Efficient calculation of self-force, internal magnetic field, & stored energy for electromagnetic coils

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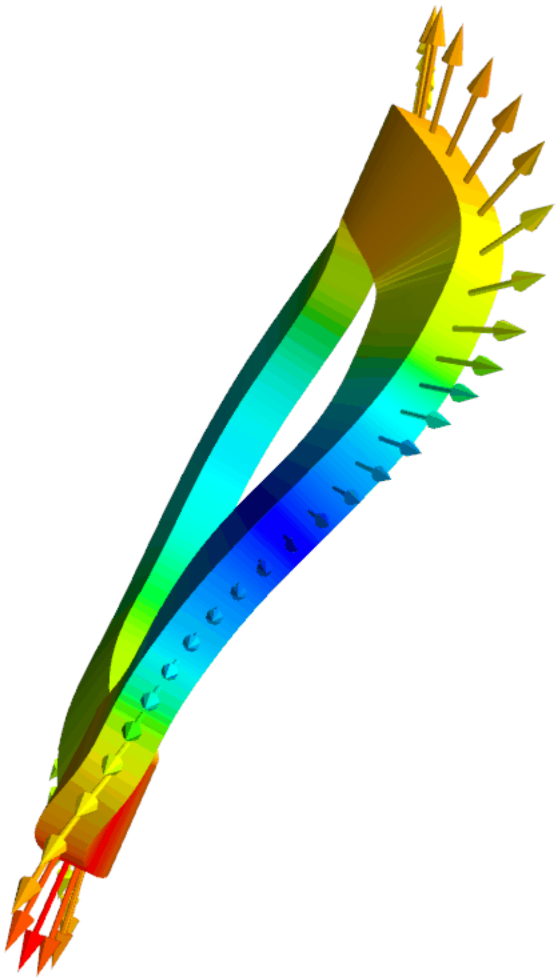
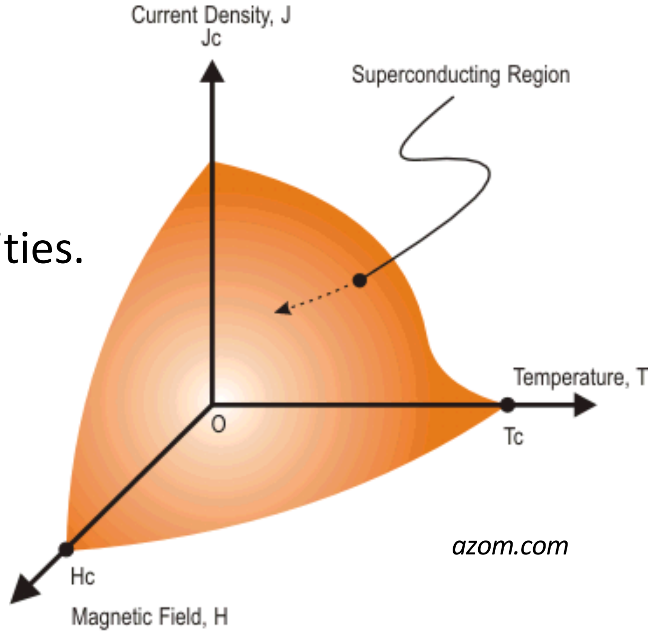
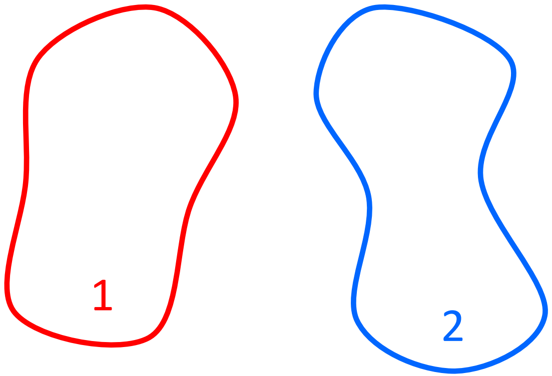


Tokamak & stellarator design requires calculations for the I x B force, internal field, and stored energy

- Forces ∝ B^2. High B limited by support structure.
- Superconductor quench limits depend on local **B**.
- Need to be able to dissipate stored energy $W = ½LI^2$.
- Coil shapes can probably be optimized for these quantities.



Field and force on coil 2 due to current in coil 1 can be computed quickly: 1D filament models are ok.



Tricky part is the self-field: singularity in Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int d\tilde{\phi}\,\frac{\frac{d\tilde{\mathbf{r}}}{d\tilde{\phi}}\times(\mathbf{r}-\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

Accurate calculation of the internal field and self-force appear to require high-dimensional integrals

Field: 3D integral

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3\tilde{r} \frac{\mathbf{J}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|^3}$$

Force per unit length: 5D integral

$$\frac{d\mathbf{F}}{d\ell} = \frac{\mu_0}{4\pi} \frac{d\phi}{d\ell} \int dx \int dy \int d^3\tilde{r} \sqrt{g} \frac{\mathbf{J}(\mathbf{r}) \times [\mathbf{J}(\tilde{\mathbf{r}}) \times (\mathbf{r} - \tilde{\mathbf{r}})]}{|\mathbf{r} - \tilde{\mathbf{r}}|^3}$$

Self-inductance & stored energy: 6D integral

$$L = \frac{\mu_0}{4\pi I^2} \int d^3 r \int d^3\tilde{r} \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|}$$

Can we simplify/approximate these integrals for fast evaluation inside an optimization loop?

