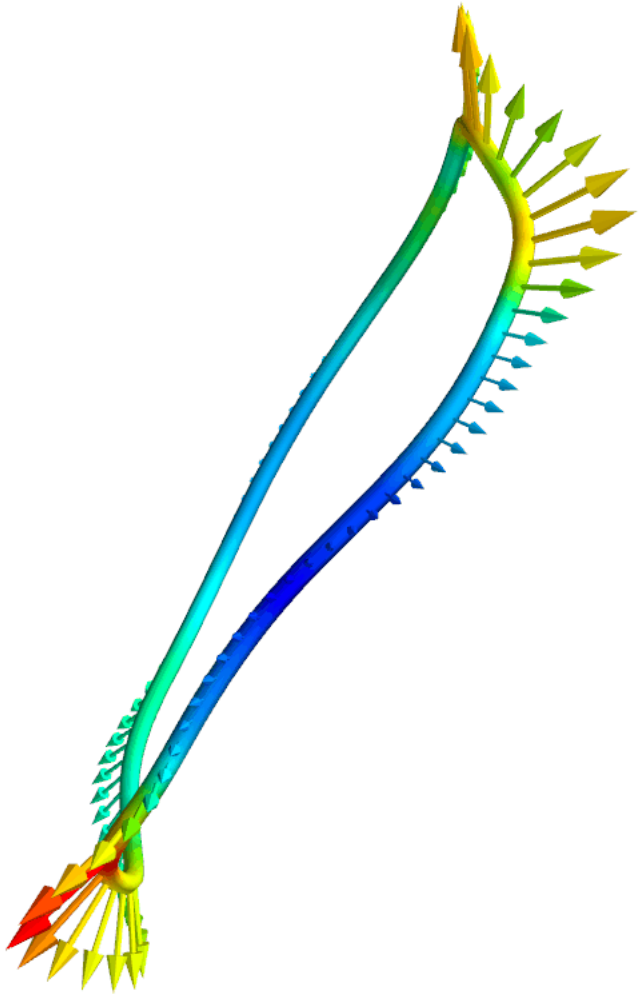
Efficient calculation of internal magnetic field & self-force for electromagnetic coils

Matt Landreman,
Siena Hurwitz,
Tom Antonsen,

University of Maryland

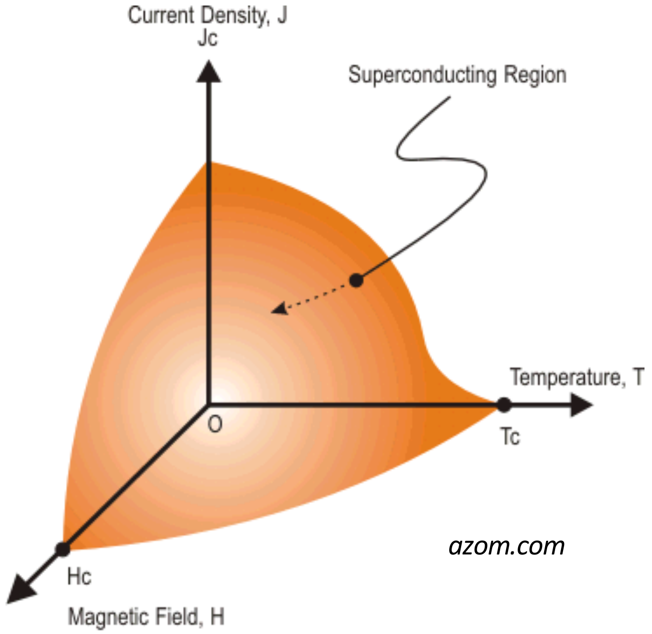


Outline

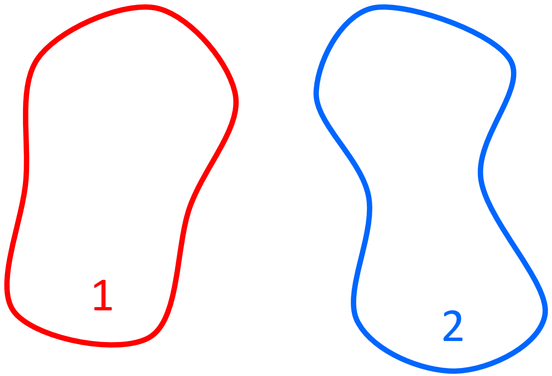
- Motivation
- Reducing 3D and 5D integrals to 1D
- Efficient quadrature for the 1D integrals

Tokamak & stellarator design requires calculations for the internal field and I x B force

- Superconductor quench limits depend on local **B**.
- Forces ∝ B^2. High B limited by support structure.
- Coil shapes can probably be optimized for force & B.



Field and force on coil 2 due to current in coil 1 can be computed quickly: 1D filament models are ok.



Tricky part is the self-field: singularity in Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int d\phi \, \frac{\frac{d\mathbf{r}'}{d\phi}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

Accurate calculation of the internal field and self-force appear to require high-dimensional integrals

Field: 3D integral

$$\mathbf{B}(x,y,\phi) = \frac{\mu_0 I}{4\pi^2 A} \int dx' \int dy' \int d\phi' \sqrt{g'} \frac{\mathbf{t}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

