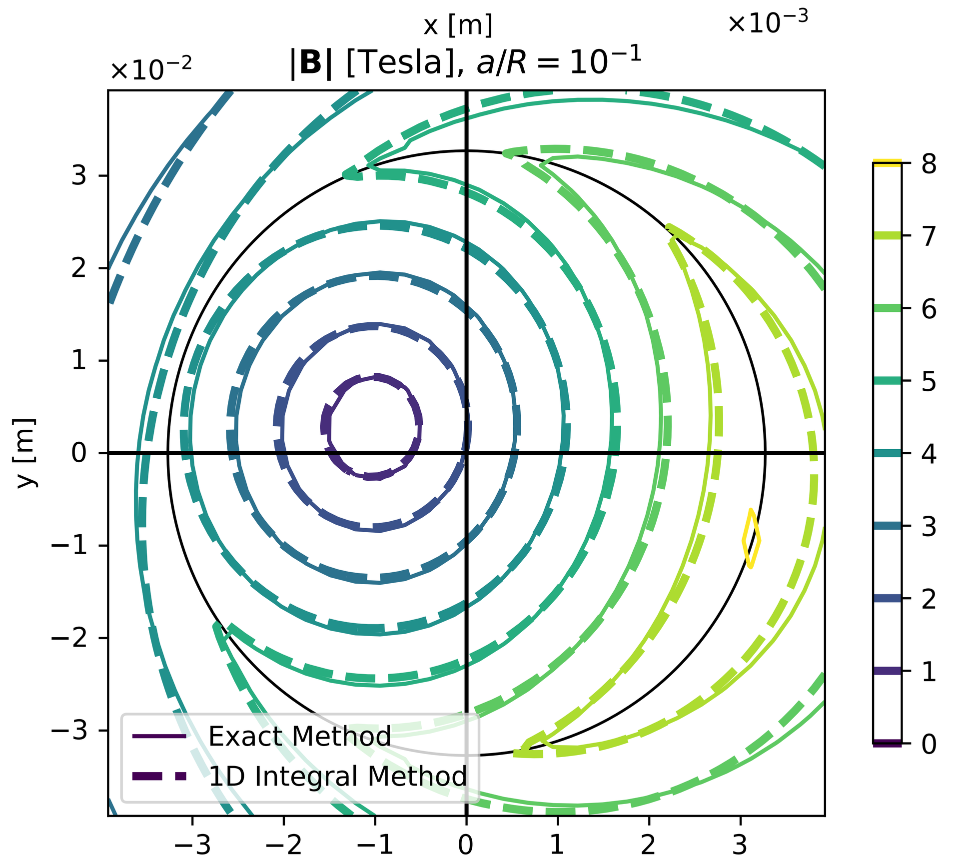
$$\mathbf{B}_{reg} \equiv \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi}\,\frac{\tilde{\mathbf{r}}_c'\times\Delta\mathbf{r}_c}{(|\Delta\mathbf{r}_c|^2 + a^2/\sqrt{e})^{3/2}}$$

$$\mathbf{B}_{cyl} \equiv (-\sin\theta\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3)\frac{\mu_0 I}{2\pi}\begin{cases} s/a^2, & s < a \\ 1/a, & s > a\end{cases}$$

$$\mathbf{B}_{hi} \equiv \frac{\mu_0 I\kappa}{8\pi}\begin{cases} -\frac{s^2\sin 2\theta}{2a^2}\mathbf{e}_2 + \left(\frac{3}{2} + \frac{s^2}{a^2}\left(\frac{\cos 2\theta}{2} - 1\right)\right)\mathbf{e}_3, & s < a \\ \left(\frac{a^2}{2s^2} - 1\right)\sin 2\theta\,\mathbf{e}_2 + \left(\frac{1}{2} - 2\ln\frac{s}{a} - \left(\frac{a^2}{2s^2} - 1\right)\cos 2\theta\right)\mathbf{e}_3, & s > a\end{cases}$$

