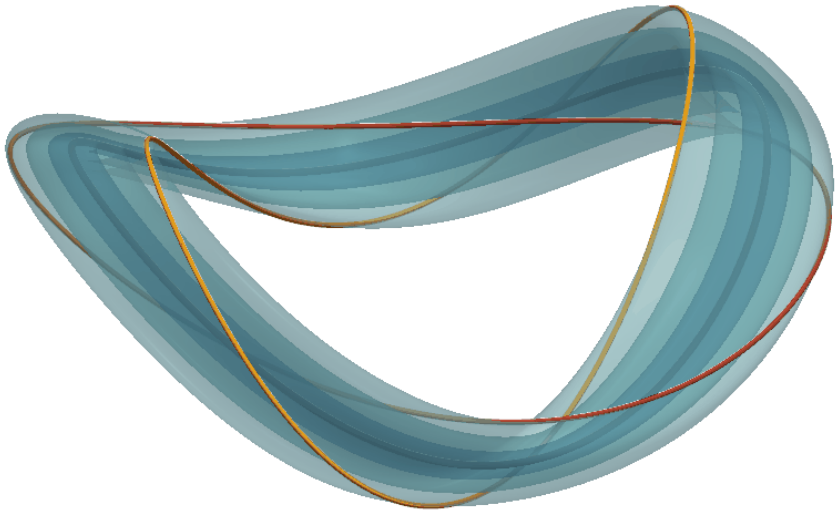
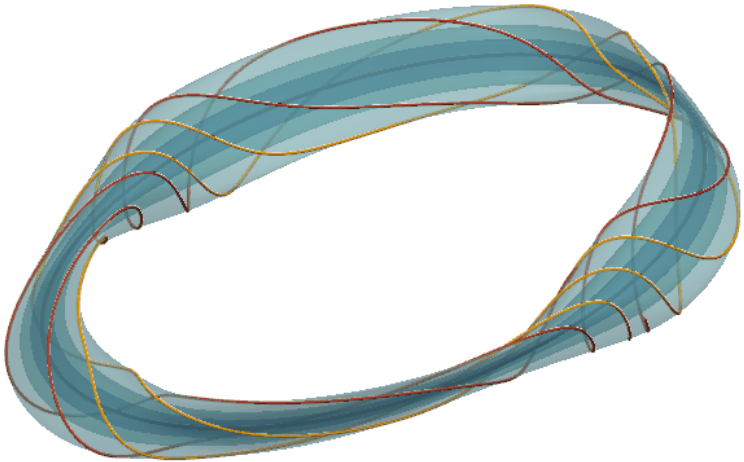
Discovering counterexamples to Grad's conjecture using AI



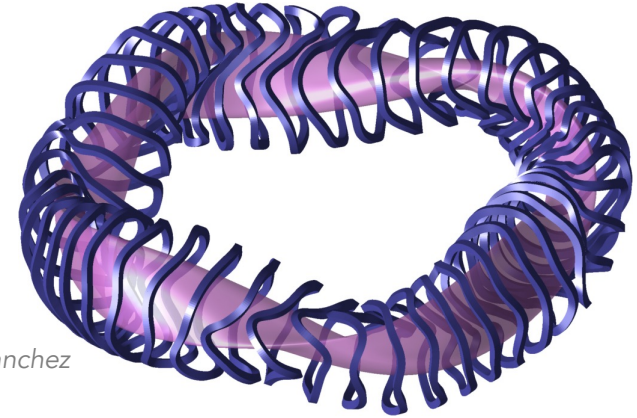


Matt Landreman, University of Maryland

arXiv:2609.26742

MHD equilibrium:

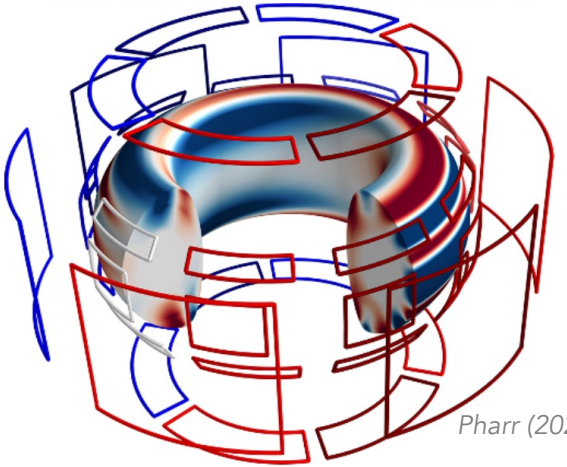
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B} \qquad \boldsymbol{J} \times \boldsymbol{B} = \nabla p$$



E Sanchez

3D solutions important for stellarators…

…& tokamaks



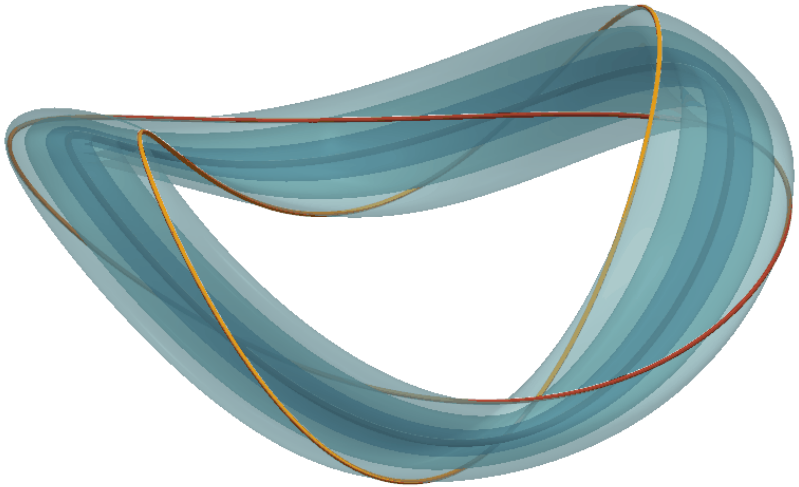
Pharr (2024)

Grad (1967), roughly: the only smooth solutions with non-constant pressure have a symmetry, e.g. axisymmetry.

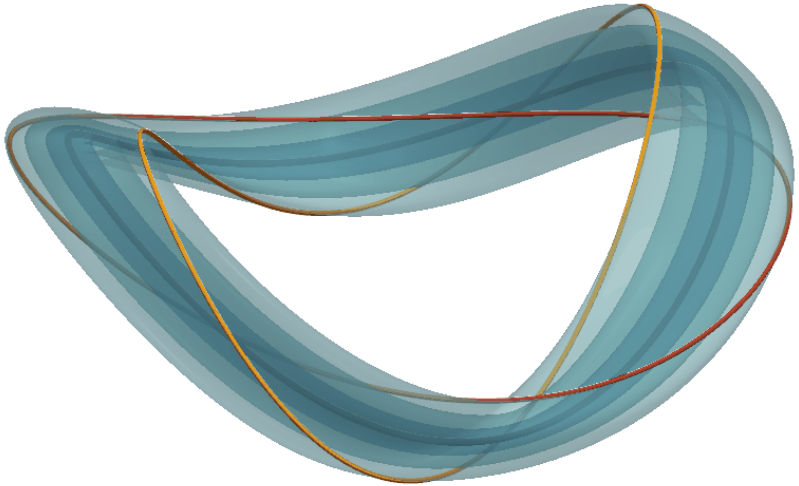
Counterexamples:

arXiv:2609.24739 Gómez-Serrano et al, Sept 21
arXiv:2609.26742 Landreman, Sept 22
arXiv:2610.07304 Issan et al, Oct 5
arXiv:2610.28466 Sengupta et al, Oct 20

- Background on Grad's conjecture
- AI's role in the discoveries
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THE PHYSICS OF FLUIDS VOLUME 10, NUMBER 1 JANUARY 1967

Toroidal Containment of a Plasma

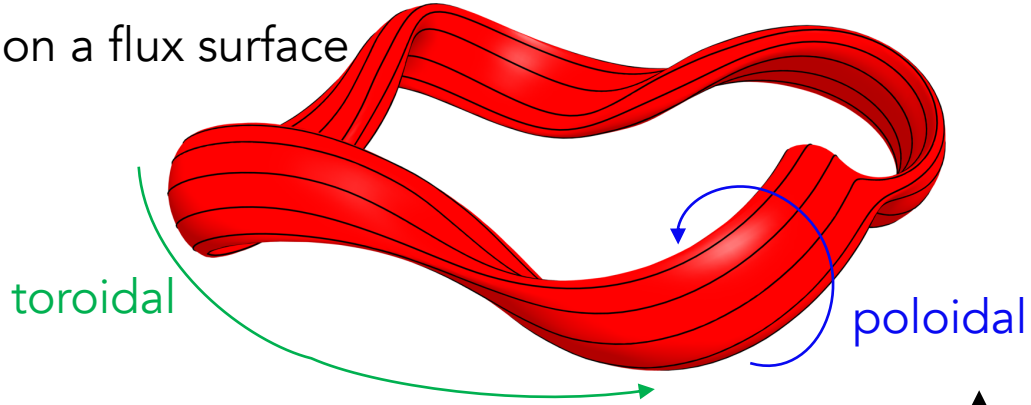
HAROLD GRAD
Courant Institute of Mathematical Sciences, New York University, New York, New York
(Received 5 July 1966; final manuscript received 10 October 1966)

The question of plasma containment in a torus is much more complicated than in an open-ended mirror system. Serious questions arise of the nonexistence of flux surfaces, of noncontained particle drifts, and of nonexistence of self-consistent equilibria at small gyroradius.