Force per unit length: 5D integral

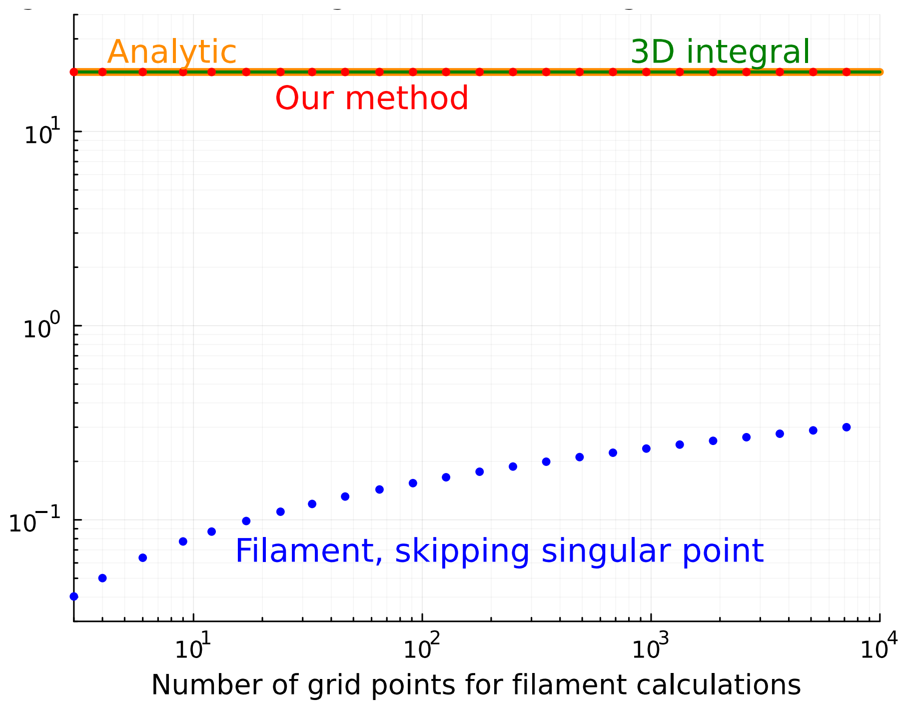
$$\frac{d\mathbf{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi A^2} \frac{d\phi}{d\ell} \int dx \int dy \int dx' \int dy' \int d\phi' \sqrt{g} \sqrt{g'} \frac{\mathbf{t} \times [\mathbf{t}' \times (\mathbf{r} - \mathbf{r}')]}{|\mathbf{r} - \mathbf{r}'|^3}$$

Can we simplify/approximate these integrals for fast evaluation inside an optimization loop?

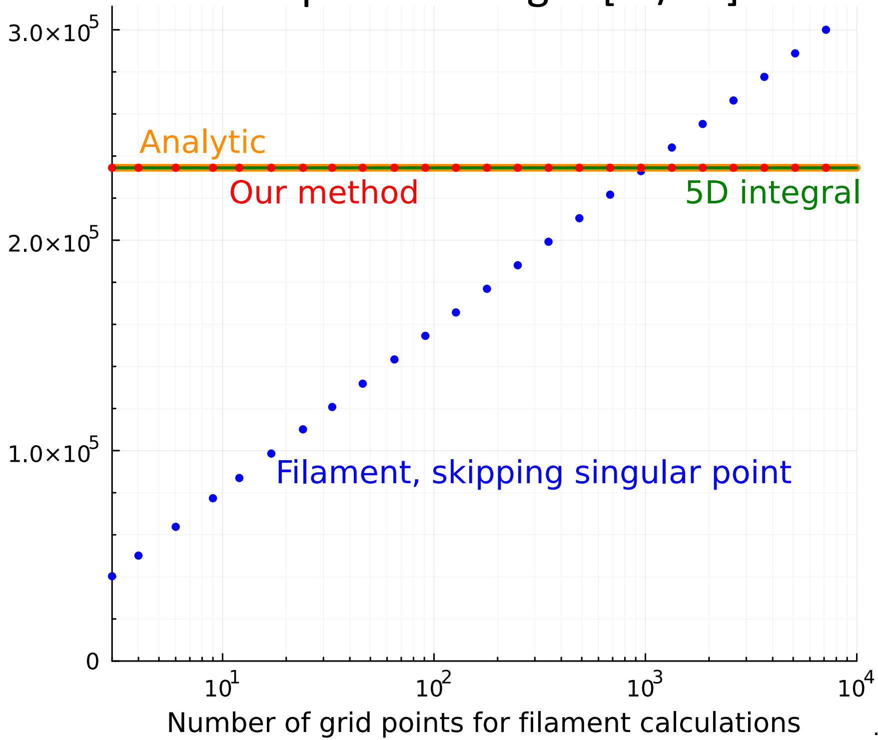
Simply skipping the singular point in a 1D filament calculation gives a non-converging result with signifcinat error

Circular coil

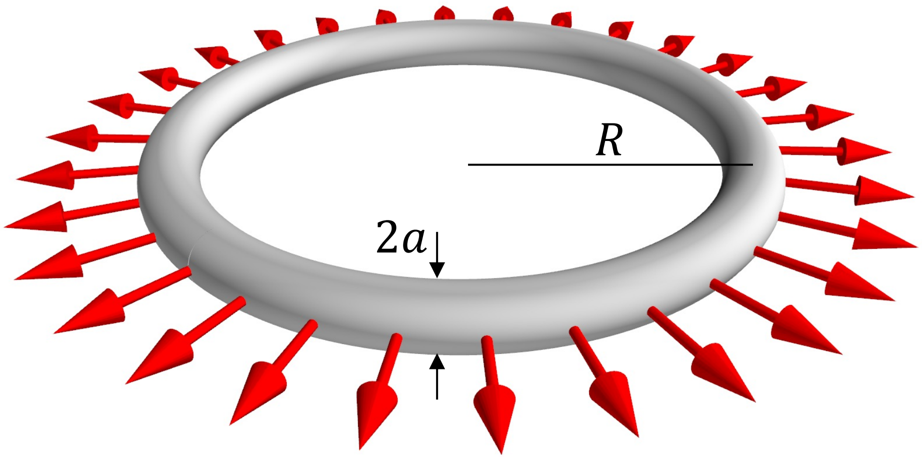
Maximum |B| in the conductor [Tesla]



Force per unit length [N / m]



The analytic formula for a circular coil shows that the finite cross-section cannot be ignored