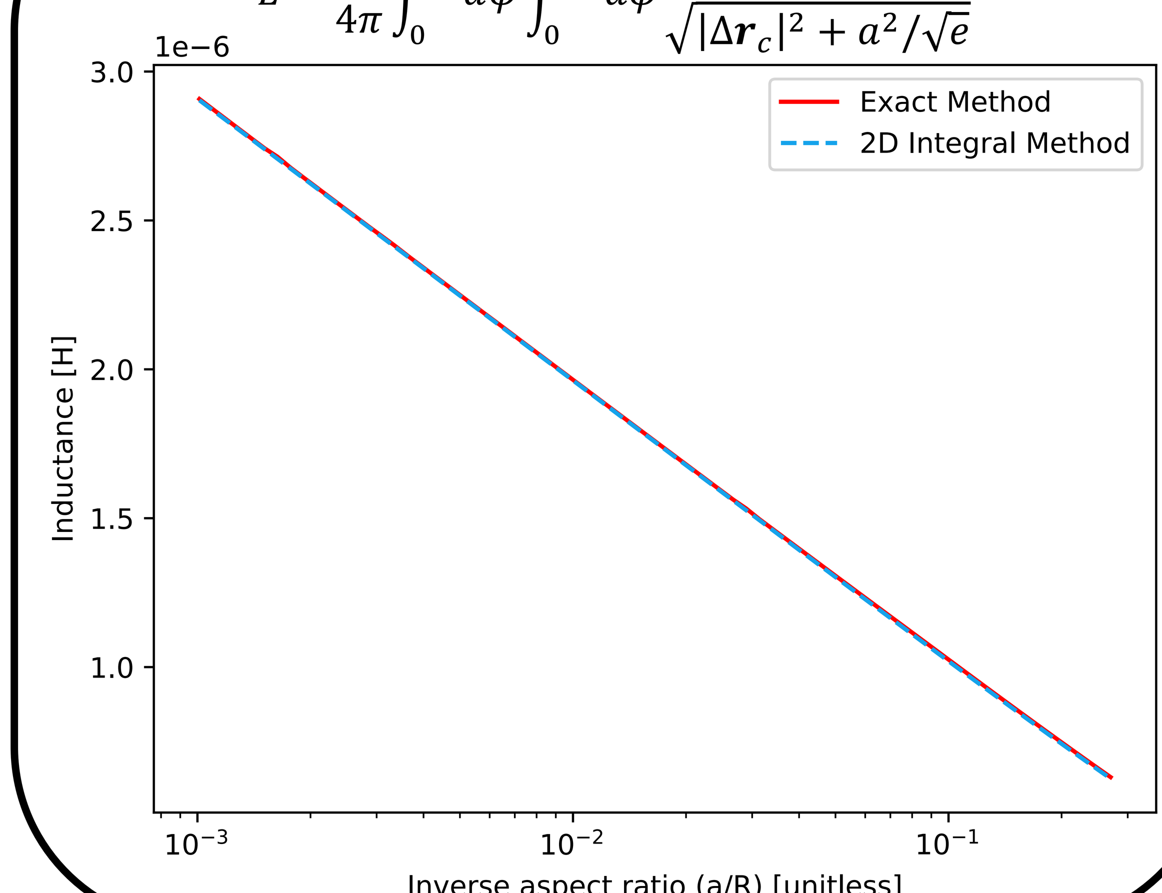
$R \equiv \int_0^{2\pi} d\phi|\mathbf{r}_c'|/2\pi$