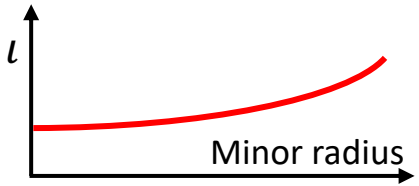
Summarizing, we find it very unlikely that there exists a general class of toroidal equilibria with smooth $p(\psi)$. In order to have a static equilibrium, $p'(\psi)$ must be zero in the neighborhood of every rational α, and flux surfaces must be relinquished.

Background: rational surfaces

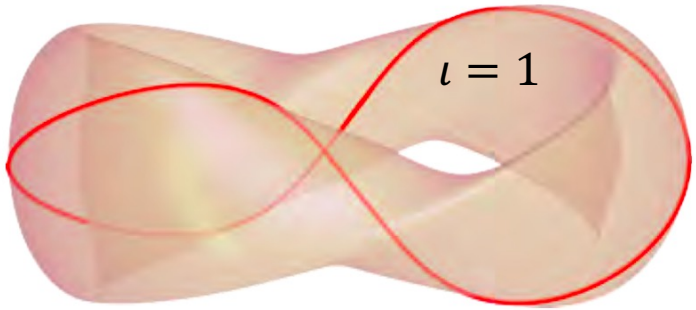
Field lines on a flux surface

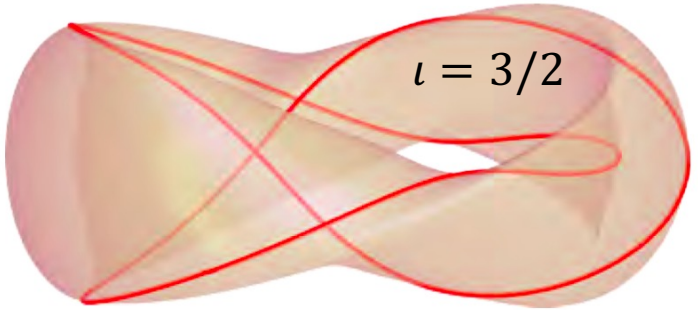


Rotational transform $\iota = \dfrac{\#\text{ poloidal turns}}{\#\text{ toroidal turns}} = \dfrac{1}{\text{Safety factor } q}$



Field lines close on rational surfaces:





Imbert-Gérard et al, "An Introduction to Stellarators"

Reasons underlying Grad's conjecture

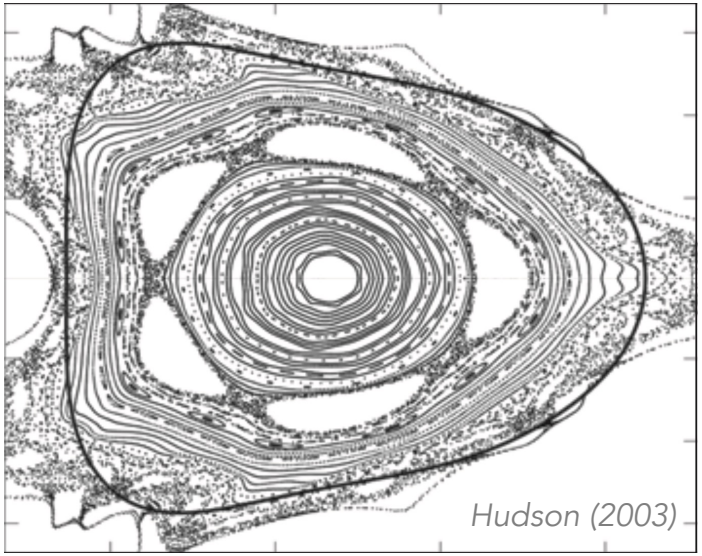
- Islands & chaotic field. $\mathbf{B} \cdot (\mathbf{J} \times \mathbf{B} = \nabla p)$
  $\Rightarrow\ \mathbf{B} \cdot \nabla p = 0$ implies $\nabla p = 0$ there.

- Constraint: $\frac{dp}{d\psi} \oint \frac{d\ell}{B}$ equal for every field line on a rational surface ("Hamada condition")

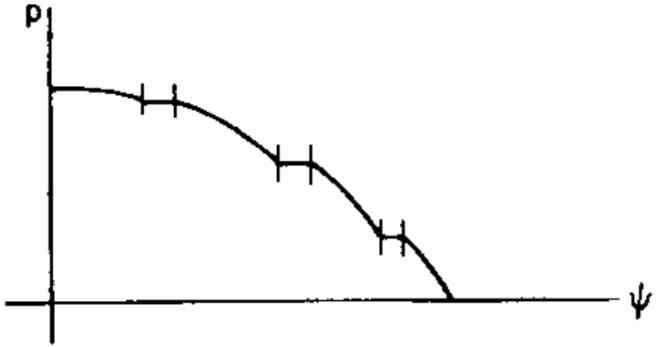
- Parallel current seems to diverge at rational surfaces, unless $\nabla p = 0$ there.

$$\left(\frac{\mathbf{J} \cdot \mathbf{B}}{B^2}\right)_{m,n} = \left(\frac{mG + nI}{n - \iota m}\right) \frac{dp}{d\psi} \left(\frac{1}{B^2}\right)_{m,n}$$

where $f(\psi, \theta, \varphi) = \sum_{m,n} f_{m,n}(\psi) e^{im\theta - in\varphi}$



Hudson (2003)



Grad (1967)

- Hard to see how pressure could be non-constant yet smooth.

International Journal of Fusion Energy, Vol. 3, No. 2, April 1985

Theory and Applications of the Nonexistence of Simple Toroidal Plasma Equilibrium[1]

by Harold Grad

There are exactly four known types of symmetry in which it is immediately evident that the pathological conclusion does not follow, and in each of these four symmetries existence theorems have been proved. In three cases (two dimensions, axial symmetry, helical symmetry), a flux function ψ exists by virtue of the essential two-dimensionality, and the real characteristics integrate out [with two "arbitrary integration constants" $f_1(\psi)$, $f_2(\psi)$] to yield a second order elliptic equation for ψ. The theory in these cases, if not complete (the elliptic equation is nonlinear), is at least very familiar. The fourth case, symmetry under reflection in a plane, implies closed magnetic lines, and also admits certain existence results (Lortz 1970). A fifth exception is with anisotropic pressure, again with a

It is virtually impossible to eliminate the possibility (which also has no practical interest) that for some isolated special domain and choice of constraints a solution may exist (with no others in its neighborhood). The proper formulation of the nonexistence statement is that, other than the stated symmetric exceptions, there are no *families* of solutions depending smoothly on a parameter. Three specific examples

A version of Grad's conjecture

In toroidal geometry,
the only smooth solutions
of the MHD equilibrium equations
with non-constant pressure
and nonzero rotational transform
are axisymmetric.