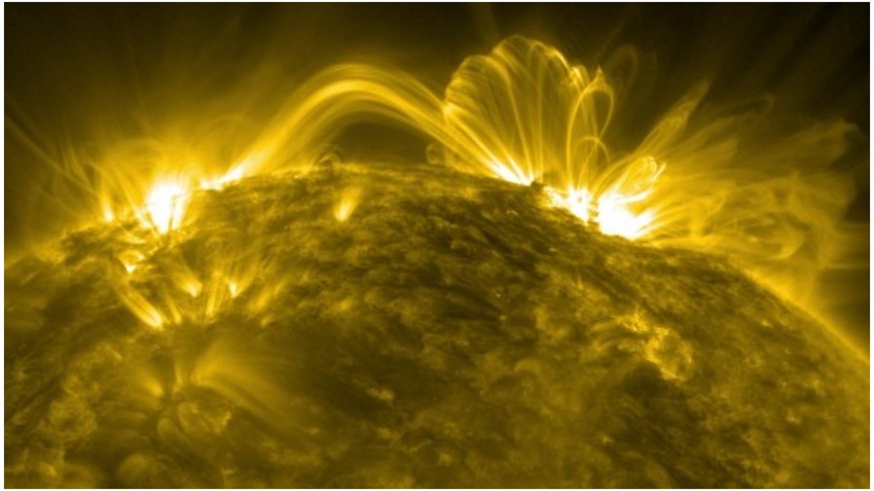


$$\frac{d\boldsymbol{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4}\right]\boldsymbol{e}_R$$

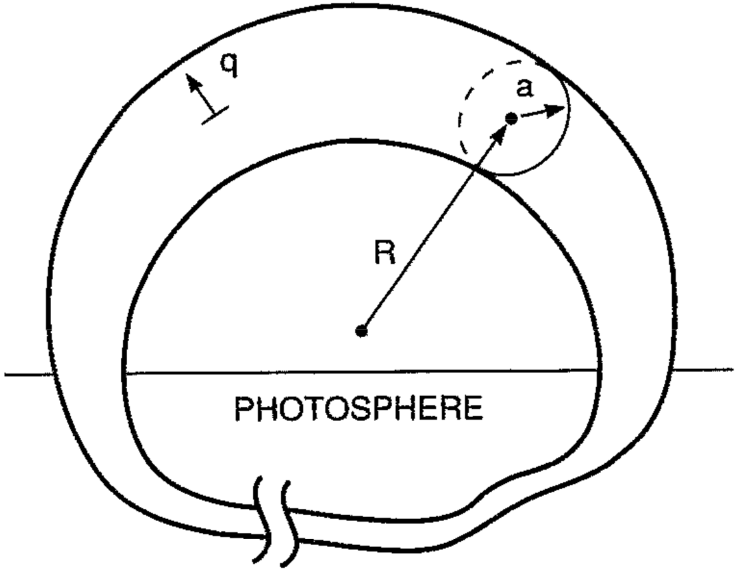
Diverges if minor radius $a \to 0$

Could a modified 1D filament model work if we supplement it with a value for a?

Calculations of internal field and self-force are also of interest for many other subjects, e.g. solar flares



NASA/SDO/Goddard



Lorentz self-forces on curved current loops

David A. Garren[a] and James Chen
Beam Physics Branch, Plasma Physics Division, Naval Research Laboratory, Washington, D.C. 20375

(Received 9 May 1994; accepted 20 June 1994)

Phys. Plasmas 1 (10), October 1994

Some related work

- Garren & Chen, *Phys. Plasmas* (1994). Looked at force but not internal field. Solution is to do a 1D integral over an incomplete loop, with a specific segment removed.
- Dengler, *Advanced Electromagnetics* (2016). Computed self-inductance using 1D integral.
- Lion, Warmer, et al *Nuclear Fusion* (2021). Computed **B** in conductor by summing analytical result for rectangular prism of **J**.
- Robin & Volpe, *Nuclear Fusion* (2022). Computed force for sheet current on a winding surface.

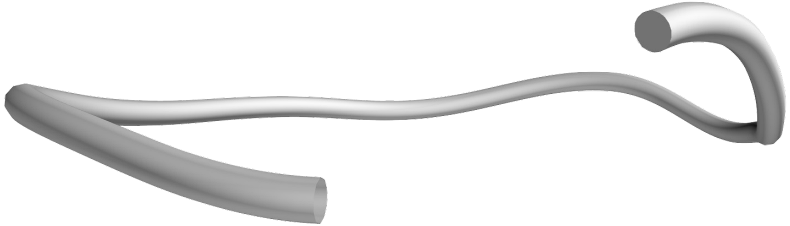
Our contribution:

- Compute both the self-force and the spatially-resolved internal field using only 1D integrals.
- Integration is over a periodic domain, so quadrature can be spectrally accurate, & can re-use points/data from other coil optimization objectives.

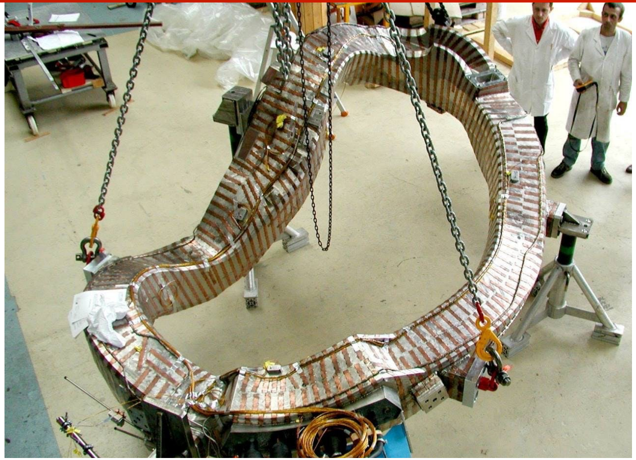
Outline

- Motivation
- Reducing 3D and 5D integrals to 1D
- Efficient quadrature for the 1D integrals

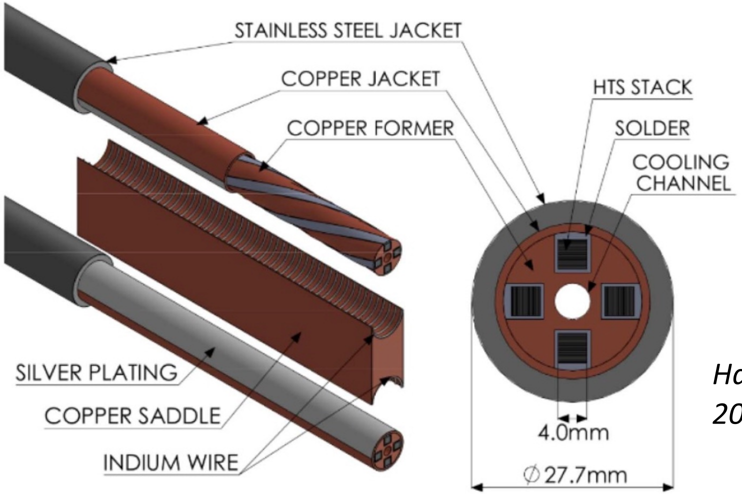
To make progress, we will consider a circular cross-section for now



Rectangular cross-section is a next step.