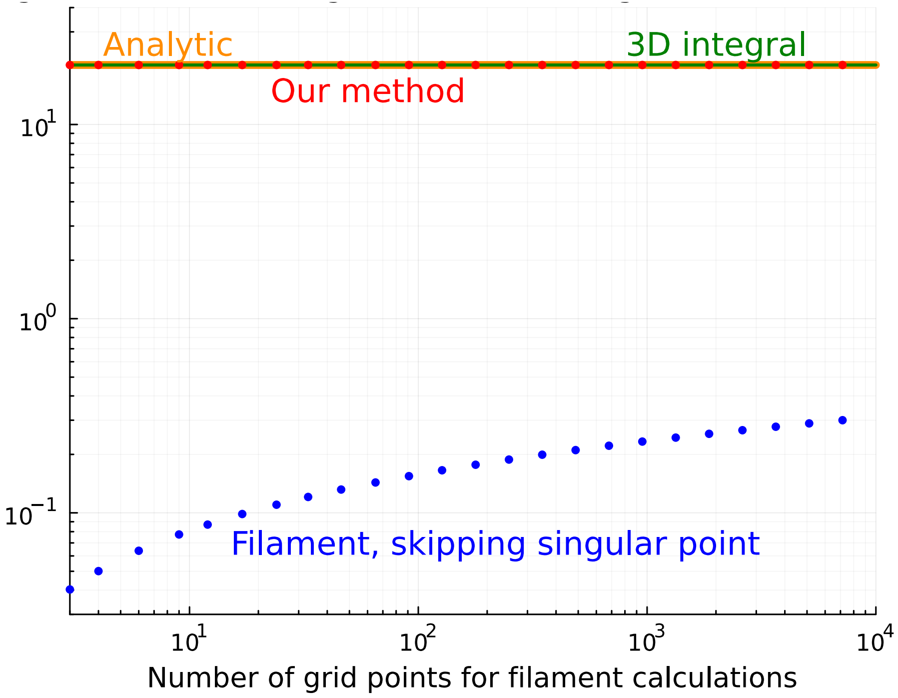
Outline

- Background
- Reducing 3D/5D/6D integrals to 1D/2D
- Efficient quadrature for the 1D/2D integrals
- Future work & conclusions

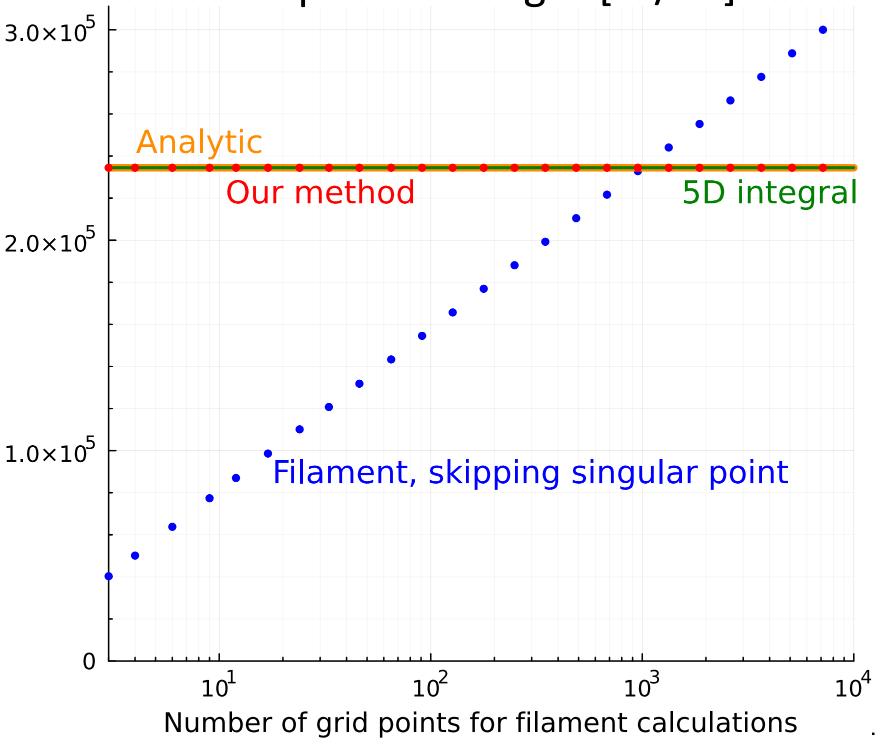
Simply skipping the singular point in a 1D filament calculation gives a non-converging result with signifciant error

Circular coil

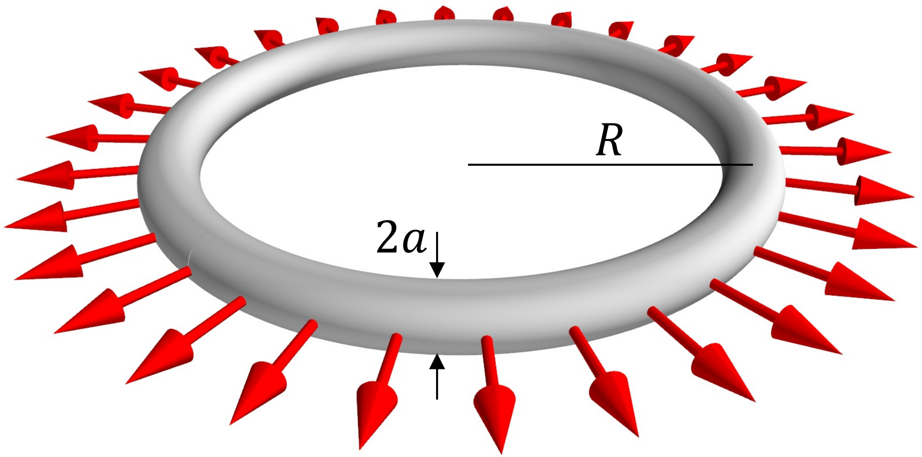
Maximum |B| in the conductor [Tesla]



Force per unit length [N / m]



Analytic formulas for a circular coil show that the finite cross-section cannot be ignored