MHD equilibrium is also of interest due to equivalence with steady incompressible Euler flow

MHD

$$\nabla \cdot \boldsymbol{B} = 0$$

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \nabla p$$

$$\Leftrightarrow \quad \boldsymbol{B} \cdot \nabla \boldsymbol{B} = \nabla \left(p + \frac{B^2}{2} \right)$$

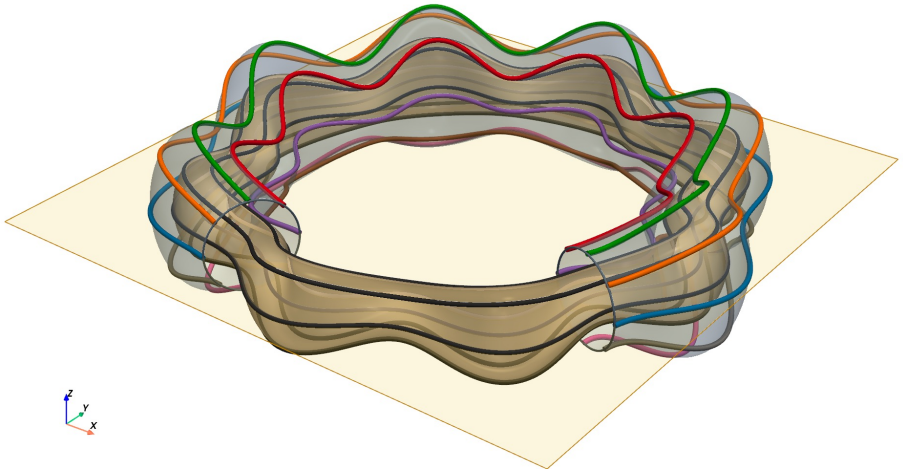
Incompressible Euler flow

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla \frac{p}{\rho}$$

Related work before September

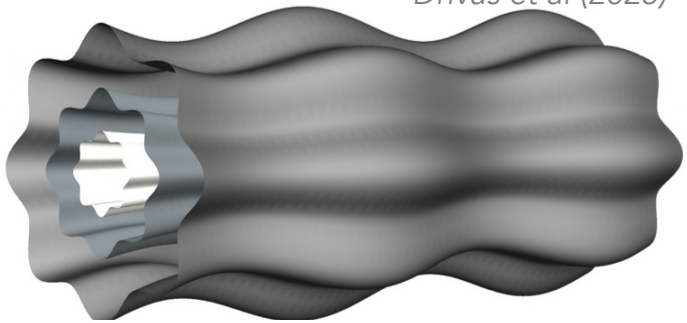
- Iota=0, mirror-reflection symmetry (Lortz 1970)
- Straight torus (Kaiser 1997, Weitzner 2020, Drivas 2025)
- Approximation: high aspect ratio (Mercier 1964, Solovev 1970, Garren 1991)
- Approximation: close to axisymmetry (Plunk 2020, Sarakina 2024)

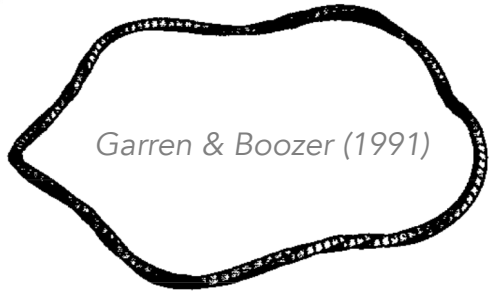


996, Dewar

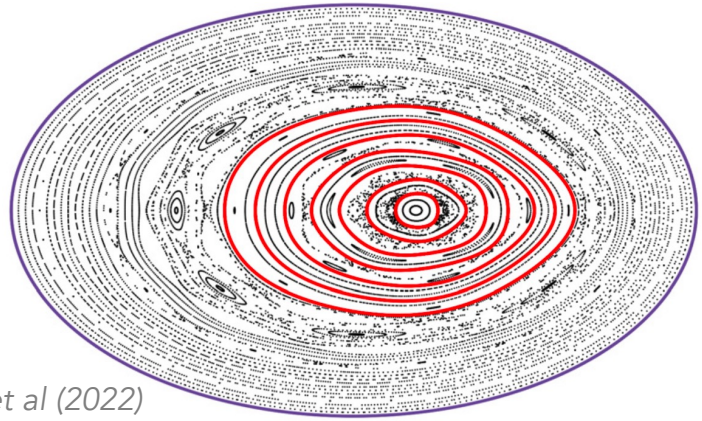
tin 2021,

Drivas et al (2025)



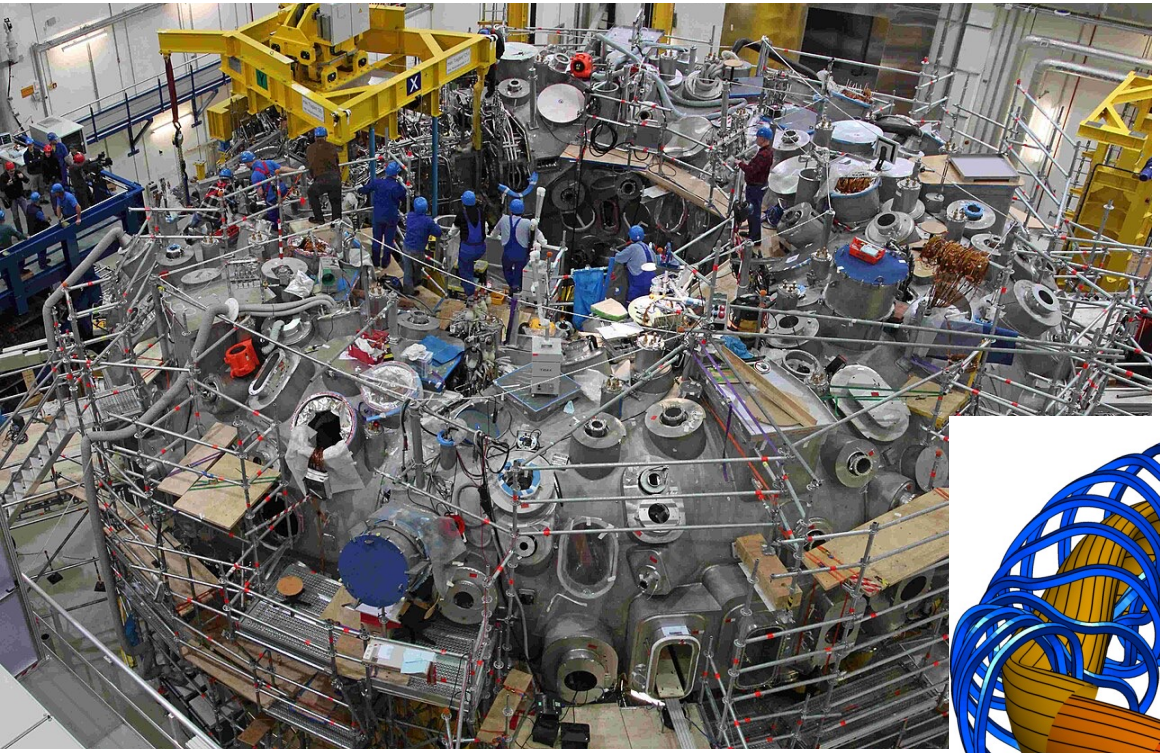


Garren & Boozer (1991)



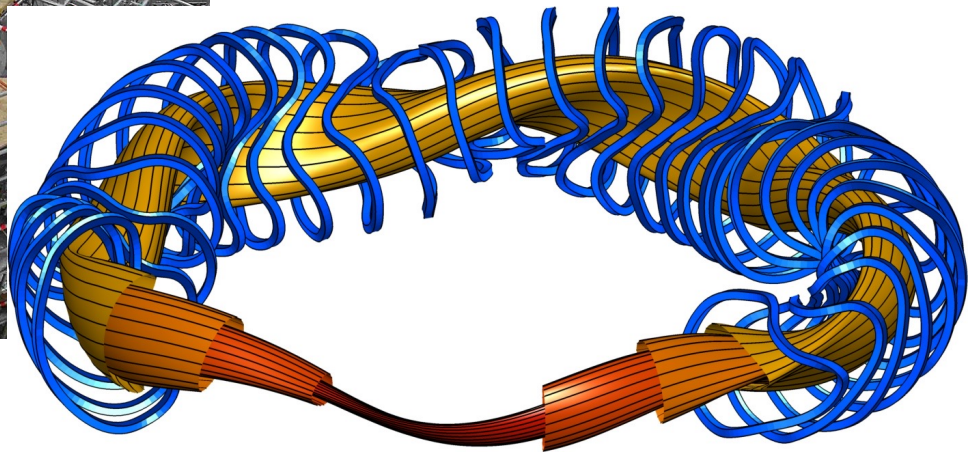
Baillod et al (2022)

In practice: use approximate solutions

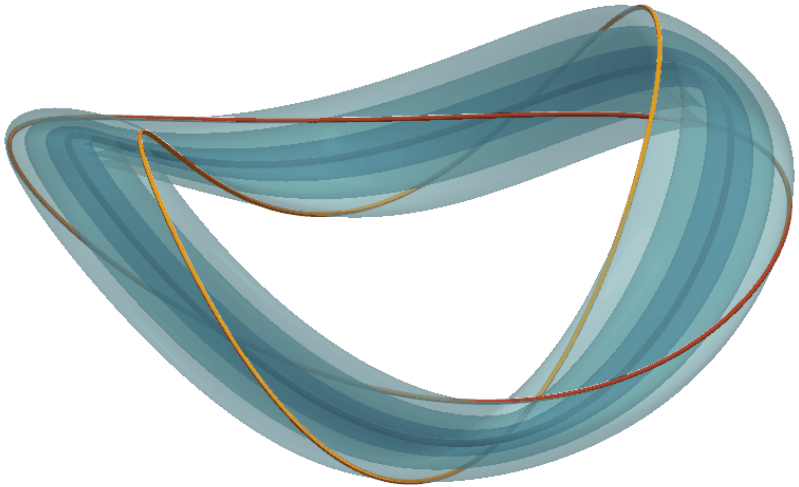


Max-Planck-Institut für Plasmaphysik

- Minimize residual $\|\boldsymbol{J}\times\boldsymbol{B}-\nabla p\|$
- Avoid rational iota values