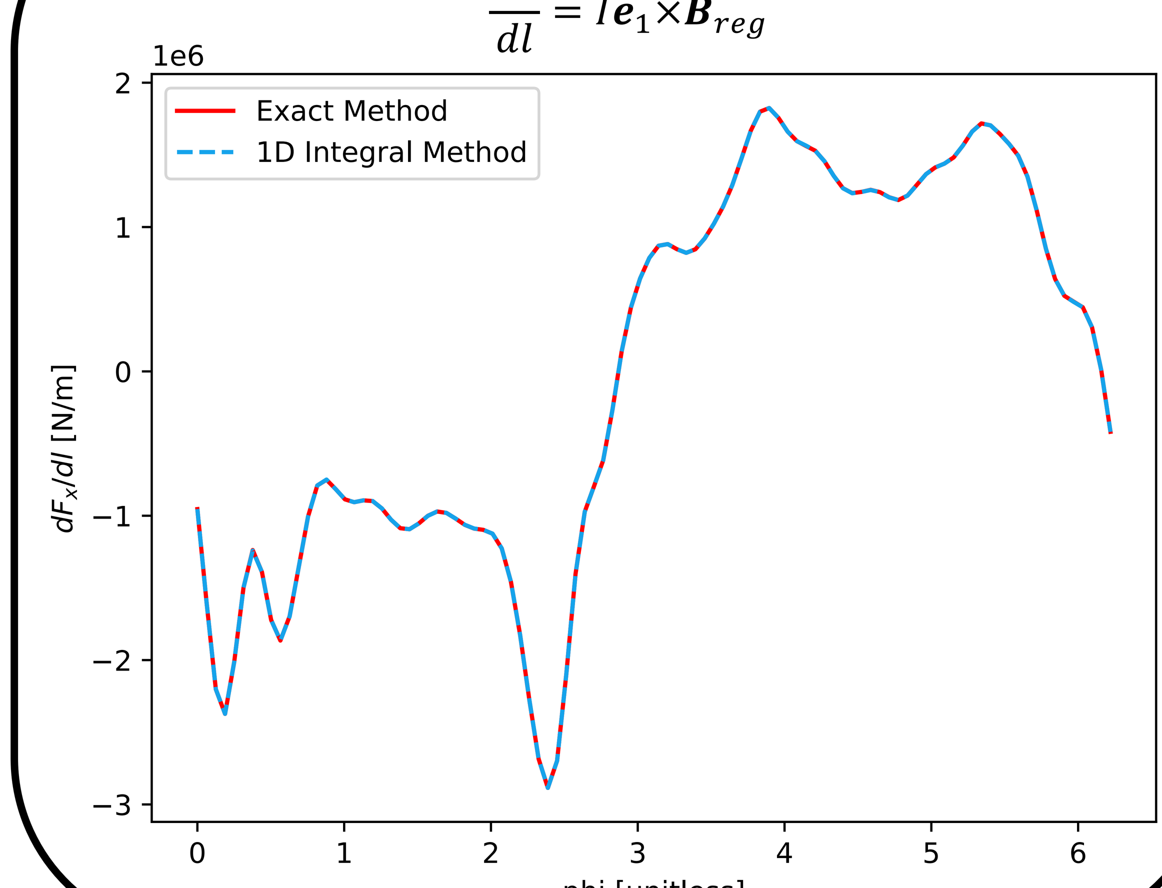
Self-Inductance

$$L = \frac{\mu_0}{4\pi}\int_0^{2\pi} d\tilde{\phi}\int_0^{2\pi} d\phi\,\frac{\mathbf{r}_c'\cdot\tilde{\mathbf{r}}_c'}{\sqrt{|\Delta\mathbf{r}_c|^2 + a^2/\sqrt{e}}}$$