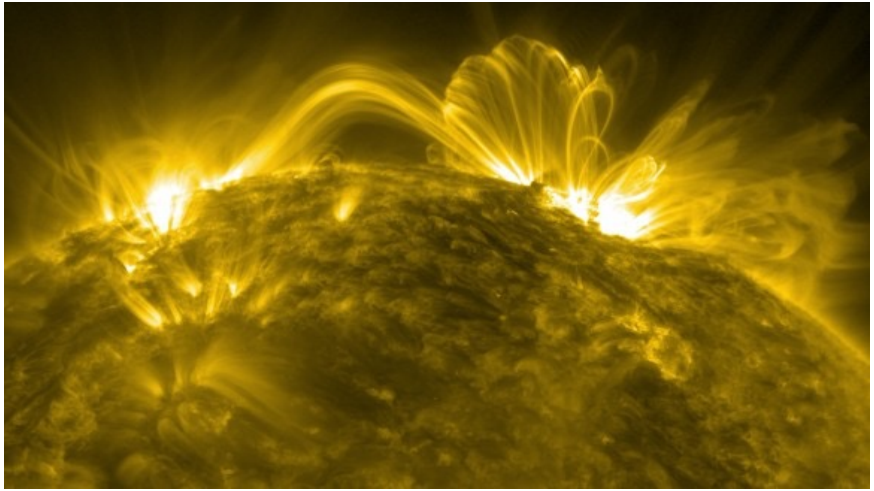


$$\frac{d\boldsymbol{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4}\right]\boldsymbol{e}_R$$

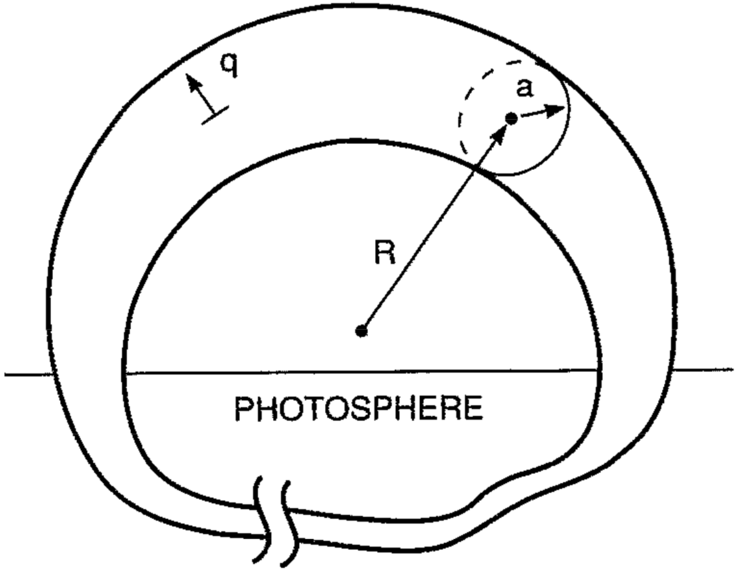
Diverges if minor radius $a \to 0$

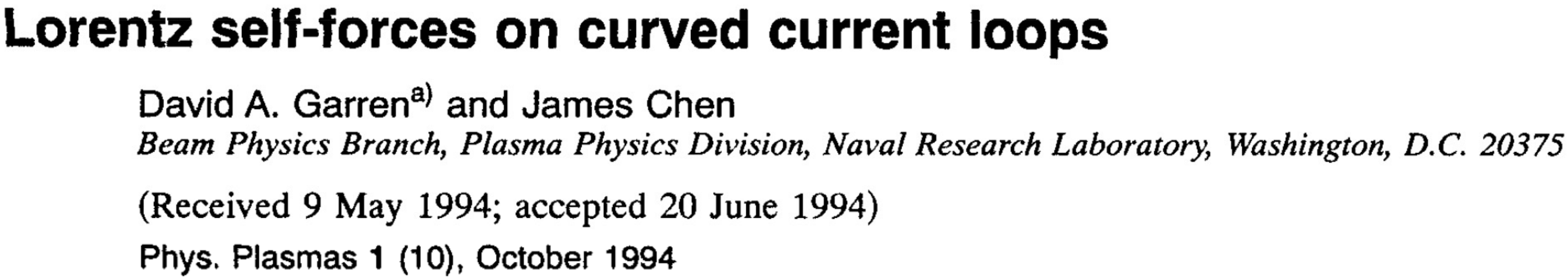
Could a modified 1D filament model work if we supplement it with a value for a?

Calculations of internal field and self-force are also of interest for many other subjects, e.g. solar flares



NASA/SDO/Goddard





Lorentz self-forces on curved current loops

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(Received 9 May 1994; accepted 20 June 1994)

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Some related work

- Garren & Chen, *Phys. Plasmas* (1994). Looked at force but not internal field. Solution is to do a 1D integral over an incomplete loop, with a specific segment removed.
- Dengler, *Advanced Electromagnetics* (2016). Computed self-inductance using 2D integral.
- Lion, Warmer, et al *Nuclear Fusion* (2021). Computed **B** in conductor by summing analytical result for rectangular prism of **J**.
- Robin & Volpe, *Nuclear Fusion* (2022). Computed force for sheet current on a winding surface.

Our contribution:

- Compute self-force, stored energy / inductance, and spatially-resolved internal field using only 1D/2D integrals.
- Integration is over a periodic domain, so quadrature can be spectrally accurate, & can re-use points/data from other coil optimization objectives.