- Background on Grad's conjecture
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Early unsuccessful attempt 1

ChatGPT 5.5 Pro, June 4

Consider the MHD (magnetohydrodynamic) equilibrium equation J cross B = grad p, where J = curl B is the current, B is a magnetic field satisfying div B = 0, and p is the pressure. We are interested in the case of nested toroidal flux surfaces, where surfaces of constant p in some domain are toroidal, except for the magnetic axis, where the set of constant p is a closed curve nested inside the toroidal surfaces. Consider the following conjecture: There is no non-axisymmetric magnetic field in a toroidal subset of R^3 that solves the MHD equilibrium equation with nested non-constant p surfaces and nonzero rotational transform and p a smooth function of position. Try to prove this conjecture false by finding a counter-example. If you cannot find one, say so, and explain the difficulty. Give your results as latex source and ensure it compiles without error. In the document, be extremely rigorous and show every step in detail.

Thought for 29m 0s >

Done — I produced a compilable LaTeX source file and a compiled PDF.

I did **not** find a self-contained exact counterexample satisfying all the stated hypotheses. ...

Early unsuccessful attempt 2

Fable 5, July 3

A magnetic field B has flux surfaces if there exists a single-valued function of position, psi, such that B dot grad psi = 0. We are interested in the case of nested toroidal flux surfaces, where surfaces of constant psi in some domain are closed, bounded, and topologically toroidal, except for the magnetic axis, where the set of constant psi is a closed curve inside the toroidal surfaces. Your task is to find an example of a non-axisymmetric magnetic field in a toroidal subset of R^3 with nested toroidal flux surfaces such that psi is non-constant and is a smooth function of position. psi should be non-negative, with the set psi=0 being the magnetic axis, corresponding to a closed curve. The constant-psi surfaces for psi in (0, delta] for some delta>0 should be bounded, closed, and topologically toroidal. The divergence of B must be 0. I specifically want an example in which the rotational transform is nonzero, so if a streamline of B (a field line) slightly away from the magnetic axis is followed around once toroidally, it comes back to a different point in the (R,Z) plane. You don't need to prove conclusively that iota is nonzero, but at least ensure there are no symmetries of the field that would guarantee iota to be 0. Rotational transform can be achieved by giving net (integrated) torsion to the magnetic axis, or by making the elongation of the constant-psi surfaces in the (R,Z) plane rotate poloidally as the toroidal angle phi increases, or through a current density J = curl B along the magnetic axis. Moreover, I would like the magnetic field to be a vacuum field (meaning curl B = 0), or a force-free field (meaning curl B is parallel to B), or a field in MHD force balance (so (curl B) cross B = grad p for some p(psi)), whichever of these three conditions enables a solution to be found. This is an open problem, so you will need to be creative to find a solution that hasn't been found by people before.

Show less

Strategy, stated up front

Any fully explicit field I can write down in closed form will satisfy *either* the integrability requirements exactly *or* the force condition exactly — if I could do both, Grad's conjecture would be settled in a chat window. So let me make a deliberate choice: I will construct a field for which **every requirement except force balance holds exactly and provably** (smooth ψ, exactly nested

Original successful prompt

Available at /github.com/landreman/analytic_3d_equilibria

ChatGPT Astra 6 Pro

Thu, Sep 10 at 6:44 AM

A magnetic field B has flux surfaces if there exists a single-valued function of position, psi, such that B dot grad psi = 0. We are interested in the case of nested toroidal flux surfaces, where surfaces of constant psi in some domain are closed, bounded, and topologically toroidal, except for the magnetic axis, where the set of constant psi is a closed curve inside the toroidal surfaces. Your task is to find an example of a non-axisymmetric magnetic field in a toroidal subset of R^3 with nested toroidal flux surfaces such that psi is non-constant and is a smooth function of position. psi should be non-negative, with the set psi=0 being the magnetic axis, corresponding to a closed curve. The constant-psi surfaces for psi in (0, delta] for some delta>0 should be bounded, closed, and topologically toroidal. The divergence of B must be 0. I specifically want an example in which the rotational transform is nonzero, so if a streamline of B (a field line) slightly away from the magnetic axis is followed around once toroidally, it comes back to a different point in the (R,Z) plane. It would be even better (though not critical) if iota is not an integer so the return map is not the identity. You don't need to prove conclusively that iota is nonzero, but at least ensure there are no symmetries of the field that would guarantee iota to be 0. Rotational transform can be achieved by giving net (integrated) torsion to the magnetic axis, or by making the elongation of the constant-psi surfaces in the (R,Z) plane rotate poloidally as the toroidal angle phi increases, or through a current density J = curl B along the magnetic axis. Moreover, I would like the magnetic field to be in MHD force balance, so (curl B) cross B = grad p for some p(psi). This is an open problem, so you will need to be persistent and creative to find a solution that hasn't been found by people before. But you have already solved many open problems in mathematics, so I know that with enough perseverance, you can do it!

Show less

Worked for 20m 34s

I found an explicit, exact family with nonconstant scalar pressure, a nonplanar magnetic axis, nested analytic flux surfaces, and rotational transform $\iota = -2$.

There is one important qualification: **its one-toroidal-turn return map is the identity.** Thus it satisfies your allowance that noninteger transform is "not critical," but it does not satisfy the stronger statement that a field line must return to a different point after one toroidal turn. The nonzero transform here is actual winding in the physical (R, Z) cross-section, not merely an angle relabeling.

Below I give the field as a function of Cartesian position and verify its properties. The construction uses the ordinary Euclidean curl and gradient, with no external force or pressure anisotropy. I have not established whether this particular family has appeared previously in the literature.

1. The explicit example

Use dimensionless coordinates and the normalization in your question, so that

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p.$$

Choose a parameter

$$0 < \epsilon < 1, \qquad a = \sqrt{1+\epsilon}, \qquad b = \sqrt{1-\epsilon}.$$

Define

$$X = \frac{x}{a}, \qquad Y = \frac{y}{b}, \qquad s = X^2 + Y^2, \qquad F = \sqrt{2s - s^2 - 4z^2},$$

Read the attachment. Your task is to generalize this analytic solution to a solution where rotational transform iota is not an integer; ideally iota would be an irrational number. You must preserve div B = 0, B dot grad psi = 0 (the existence of nested toroidal flux surfaces), and (curl B) cross B = grad p (MHD equilibrium), for a smooth pressure p satisfying grad p \ne 0 everywhere except the magnetic axis. The solution must remain non-axisymmetric. The surfaces of constant psi in some domain must remain closed, bounded, and topologically toroidal, except for the magnetic axis, where the set of constant psi is a closed curve inside the toroidal surfaces. You don't need to prove conclusively that iota is nonzero, but at least ensure there are no symmetries of the field that would guarantee iota to be 0. This is an open problem, so you will need to be persistent and creative to find a solution that hasn't been found by people before. But you have already solved many open problems in mathematics, so I know that with enough perseverance, you can do it!