Self-Force

$$\frac{d\mathbf{F}}{dl} = I\mathbf{e}_1\times\mathbf{B}_{reg}$$



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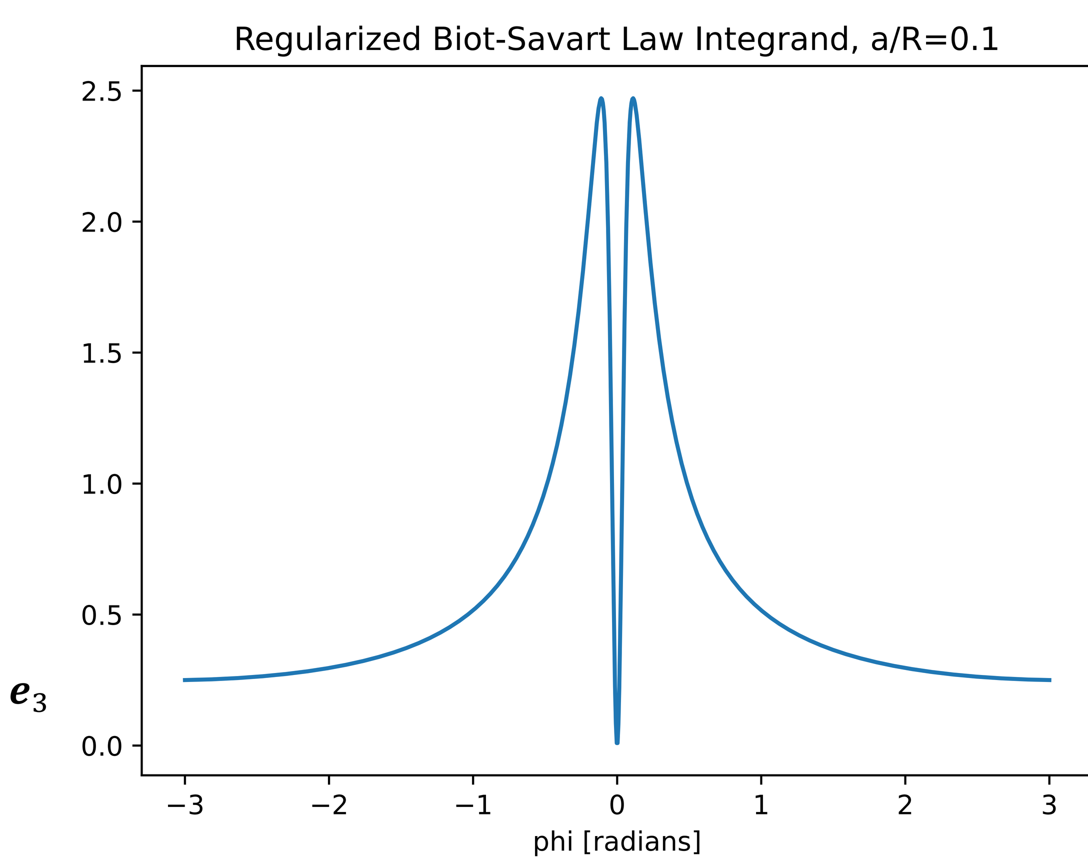
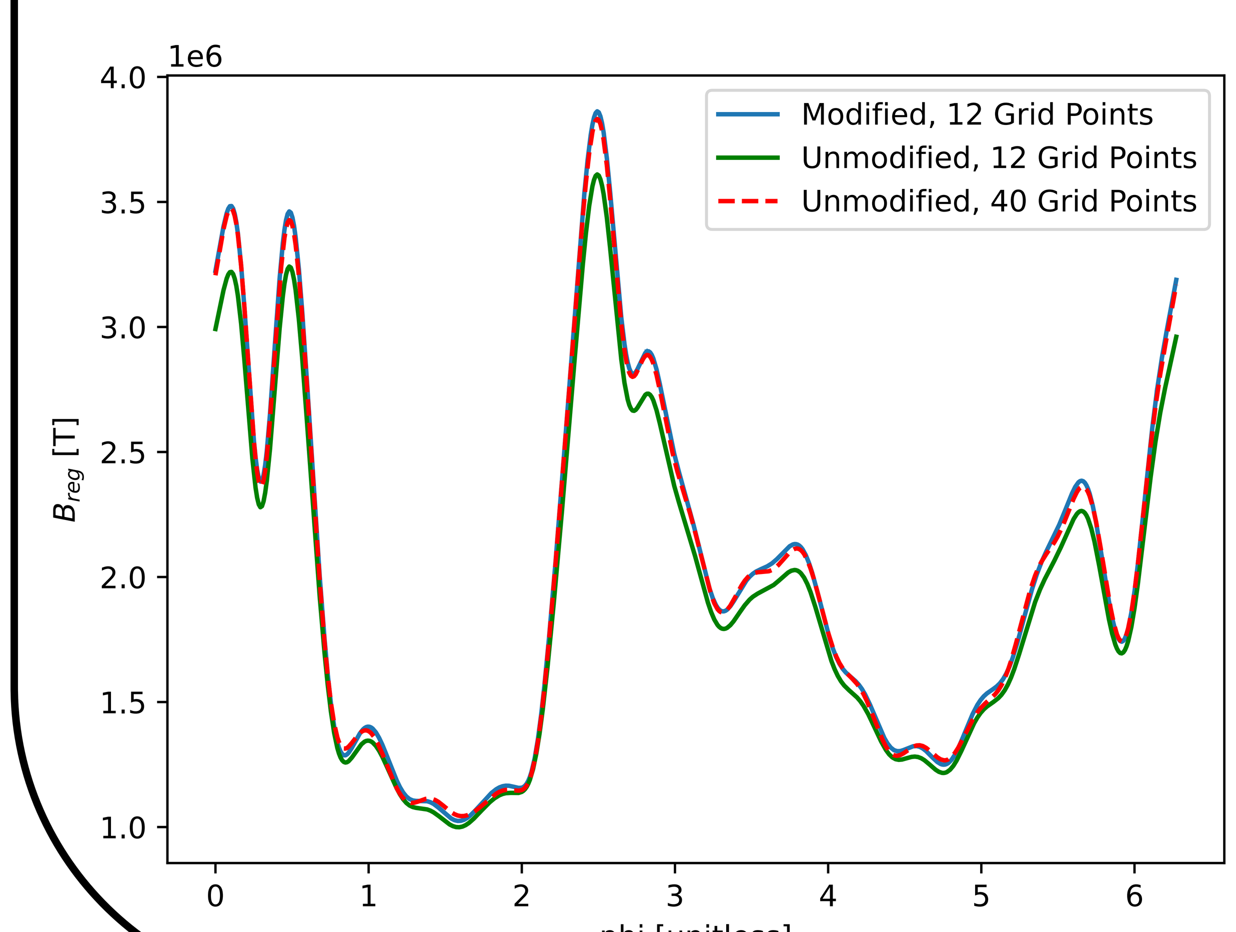
Modified Methods for Self-Force and Self-Inductance