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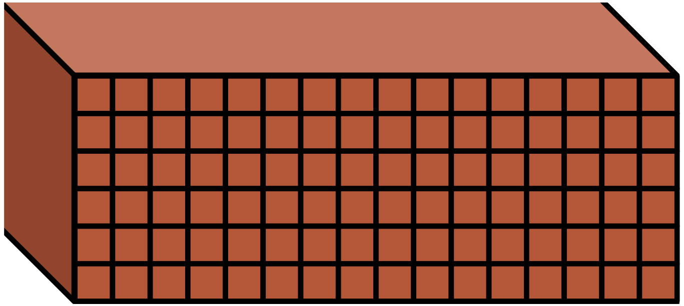
Assumption: current density *J* is uniform

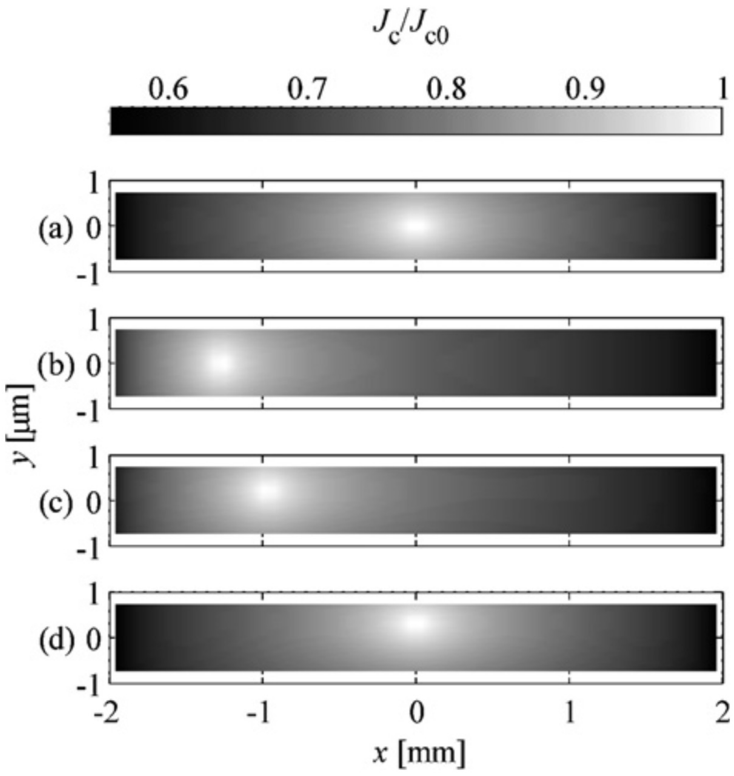
$$\boldsymbol{J} = \frac{I}{A}\boldsymbol{t}$$

I = current
A = x-sectional area
$\boldsymbol{t}$ = unit tangent along conductor

- Ok if multiple turns in both dimensions of the x-section.
- Not necessarily accurate for superconductors, particularly HTS tapes.
- Good enough for optimization?

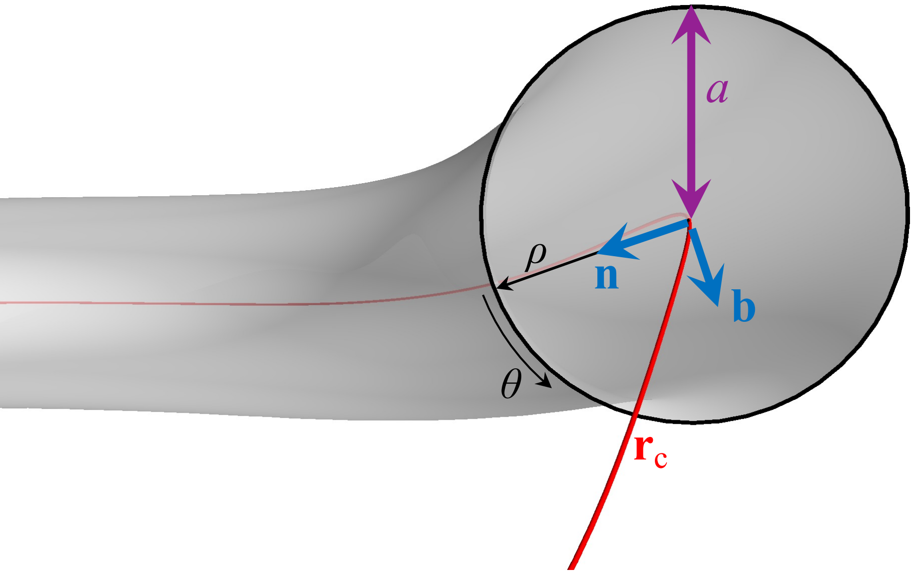


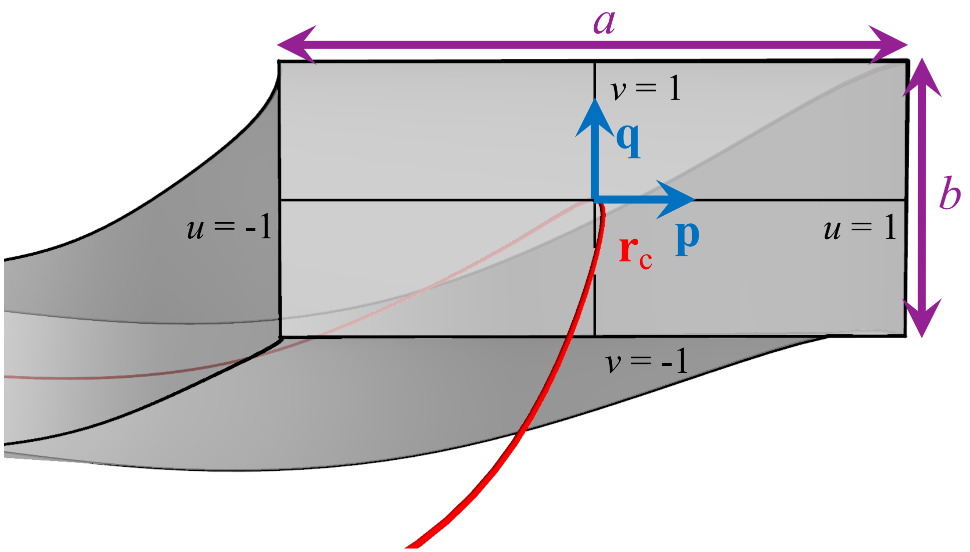
J in YBCO



Rostila et al, (2007)

We can do the calculations for cross-sections that are either circular or rectangular





$$\mathbf{r}(\rho,\theta,\phi)=\mathbf{r}_c(\phi)+\rho a\cos\theta\,\mathbf{n}(\phi)+\rho a\sin\theta\,\mathbf{b}(\phi)$$