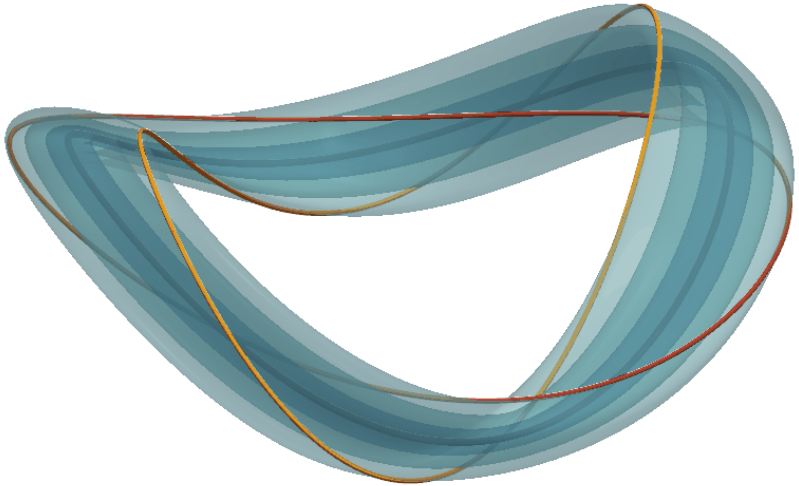
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AI played a key role in the other counterexample family too

arXiv:2609.24739 Gómez-Serrano et al

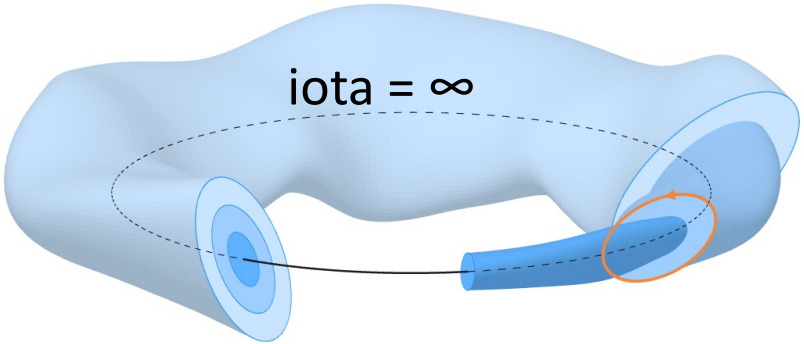
LLM usage disclosure: The work on this project began in July 2026. In July, the authors set up the problem and provided a detailed roadmap to construct the solutions. After this, they used a combination of the LLMs GPT-5.6 Sol, Claude Fable 5, and Claude Opus 5 to fill in the technical details, assist with calculations, and provide mathematical feedback on the parts of the blueprint that were not detailed enough. The Lean code was produced by the same models with continuous guidance from the authors. During the latter stages of the Lean verification, the authors switched to the newer models GPT-6 Astra and Claude Fable 5.1. These models were also used for final proofreading but did not contribute to any of the mathematical content of the article. All mathematical statements and proofs have been checked by the authors, who take full responsibility for their correctness.

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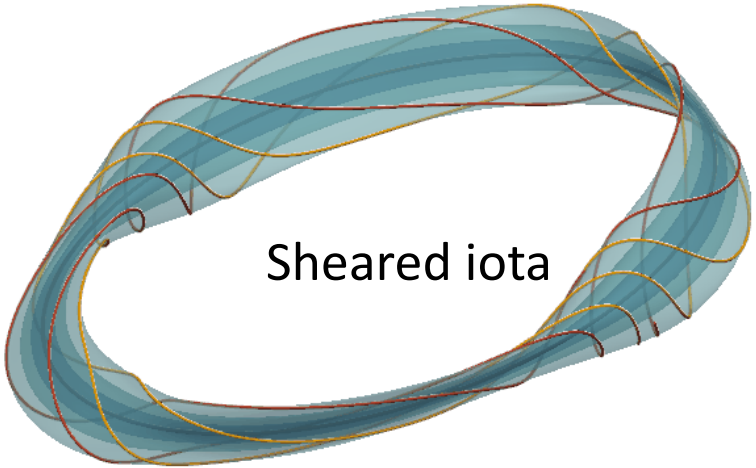


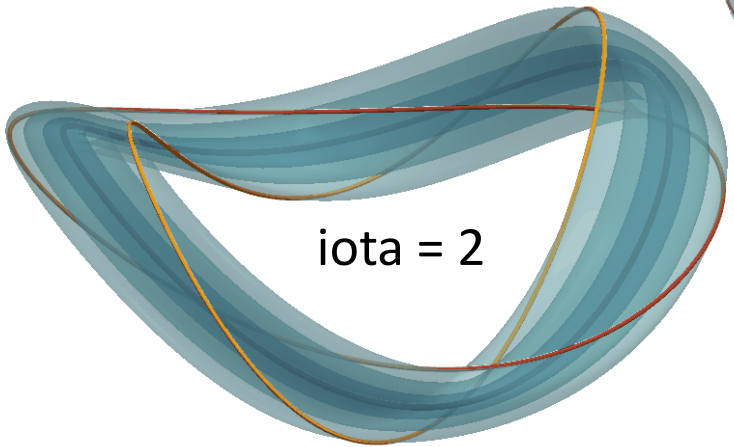
We know 3 quite different families of counterexamples

arXiv:2609.24739 Gómez-Serrano et al



arXiv:2609.26742 Landreman





Iota=2 counterexamples, in Mathematica

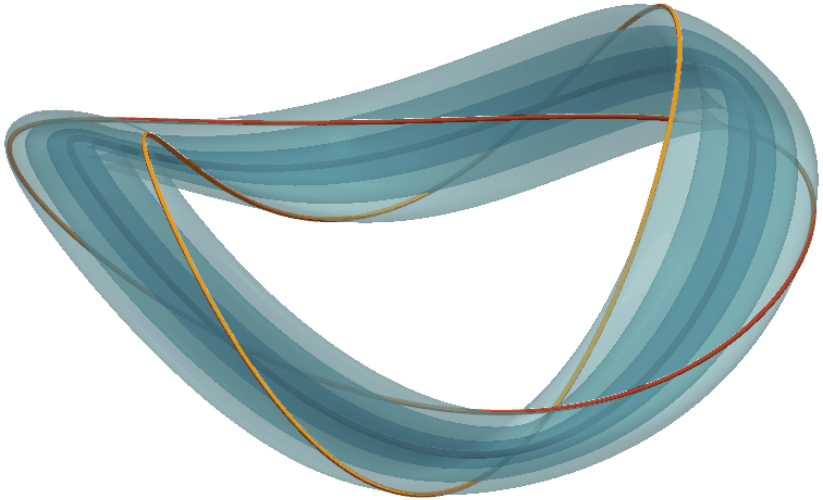
```
a = Sqrt[1 + ϵ];
b = Sqrt[1 - ϵ];
s = (x / a)^2 + (y / b)^2;
F = Sqrt[2 s - s^2 - 4 z^2];
Bx = (2 z x - (a / b) F y) / s;
By = (2 z y + (b / a) F x) / s;
Bz = 1 - s;
B = {Bx, By, Bz};
modB2 = Bx^2 + By^2 + Bz^2;
ψ = (x^2 + y^2 + 4 z^2 + modB2 - 2 - ϵ^2) / 4;
p = pa - 2 ψ;
J = Curl[B, {x, y, z}];

Simplify[Div[B,{x,y,z}]]

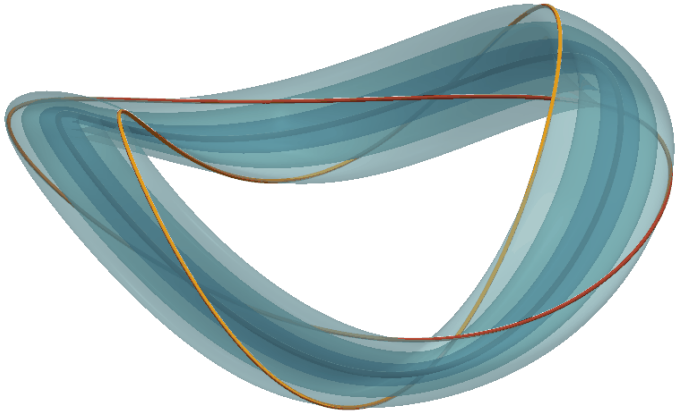
0                                   ∇·B = 0

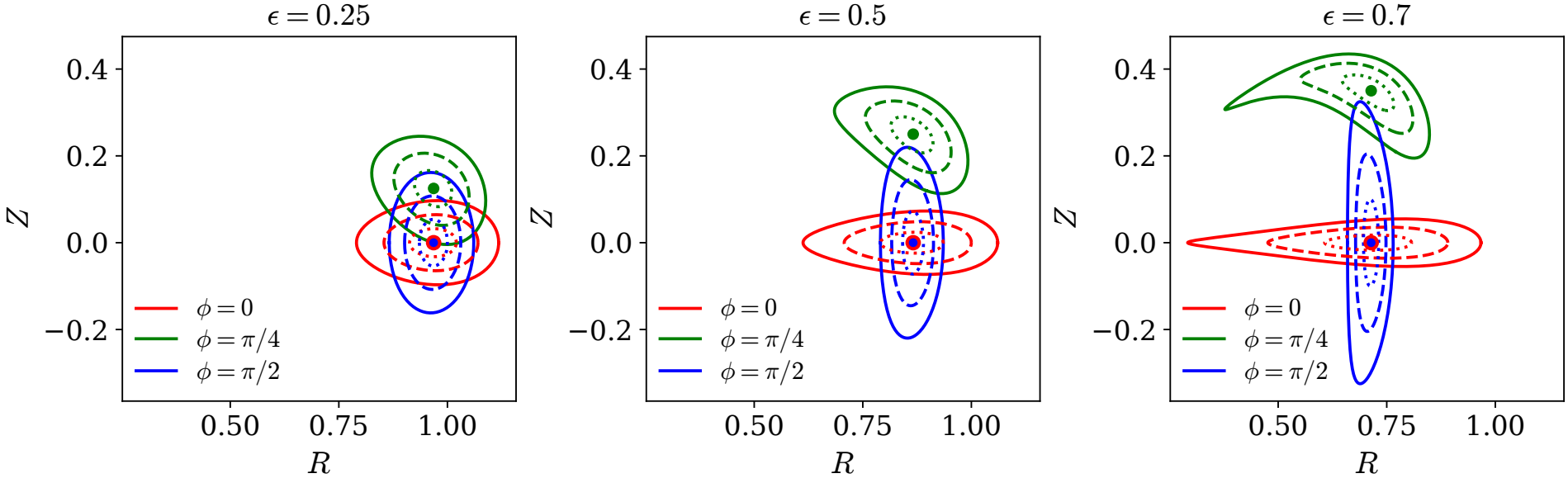
Simplify[Cross[J, B] - Grad[p, {x, y, z}]]

{0, 0, 0}                           (∇×B)×B - ∇p = 0
```