Some real conductors have more complicated geometry, e.g. HTS VIPER cable…



Hartwig et al 2020

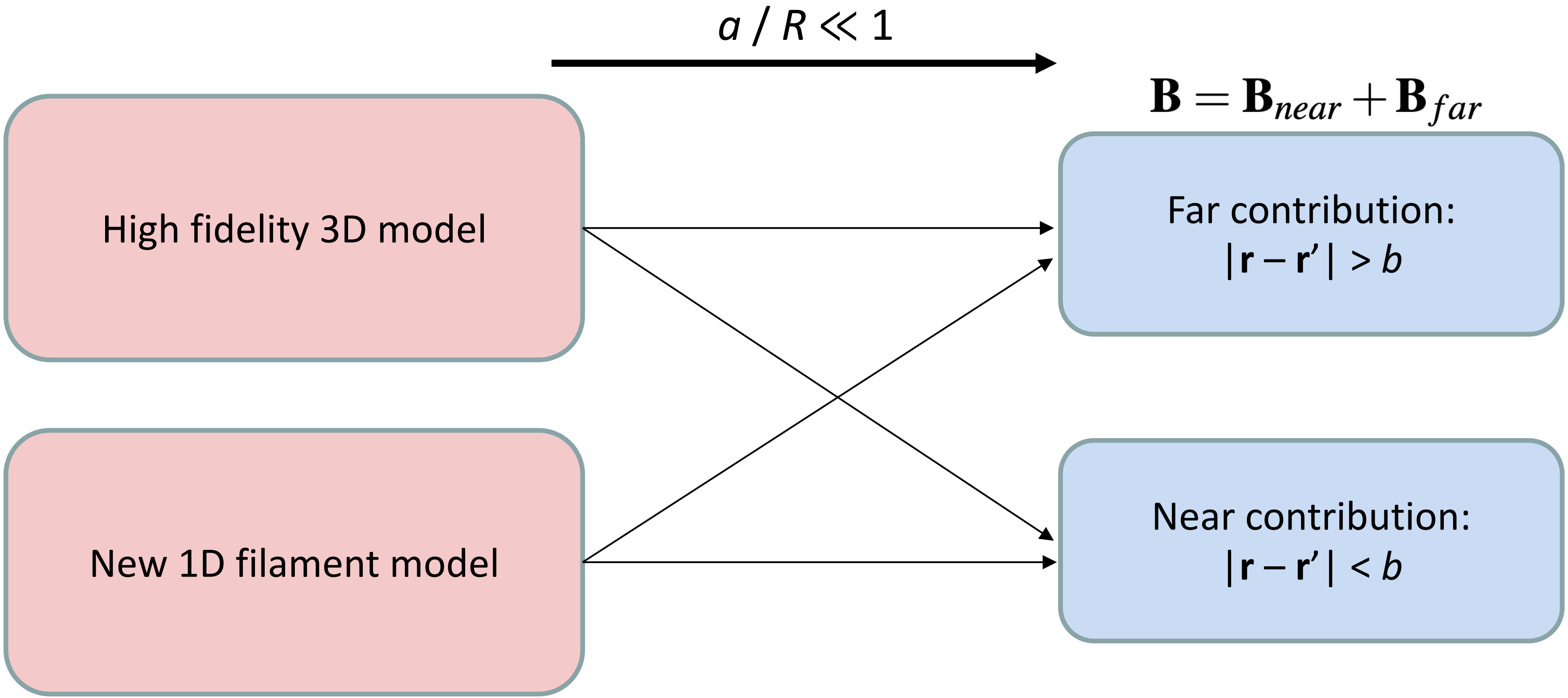
Methodology for finding an accurate reduced model

- Parameterize the coil volume:

$$\mathbf{r}(\rho,\theta,\phi) = \underbrace{\mathbf{r}_c(\phi)}_{\text{centerline}} + \rho a \cos\theta\, \mathbf{n}(\phi) + \rho a \sin\theta\, \mathbf{b}(\phi)$$

- Expansion parameter: $a / R \ll 1$, where $R \sim$ scales of curve centerline.
- Introduce intermediate scale b, with $a \ll b \ll R$.
- Split integrals into "near part" + "far part".
- Far part defined by $|\mathbf{r} - \mathbf{r}'| > b$. Finite cross-section can be neglected.
- Near part defined by $|\mathbf{r} - \mathbf{r}'| < b$. Coil centerline can be Taylor-expanded, so integrals can be done explicitly.
- Identify a 1D integral that has the same near part and far part as the above "high fidelity" calculation for $a / R \ll 1$.