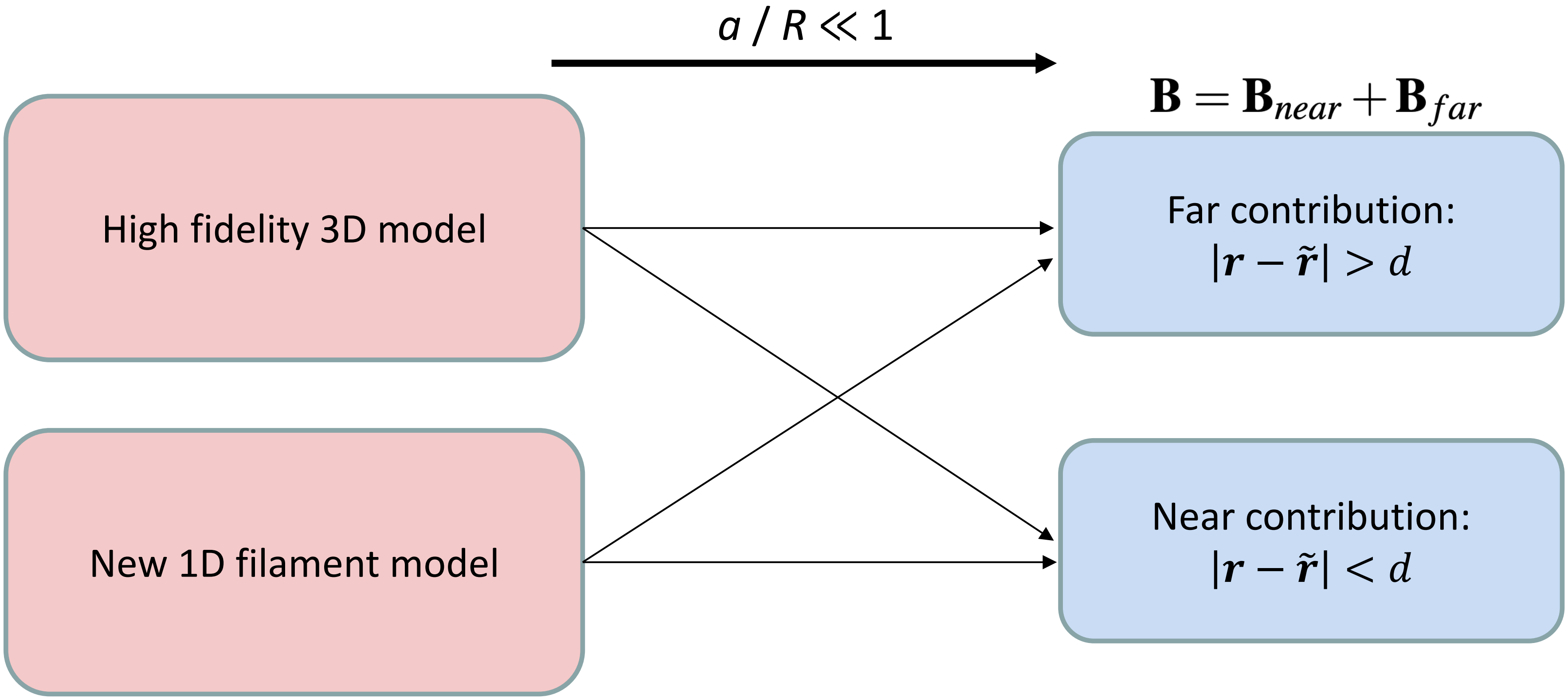
$$\mathbf{r}(u,v,\phi)=\mathbf{r}_c(\phi)+\frac{au}{2}\mathbf{p}(\phi)+\frac{bv}{2}\mathbf{q}(\phi)$$

Methodology for finding an accurate reduced model

- Parameterize the coil volume: $$\mathbf{r}(u,v,\phi) = \mathbf{r}_c(\phi) + \frac{au}{2}\mathbf{p}(\phi) + \frac{bv}{2}\mathbf{q}(\phi)$$
  centerline

- Expansion parameter: $a / R \ll 1$, where $R \sim$ scales of curve centerline, and $b \sim a$.
- Introduce intermediate scale d, with $a \ll d \ll R$.
- Split integrals into "near part" + "far part".
- Far part defined by $|\boldsymbol{r} - \tilde{\boldsymbol{r}}| > d$. Finite cross-section can be neglected.
- Near part defined by $|\boldsymbol{r} - \tilde{\boldsymbol{r}}| < d$. Coil centerline can be Taylor-expanded, so integrals can be done explicitly.
- Identify a 1D integral that has the same near part and far part as the above "high fidelity" calculation for $a / R \ll 1$.

Methodology for finding an accurate reduced model



Limit of the 3D integral for the internal field for $a / R \ll 1$

$$\mathbf{B} = \mathbf{B}_{near} + \mathbf{B}_{far}$$

$$\mathbf{r}(r,\theta,\phi) = \mathbf{r}_c(\phi) + r\cos\theta\,\mathbf{n}(\phi) + r\sin\theta\,\mathbf{b}(\phi)$$

$$\mathbf{B}_{far} = \frac{\mu_0 I}{4\pi}\int_{\phi+\phi_0}^{2\pi+\phi-\phi_0} d\tilde{\phi}\left|\frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}}\right|\frac{\tilde{\mathbf{t}}\times(\mathbf{r}_c-\tilde{\mathbf{r}}_c)}{|\mathbf{r}_c-\tilde{\mathbf{r}}_c|^3}$$

$$\phi_0 = d/R$$

$$\mathbf{B}_{near} = \boxed{\frac{\mu_0 I\rho}{2\pi a}\left[-\mathbf{n}\sin\theta + \mathbf{b}\cos\theta\right]} + \frac{\mu_0 I\kappa}{8\pi}\left[-\frac{\rho^2}{2}\sin 2\theta\,\mathbf{n} + \left(2\ln 2 - \rho^2 + \frac{\rho^2}{2}\cos 2\theta + 2\ln\left(\frac{1}{a}\left|\frac{d\mathbf{r}_c}{d\phi}\right|\right) + 2\ln\phi_0\right)\mathbf{b}\right]$$

Intuition:

- Leading order near-field is same as a straight wire. But corrections contribute to the force.