Iota=2 counterexamples





Map from flux coordinates to position is also explicit

$(\psi, \alpha, \zeta) \to (x, y, z)$:

$$u = -\frac{\epsilon}{2} + \sqrt{\psi}\cos\alpha, \quad v = \sqrt{\psi}\sin\alpha$$

$$q^2 = u^2 + v^2, \qquad L = \left(\frac{1+\sqrt{1-4q^2}}{2}\right)^{1/2},$$

$$x = a\left[L\cos\zeta + \frac{u\cos\zeta + v\sin\zeta}{L}\right],$$

$$y = b\left[L\sin\zeta + \frac{v\cos\zeta - u\sin\zeta}{L}\right],$$

$$z = v\cos 2\zeta - u\sin 2\zeta,$$

Also volume, volume-averaged |B| and beta, …

A little intuition about why this example works

If $\mathbf{B} \cdot \nabla \mathbf{B} = -\nabla Q$ for some Q then

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \left(\frac{B^2}{2} \right) = \nabla \underbrace{\left(-Q - \frac{B^2}{2} \right)}_{p}$$

Axisymmetric field $\mathbf{B}_0$ satisfies $\mathbf{B}_0 \cdot \nabla \mathbf{B}_0 = -D\mathbf{x} = -\nabla Q$,

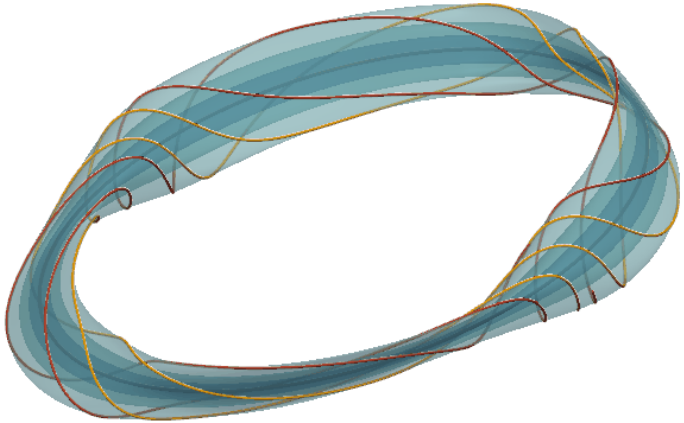
$$D = \mathrm{diag}(1, 1, 4) \qquad Q = \frac{x^2 + y^2 + 4z^2}{2},$$

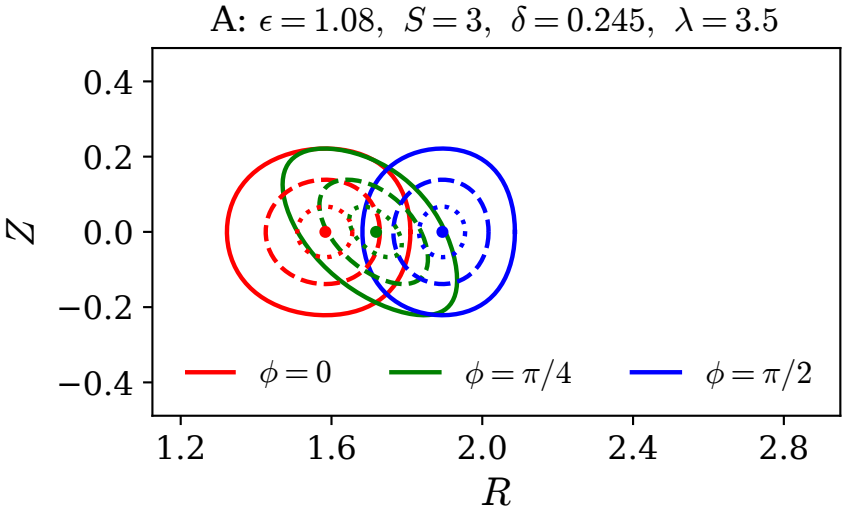
3D deformation: $\mathbf{B}(\mathbf{x}) = A\mathbf{B}_0(A^{-1}\mathbf{x}), \qquad A = \mathrm{diag}(a, b, 1)$

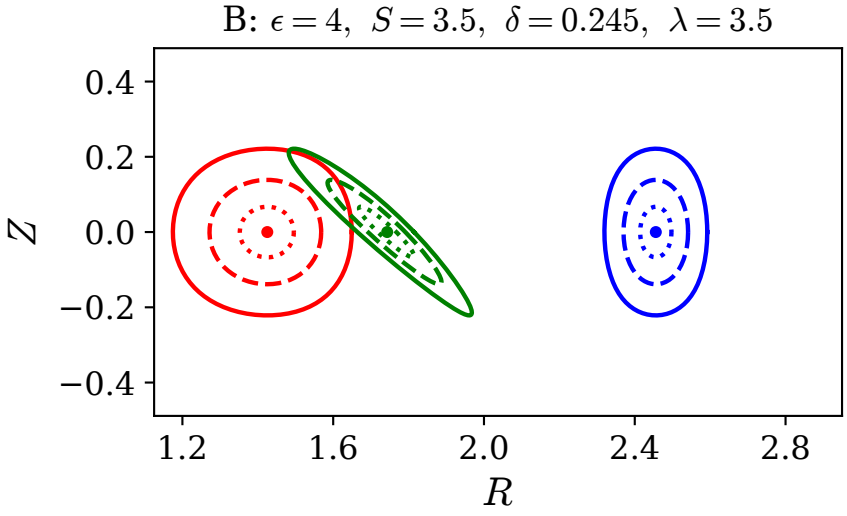
$$\mathbf{B} \cdot \nabla \mathbf{B} = A \left[\mathbf{B}_0 \cdot \nabla \mathbf{B}_0 \right] \left(A^{-1}\mathbf{x} \right) = -ADA^{-1}\mathbf{x} = -D\mathbf{x} = -\nabla Q$$

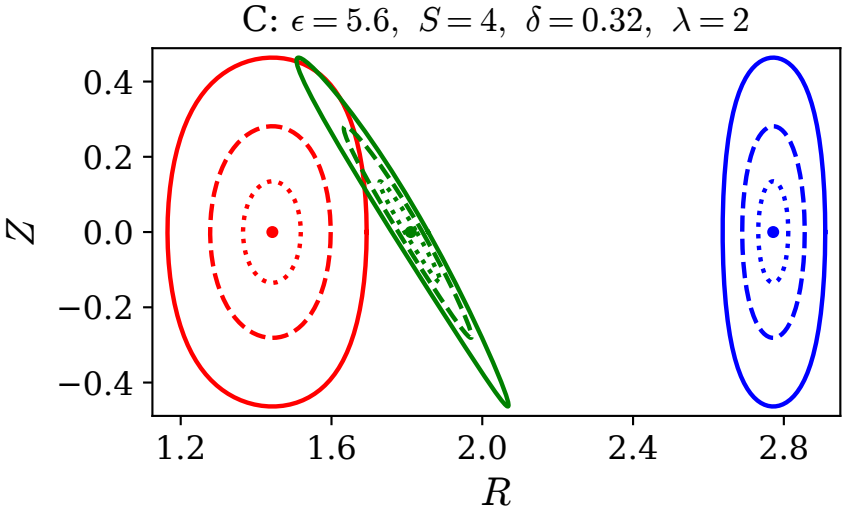
since A and D commute

Sheared iota counterexamples









Sheared iota counterexamples

Parameters: $\epsilon, \lambda, S, p_a, \delta$

$$\omega = x + iy, \qquad K = \bar{\omega}\sqrt{1 + \frac{\epsilon}{\bar{\omega}^2}}, \qquad \Xi = \omega K + \frac{\pi}{2} - S$$