Methodology for finding an accurate reduced model



Limit of the 3D integral for the internal field for *a* / *R* ≪ 1

$$\mathbf{B} = \mathbf{B}_{near} + \mathbf{B}_{far}$$

$$\mathbf{r}(r,\theta,\phi) = \mathbf{r}_c(\phi) + r\cos\theta\,\mathbf{n}(\phi) + r\sin\theta\,\mathbf{b}(\phi)$$

$$\mathbf{B}_{far} = \frac{\mu_0 I}{4\pi}\int_{\phi+\phi_0}^{2\pi+\phi-\phi_0} d\phi' \left|\frac{d\mathbf{r}_c'}{d\phi'}\right| \frac{\mathbf{t}' \times (\mathbf{r}_c - \mathbf{r}_c')}{|\mathbf{r}_c - \mathbf{r}_c'|^3}$$

$$\phi_0 = b/R$$

$$\begin{aligned}\mathbf{B}_{near} &= \boxed{\frac{\mu_0 I \rho}{2\pi a}\left[-\mathbf{n}\sin\theta + \mathbf{b}\cos\theta\right]} \\ &+ \frac{\mu_0 I \kappa}{8\pi}\left[-\frac{\rho^2}{2}\sin 2\theta\,\mathbf{n} + \left(2\ln 2 - \rho^2 + \frac{\rho^2}{2}\cos 2\theta + 2\ln\left(\frac{1}{a}\left|\frac{d\mathbf{r}_c}{d\phi}\right|\right) + 2\ln\phi_0\right)\mathbf{b}\right]\end{aligned}$$

Intuition:

- Leading order near-field is same as a straight wire. But corrections contribute to the force.

Our new 1D filament model reproduces the same limit as the original 3D integral

$$\mathbf{B} = \mathbf{B}_{local} + \mathbf{B}_{reg}$$

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\phi' \left| \frac{d\mathbf{r}'_c}{d\phi'} \right| \frac{\mathbf{t}' \times (\mathbf{r}_c - \mathbf{r}'_c)}{\left(|\mathbf{r}_c - \mathbf{r}'_c|^2 + a^2/\sqrt{e} \right)^{3/2}}$$

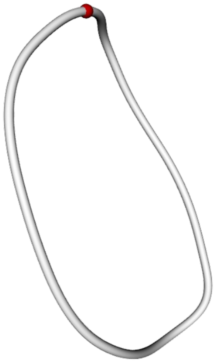
$$\mathbf{B}_{local} = \frac{\mu_0 I \rho}{2\pi a} [-\mathbf{n} \sin\theta + \mathbf{b} \cos\theta] + \frac{\mu_0 I \kappa}{8\pi} \left[-\frac{\rho^2}{2} \sin 2\theta \mathbf{n} + \left(\frac{3}{2} - \rho^2 + \frac{\rho^2}{2} \cos 2\theta \right) \mathbf{b} \right]$$

Intuition:

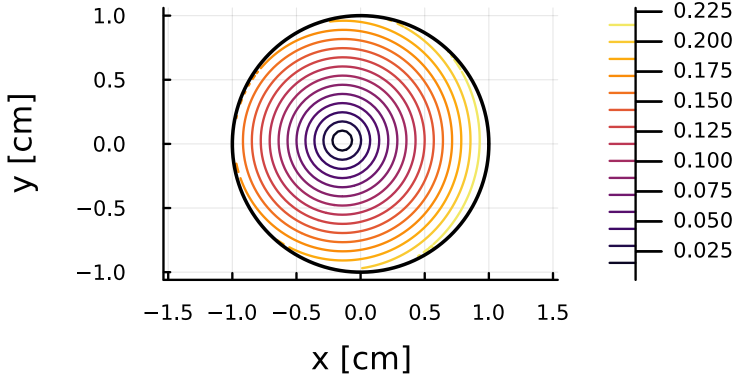
- Regularization added to Biot-Savart. Makes a difference when source and evaluation points are as close as the coil radius.

The new filament model agrees with the high-fidelity 3D integral for **B** in stellarator coils

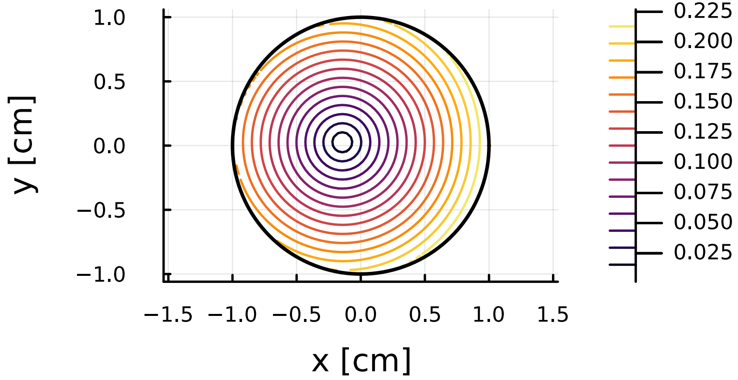
HSX coil 1



|B| [T], High fidelity

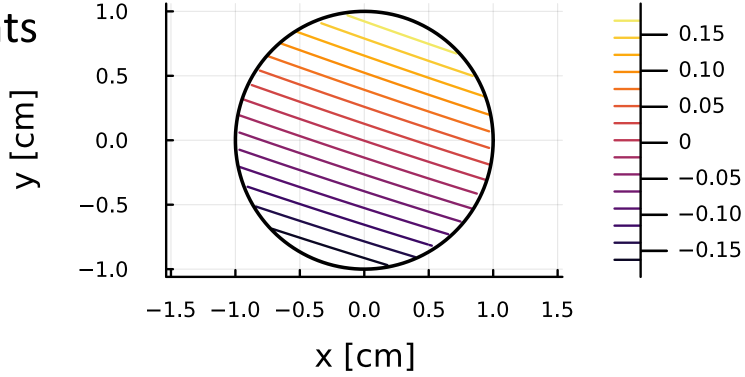


|B| [T], Filament

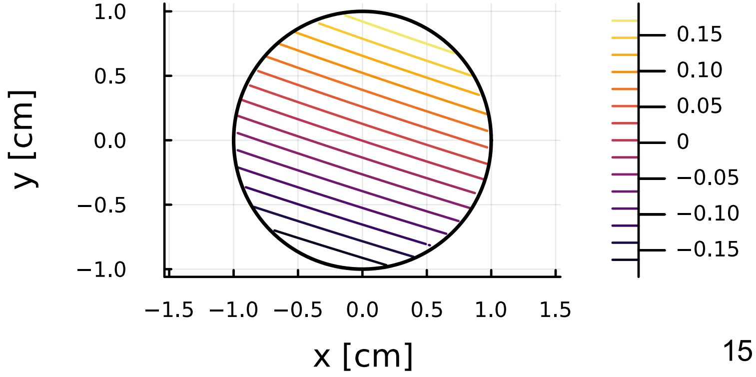


The individual **B** components also agree:

B_z [T], High fidelity



B_z [T], Filament



If curve centerline is a circle, the new filament model matches analytic formula for B

$$\begin{aligned}\mathbf{B} \quad &= \quad \frac{\mu_0 I \rho}{2\pi a}\left[\mathbf{e}_x \sin\theta - \mathbf{e}_z \cos\theta\right] \\ &+ \frac{\mu_0 I}{8\pi R_0}\left[\left(-\rho^2 \sin 2\theta\right)\mathbf{e}_x + \left(6\ln 2 - \rho^2 + \frac{\rho^2}{2}\cos 2\theta + 2\ln\frac{R_0}{a}\right)\mathbf{e}_z\right]\end{aligned}$$

Integrating the J x B force over the conductor cross-section, our method reduces the 5D integral for the self-force to a 1D integral

$$\frac{d\mathbf{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi^3}\int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^1 d\rho' \int_0^{2\pi} d\theta' \int_0^{2\pi} d\phi' \rho\rho' \left(1-\kappa\rho a\cos\theta\right)\left(1-\kappa'\rho' a\cos\theta'\right)\left|\frac{d\mathbf{r}'_c}{d\phi'}\right| \frac{\mathbf{t}\times[\mathbf{t}'\times(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\frac{d\mathbf{F}}{d\ell} = I\mathbf{t}\times\mathbf{B}_{reg}, \qquad \mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \left|\frac{d\mathbf{r}'_c}{d\phi'}\right| \frac{\mathbf{t}'\times(\mathbf{r}_c-\mathbf{r}'_c)}{\left(|\mathbf{r}_c-\mathbf{r}'_c|^2 + a^2/\sqrt{e}\right)^{3/2}}$$

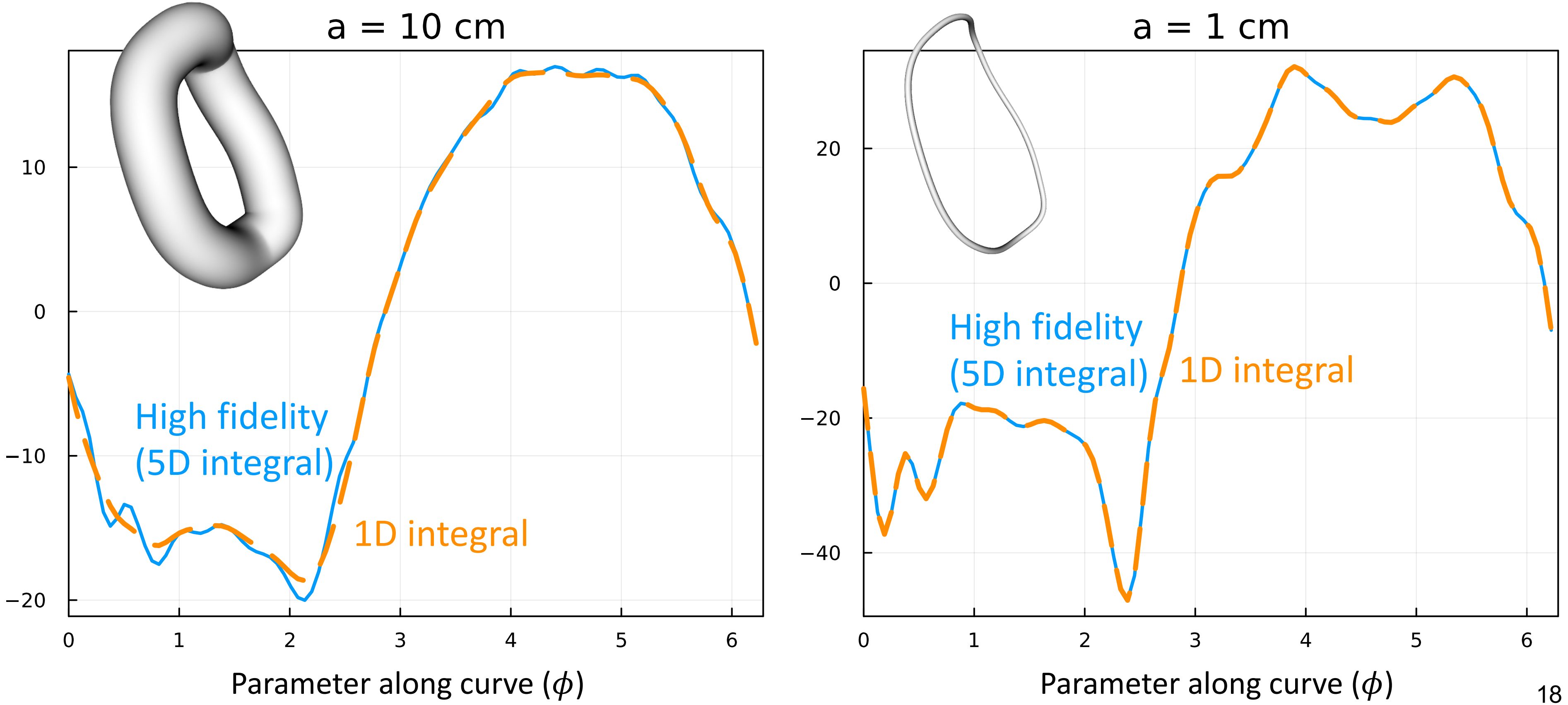
If $\mathbf{r}_c$ is a circle

$$\frac{d\boldsymbol{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4}\right]\boldsymbol{e}_R$$

Same as analytic result

The 1D integral accurately approximates high-fidelity calculations for the self-force in stellarator coils

$dF_x/d\ell$ [kN / m] for HSX coil 1, @ 150 kA



Outline

- Motivation
- Reducing 3D and 5D integrals to 1D
- Efficient quadrature for the 1D integrals