Our new 1D filament model reproduces the same limit as the original 3D integral

$$\mathbf{B} = \mathbf{B}_{local} + \mathbf{B}_{reg}$$

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\tilde{\phi} \left| \frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}} \right| \frac{\tilde{\mathbf{t}} \times (\mathbf{r}_c - \tilde{\mathbf{r}}_c)}{\left(|\mathbf{r}_c - \tilde{\mathbf{r}}_c|^2 + a^2/\sqrt{e} \right)^{3/2}}$$

$$\mathbf{B}_{local} = \frac{\mu_0 I \rho}{2\pi a} [-\mathbf{n} \sin\theta + \mathbf{b}\cos\theta] + \frac{\mu_0 I \kappa}{8\pi} \left[-\frac{\rho^2}{2} \sin 2\theta \mathbf{n} + \left(\frac{3}{2} - \rho^2 + \frac{\rho^2}{2} \cos 2\theta \right) \mathbf{b} \right]$$

Intuition:

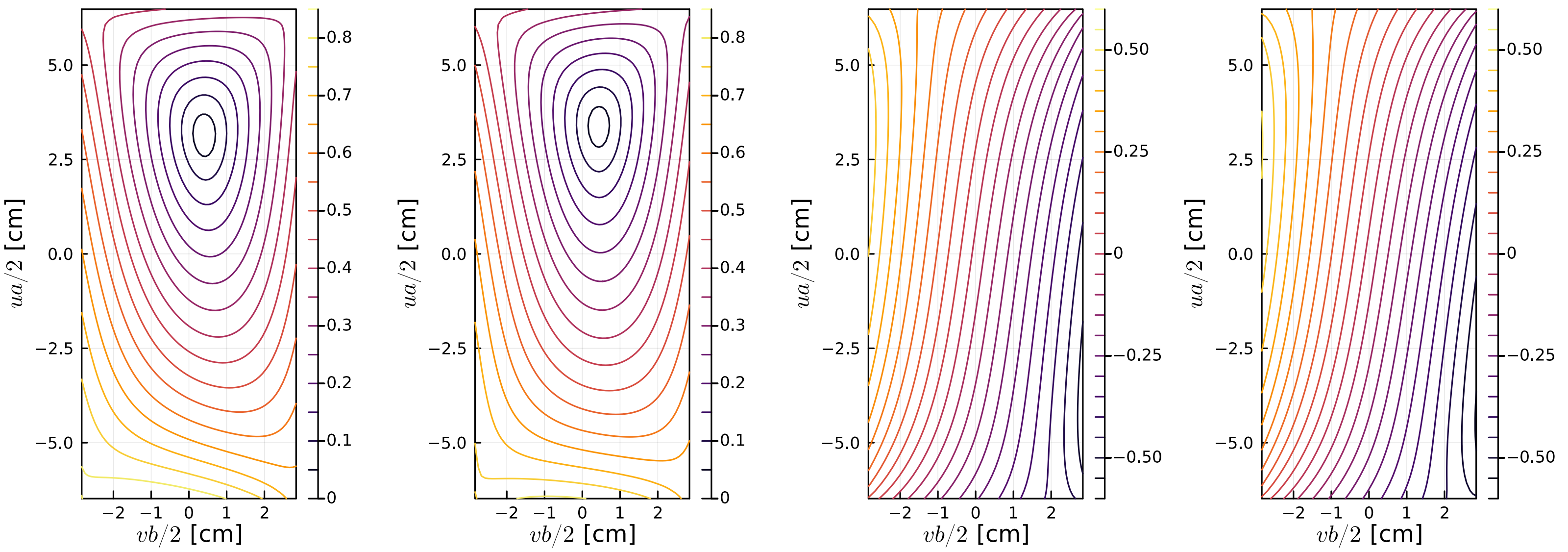
- Regularization added to Biot-Savart. Makes a difference when source and evaluation points are as close as the coil radius.

The new filament model agrees with the high-fidelity 3D integral for B in stellarator coils

The individual **B** components also agree:

HSX coil 1

High fidelity $|B|$ [Tesla] Reduced model $|B|$ [Tesla] High fidelity B_z [Tesla] Reduced model B_z [Tesla]



If curve centerline is a circle, the new filament model matches analytic formula for **B**

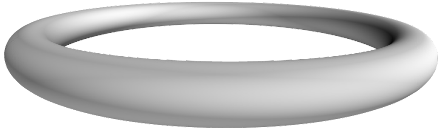
$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I \rho}{2\pi a}\left[\mathbf{e}_x \sin\theta - \mathbf{e}_z \cos\theta\right] \\
&+ \frac{\mu_0 I}{8\pi R_0}\left[\left(-\rho^2 \sin 2\theta\right)\mathbf{e}_x + \left(6\ln 2 - \rho^2 + \frac{\rho^2}{2}\cos 2\theta + 2\ln\frac{R_0}{a}\right)\mathbf{e}_z\right]
\end{aligned}
$$

Integrating the J x B force over the conductor cross-section, our method reduces the 5D integral for the self-force to a 1D integral

$$\frac{d\mathbf{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi^3} \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^1 d\tilde{\rho} \int_0^{2\pi} d\tilde{\theta} \int_0^{2\pi} d\tilde{\phi}\, \rho\tilde{\rho}\, (1-\kappa\rho a\cos\theta)\left(1-\tilde{\kappa}\tilde{\rho} a\cos\tilde{\theta}\right) \left|\frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}}\right| \frac{\mathbf{t}\times\left[\tilde{\mathbf{t}}\times(\mathbf{r}-\tilde{\mathbf{r}})\right]}{|\mathbf{r}-\tilde{\mathbf{r}}|^3}$$

$$\frac{d\mathbf{F}}{d\ell} = I\mathbf{t}\times\mathbf{B}_{reg}, \qquad \mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi} \left|\frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}}\right| \frac{\tilde{\mathbf{t}}\times(\mathbf{r}_c-\tilde{\mathbf{r}}_c)}{\left(|\mathbf{r}_c-\tilde{\mathbf{r}}_c|^2 + a^2/\sqrt{e}\right)^{3/2}}$$