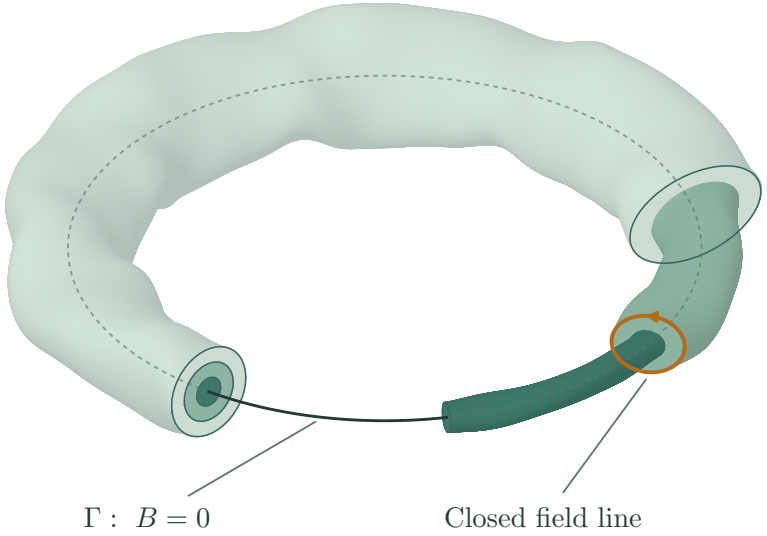
$$B_x = \mathrm{Re}\left(\frac{ie^{-i\lambda z}}{2K}\sin\Xi\right), \qquad B_y = \mathrm{Im}\left(\frac{ie^{-i\lambda z}}{2K}\sin\Xi\right),$$

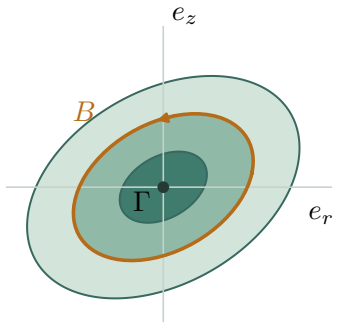
$$B_z = \frac{1}{\lambda}\mathrm{Re}\left(e^{-i\lambda z}\cos\Xi\right),$$

$$\psi = \frac{1}{2}\left[\sin^2(\lambda z) + \left\{\mathrm{Re}\left(e^{-i\lambda z}\cos\Xi\right)\right\}^2\right], \qquad p = p_a - \frac{1}{\lambda^2}\psi,$$

Inverse representation is again explicit

Gómez-Serrano family of counterexamples





$P = \psi_a - r^2$

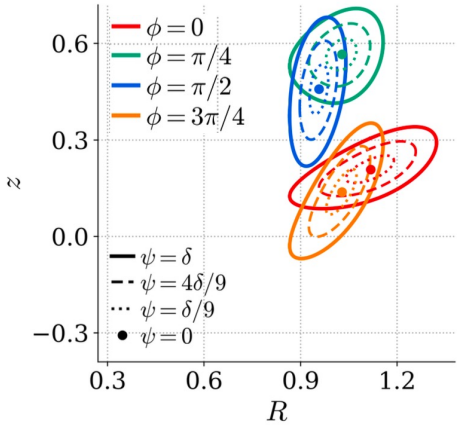
$r = 0.32, 0.66, 1$

- Field lines close poloidally.
- B vanishes on magnetic axis.
- Large number of field periods.
- A certain nonlinear function is proved to exist, but it is not known explicitly.
- Includes proof in Lean.

arXiv:2609.24739 Gómez-Serrano et al

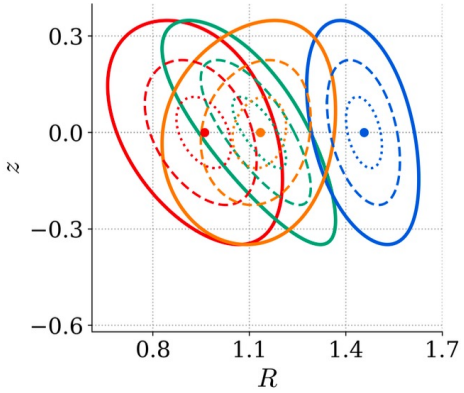
Extension: breaking stellarator symmetry

arXiv:2610.07304 Issan, Citrin, ML, & Tracey,



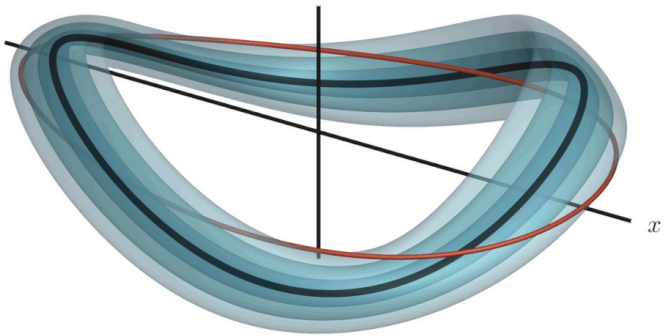
ι = 2 family

DeepMind, using Gemini 4 Argon

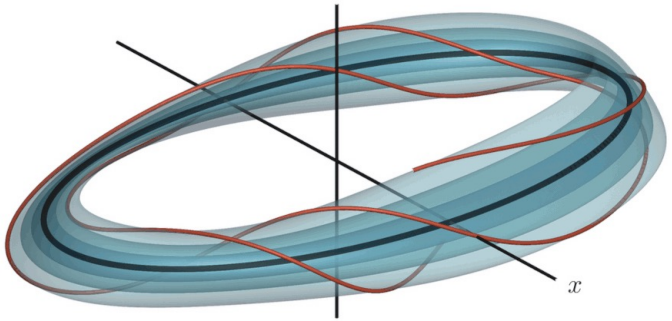


Sheared ι family

non-stellarator symmetric $\tau_0 = 0.5$



non-stellarator symmetric $\tau_1 = 0.8$



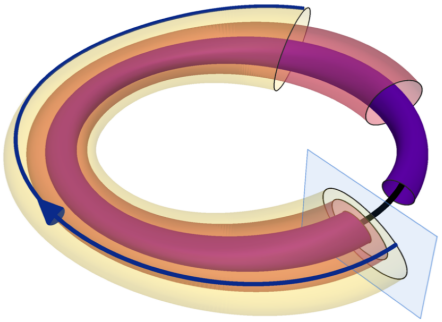
Figures by O Issan

Vacuum fields with surfaces

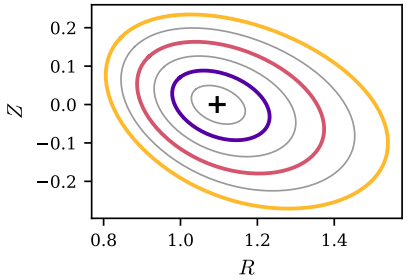
arXiv:2610.28466 Sengupta et al

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = 0 \qquad \boldsymbol{B} \cdot \nabla \psi = 0$$

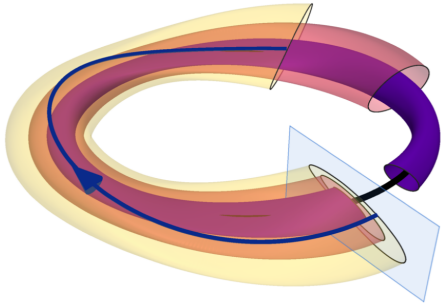
(a) Ellipsoidal, $\iota_0 \simeq 0.0423$ (reference)



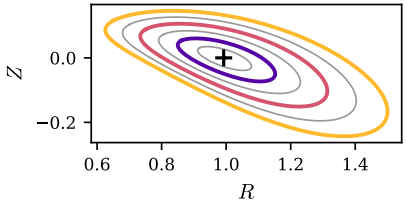
(b) section $\phi = \pi/4$



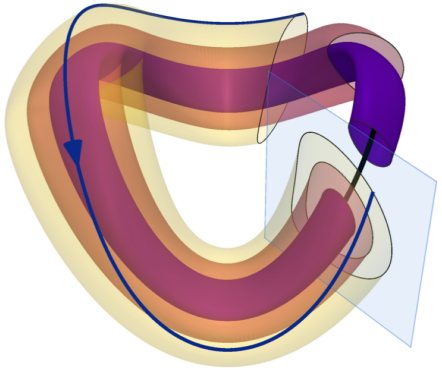
(c) Ellipsoidal, $\iota_0 = \sqrt{2}/7 \approx 0.202$