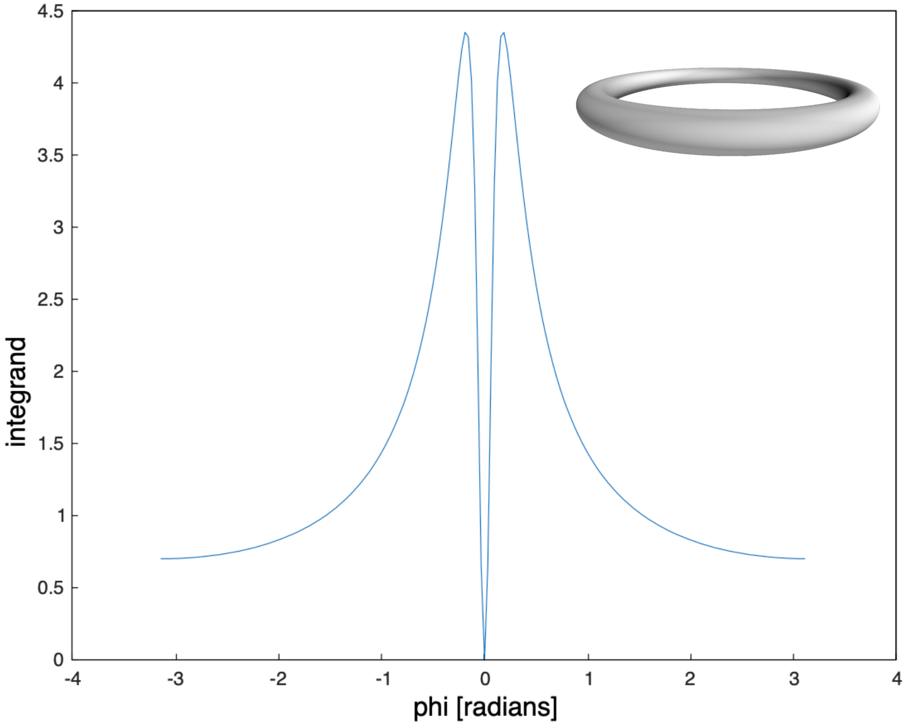
Remaining 1D integral is still tricky since integrand has fine structure

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi} \int d\phi' \left| \frac{d\mathbf{r}'_c}{d\phi'} \right| \frac{\mathbf{t}' \times (\mathbf{r}_c - \mathbf{r}'_c)}{\left(|\mathbf{r}_c - \mathbf{r}'_c|^2 + a^2/\sqrt{e} \right)^{3/2}}$$

A solution: subtract and add a function to the integrand with the same near-singular behavior that can be integrated analytically.

Also examining a partition-of-unity method like Malhotra et al.

Circular coil, a/R = 0.1



To make integrand smooth, we subtract and add a function with the same singular behavior that can be integrated analytically.

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \left[\frac{1}{\left(|\mathbf{r}\text{-}\mathbf{r}'|^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}} \frac{d\mathbf{r}'}{d\phi'} \times (\mathbf{r}-\mathbf{r}') - \mathbf{Q}(\phi') \right] + \frac{\mu_0 I}{4\pi}\int d\phi' \mathbf{Q}(\phi')$$

Compute **Q** by Taylor expansion of integrand about $\phi' = \phi$.

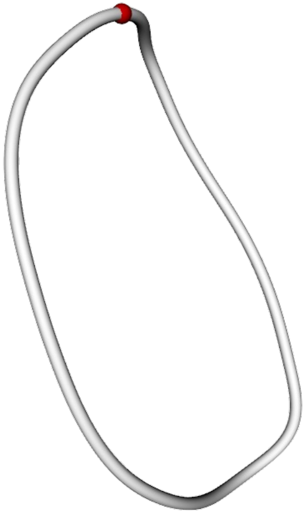
Result:

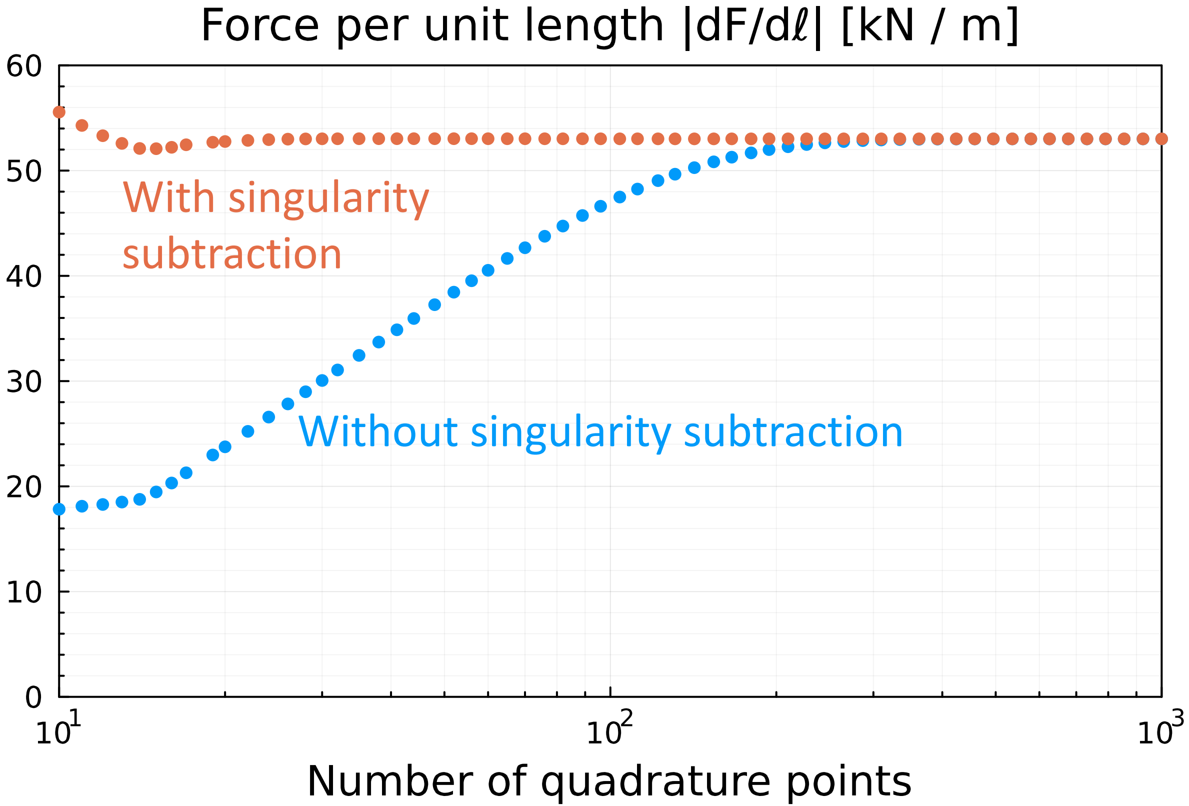
$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \left[\frac{1}{\left(|\mathbf{r}\text{-}\mathbf{r}'|^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}} \frac{d\mathbf{r}'}{d\phi'} \times (\mathbf{r}-\mathbf{r}') + \frac{1}{2}\frac{d^2\mathbf{r}}{d\phi^2} \times \frac{d\mathbf{r}}{d\phi} \frac{2-2\cos(\phi'-\phi)}{\left([2-2\cos(\phi'-\phi)]\left(\frac{d\mathbf{r}}{d\phi}\right)^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}} \right]$$