If $\mathbf{r}_c$ is a circle

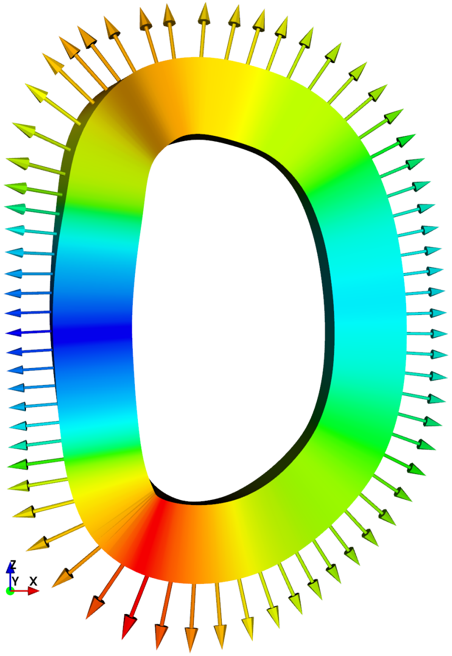


$$\frac{d\boldsymbol{F}}{d\ell} = \frac{\mu_0 I^2}{4\pi R}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4}\right]\boldsymbol{e}_R$$

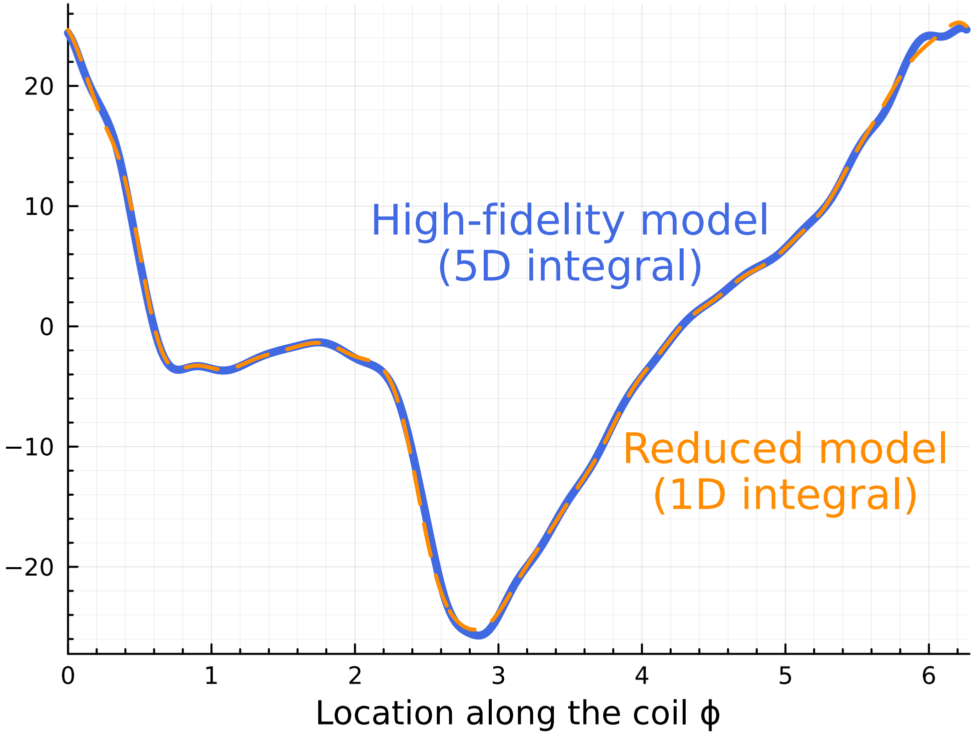
Same as analytic result

The 1D integral accurately approximates high-fidelity calculations for the self-force in stellarator coils

HSX coil 1



z component of self-force per length, $dF_z/d\ell$ [kN/m]



~ 18,000x speed-up for given precision

Similarly, the inductance & stored energy can be computed accurately with only a 2D integral

6D: $$L = \frac{\mu_0}{4\pi I^2}\int d^3r \int d^3\tilde{r}\frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|}$$