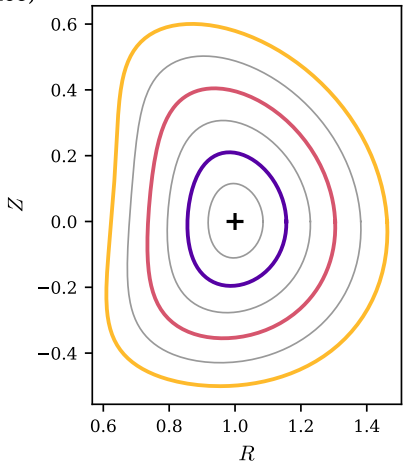
(d) section $\phi = \pi/4$



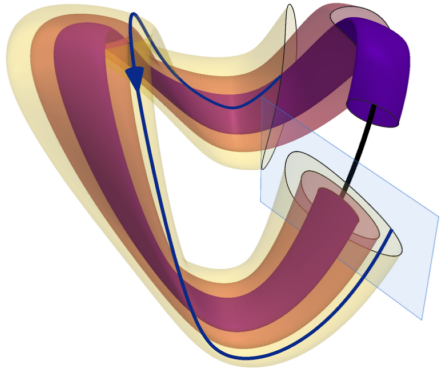
(e) Paraboloidal, $\iota = \sqrt{2}/100 \approx 0.0141$ (reference)



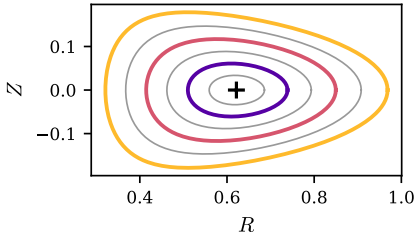
(f) section $\phi = 0$



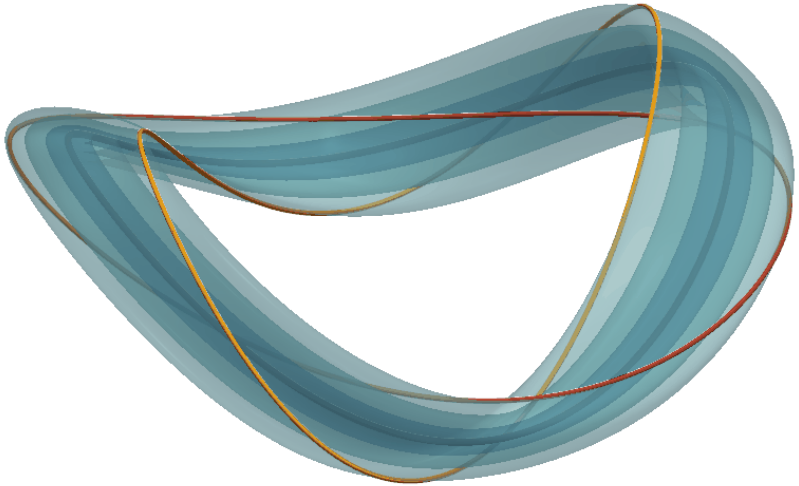
(g) Paraboloidal, $\iota = 1/\sqrt{5} \approx 0.447$



(h) section $\phi = 0$



- Background on Grad's conjecture
- AI's role in the discoveries
- The counterexamples
- Outlook



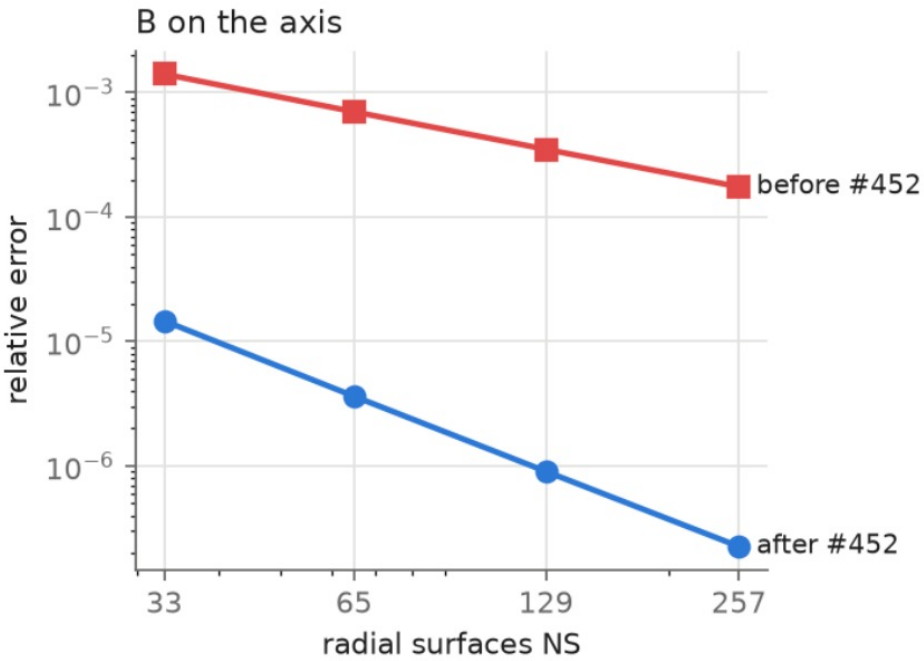
These solutions have practical use for testing codes

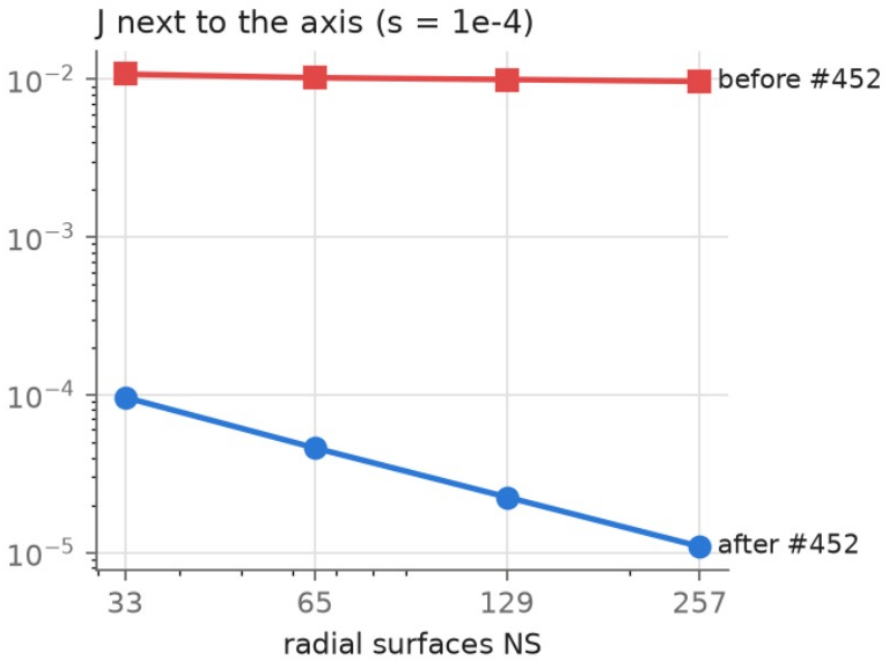
Supplemental material: `github.com/landreman/analytic_3d_equilibria`

```
PASS  B vector / analytic RMS B                                  1.464e-06
PASS  Current vector / analytic RMS current                      8.459e-05
PASS  Flux-surface label error / delta                           2.580e-05
PASS  Pressure error / axis-to-edge pressure drop                2.580e-05
PASS  Pressure gradient / analytic RMS gradient                  7.528e-04
PASS  Boundary psi error / delta                                 1.187e-06
PASS  Numerical B normal to numerical boundary / RMS B           1.670e-16
PASS  Axis R error / minor-radius scale                          3.540e-05
PASS  Nonplanar axis Z error / minor-radius scale                2.891e-05
PASS  Axis B vector (T)                                          1.107e-07
PASS  Axis parallel-current identity, relative                   1.318e-06
PASS  Integrated toroidal flux, relative                         2.920e-16
PASS  Total volume, relative                                     1.282e-12
PASS  Volume-average B squared, relative                         1.049e-10
PASS  Volume-average pressure, relative                          4.909e-11
PASS  Volume-average beta (ratio of averages), relative          5.584e-11
PASS  Line 1: analytic u,v drift / minor-radius scale            5.801e-07
PASS  Line 1: trajectory vs exact line / radius                  5.942e-07
PASS  Line 1: analytic flux drift / delta                        1.262e-07
…
```

These solutions have practical use for testing codes

`github.com/rogeriojorge/vmex-benchmark-analytical`





Outlook

- What more can we learn from these exact MHD solutions?

- Surely there are more of these solutions to be found.
  - Force-free? Match to a vacuum solution outside? Quasisymmetry? …

- Grad's basic objections still stand. What are better equations to solve for 3D equilibrium?

- What other problems could we tackle with this type of AI use?

Extra slides