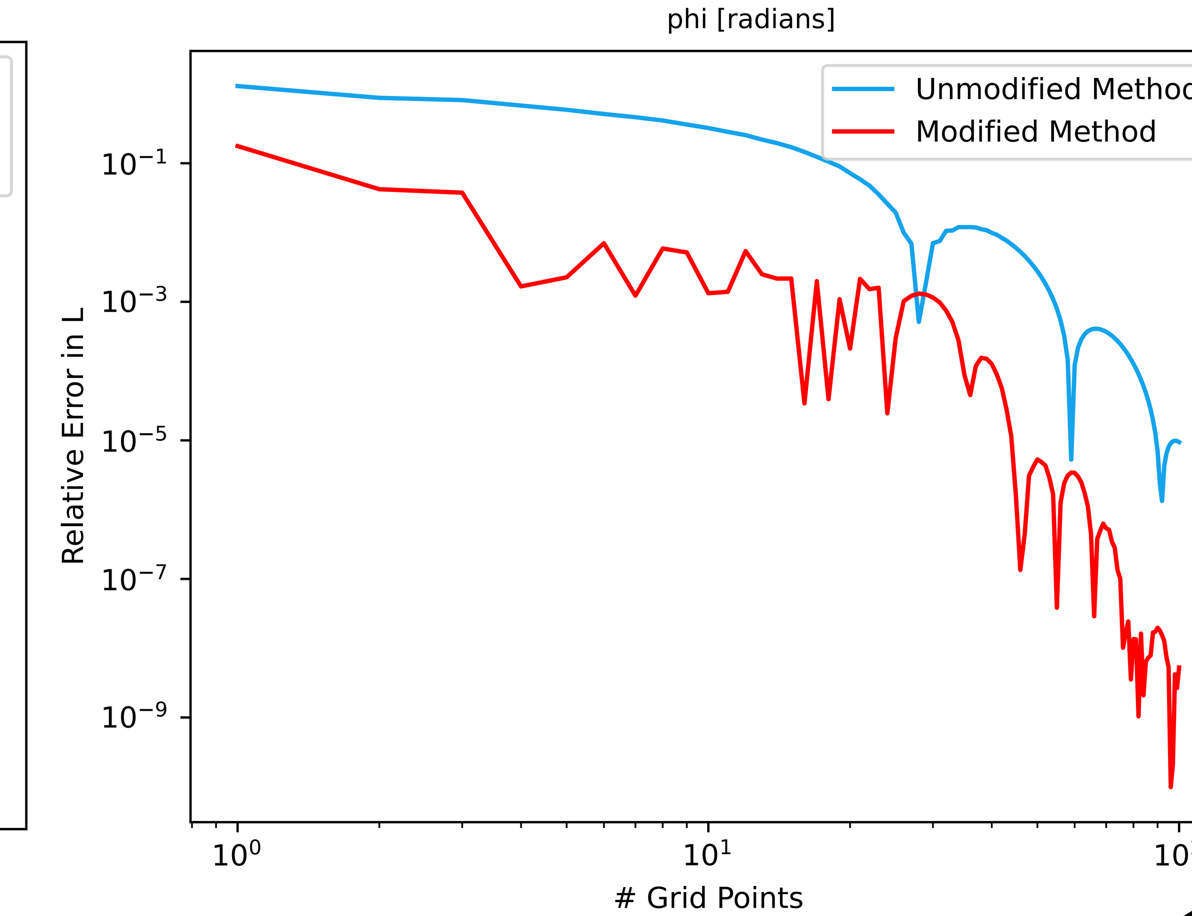
- Evaluating $d\mathbf{F}/dl$ and L by quadrature for $a/\mathcal{L} \ll 1$ is inefficient as the respective integrands become sharp as $a \to 0$.
- Instead, modify the equations for $\mathbf{B}_{reg}$ and L by:
  1. finding a quantity, $\mathbf{\Lambda}$, with a known antiderivative that fits the integrand well near $\tilde{\mathbf{r}} = \mathbf{r}$.
  2. subtracting $\mathbf{\Lambda}$ from the integrand, then adding it back in separately and analytically integrating this piece
- For the force,

$$\mathbf{\Lambda} \equiv \frac{\kappa(1-\cos\Delta\phi)\mathbf{e}_3}{2^{3/2}\left((1-\cos\Delta\phi) + a^2/2\sqrt{e}|\mathbf{r}_c'|^2\right)^{3/2}}$$

$$\Rightarrow \mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi}\left(\frac{\tilde{\mathbf{r}}_c'\times\Delta\mathbf{r}_c}{(|\Delta\mathbf{r}_c|^2 + a^2/\sqrt{e})^{3/2}} - \mathbf{\Lambda}\right) + \frac{\mu_0 I\kappa}{4\pi}\left(\ln\frac{8|\mathbf{r}_c'|}{a} - \frac{3}{4}\right)\mathbf{e}_3$$







References

Hurwitz, S., Landreman, M., & Antonsen Jr, T. M. (2023). Efficient calculation of self magnetic field, self-force, and self-inductance for electromagnetic coils. *arXiv preprint arXiv:2310.12087*.

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