$$+\frac{\mu_0 I}{4\pi}\left[\frac{d^2\mathbf{r}}{d\phi^2} \times \frac{d\mathbf{r}}{d\phi}\right] \left|\frac{d\mathbf{r}}{d\phi}\right|^{-3} \left[\frac{3}{4} - \ln\left(\frac{8}{a}\left|\frac{d\mathbf{r}}{d\phi}\right|\right)\right].$$

The singularity-subtraction method allows **B** and the force to be evaluated with very few quadrature points.

HSX coil 1





Summary of main results

Internal field:

$$\mathbf{B} = \mathbf{B}_{local} + \mathbf{B}_{reg}$$

$$\mathbf{r}(\rho,\theta,\phi) = \mathbf{r}_c(\phi) + \rho a \cos\theta \mathbf{n}(\phi) + \rho a \sin\theta \mathbf{b}(\phi)$$

$$\mathbf{B}_{local} = \frac{\mu_0 I \rho}{2\pi a}\left[-\mathbf{n}\sin\theta + \mathbf{b}\cos\theta\right] + \frac{\mu_0 I \kappa}{8\pi}\left[-\frac{\rho^2}{2}\sin 2\theta \mathbf{n} + \left(\frac{3}{2} - \rho^2 + \frac{\rho^2}{2}\cos 2\theta\right)\mathbf{b}\right]$$

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \left[\frac{1}{\left(|\mathbf{r}\text{-}\mathbf{r}'|^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}}\frac{d\mathbf{r}'}{d\phi'} \times (\mathbf{r} - \mathbf{r}') + \frac{1}{2}\frac{d^2\mathbf{r}}{d\phi^2} \times \frac{d\mathbf{r}}{d\phi}\frac{2 - 2\cos(\phi' - \phi)}{\left([2 - 2\cos(\phi'-\phi)]\left(\frac{d\mathbf{r}}{d\phi}\right)^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}}\right]$$

$$+ \frac{\mu_0 I}{4\pi}\left[\frac{d^2\mathbf{r}}{d\phi^2} \times \frac{d\mathbf{r}}{d\phi}\right]\left|\frac{d\mathbf{r}}{d\phi}\right|^{-3}\left[\frac{3}{4} - \ln\left(\frac{8}{a}\left|\frac{d\mathbf{r}}{d\phi}\right|\right)\right].$$

Self-force:

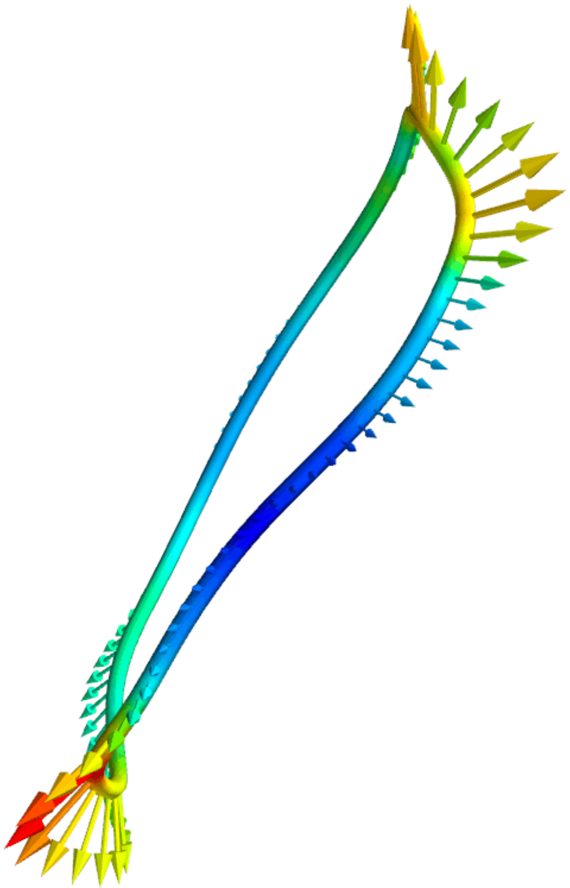
$$\frac{d\mathbf{F}}{d\ell} = I\mathbf{t} \times \mathbf{B}_{reg}$$

Conclusions & future work

- To compute internal **B** field and self-force of a coil, the finite cross-section matters, & it is not accurate to just drop the singular point.
- These quantities can be computed using just a 1D integral if formulated carefully.
- New method agrees with high-fidelity finite-cross-section calculations & analytic results for a circular coil.

<u>Next steps:</u>

- Extend to rectangular cross-section
- Apply in coil optimization
- Would welcome collaboration with this!



Extra slides