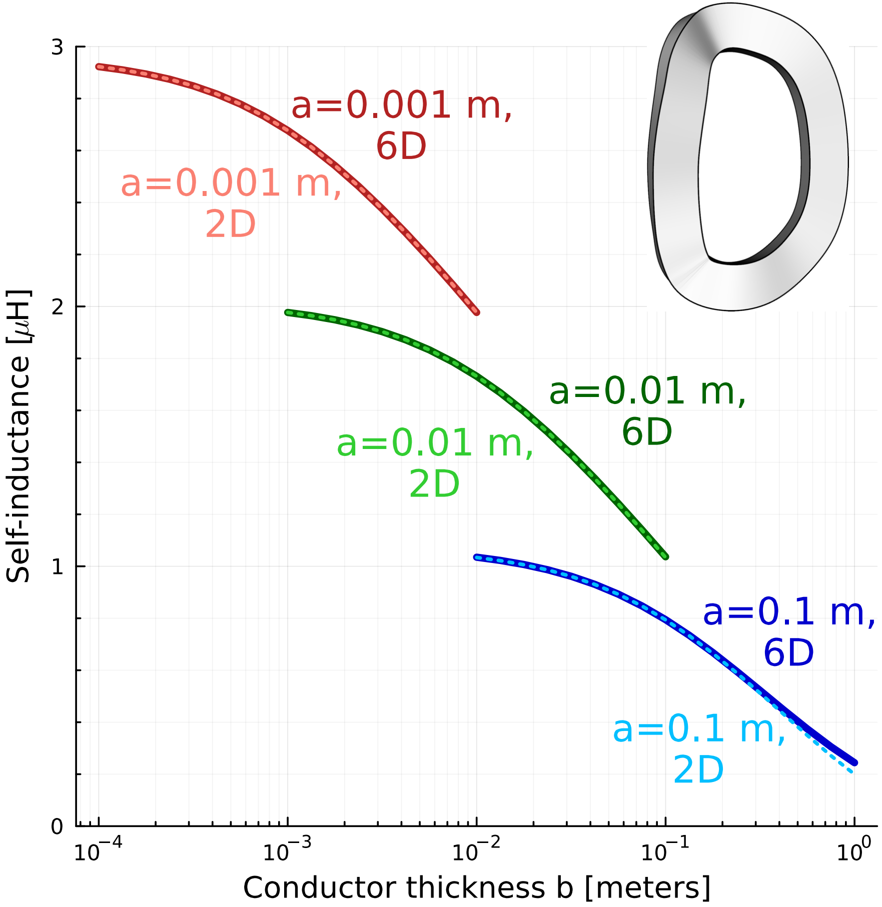
2D: $$L = \frac{\mu_0}{4\pi}\int d\phi \int d\tilde{\phi}\frac{1}{\sqrt{|\mathbf{r}_c-\tilde{\mathbf{r}}_c|^2+\Delta}}\frac{d\mathbf{r}_c}{d\phi}\cdot\frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}}$$

$\Delta = a^2/\sqrt{e}$ for circular x-section,

$$\Delta = ab\exp\left(-\frac{25}{6}+\frac{4b}{3a}\tan^{-1}\frac{a}{b}+\frac{4a}{3b}\tan^{-1}\frac{b}{a} + \frac{b^2}{6a^2}\ln\frac{b}{a}+\frac{a^2}{6b^2}\ln\frac{a}{b}-\frac{a^4-6a^2b^2+b^4}{6a^2b^2}\ln\left(\frac{a}{b}+\frac{b}{a}\right)\right)$$

for rectangular x-section.

For circular centerline, matches analytic result by Weinstein, *Annalen der Physik* (1884)



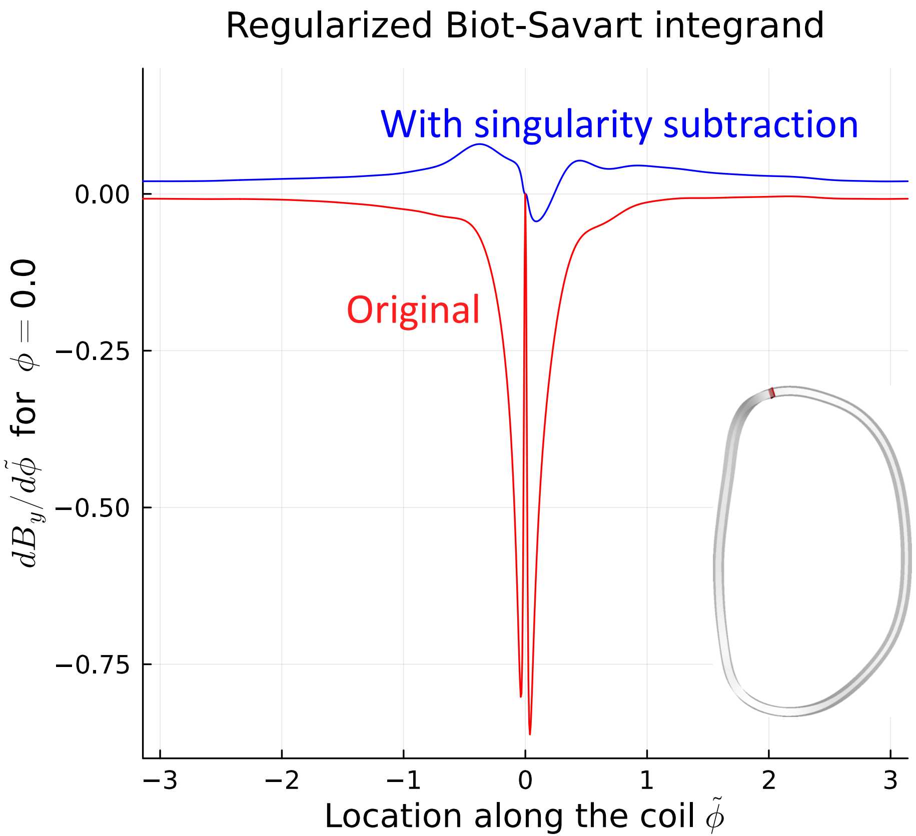
Outline

- Motivation
- Reducing 3D/5D/6D integrals to 1D/2D
- Efficient quadrature for the 1D/2D integrals
- Future work & conclusions

Remaining 1D integral is still tricky since integrand has fine structure

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\tilde{\phi} \left| \frac{d\tilde{\mathbf{r}}_c}{d\tilde{\phi}} \right| \frac{\tilde{\mathbf{t}} \times (\mathbf{r}_c - \tilde{\mathbf{r}}_c)}{\left(|\mathbf{r}_c - \tilde{\mathbf{r}}_c|^2 + \Delta \right)^{3/2}}$$

A solution: subtract and add a function to the integrand with the same near-singular behavior that can be integrated analytically.



To make integrand smooth, we subtract and add a function with the same singular behavior that can be integrated analytically.

$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi}\left[\frac{1}{\left(|\mathbf{r}\text{-}\tilde{\mathbf{r}}|^2+\Delta\right)^{3/2}}\frac{d\tilde{\mathbf{r}}}{d\tilde{\phi}}\times(\mathbf{r}-\tilde{\mathbf{r}}) - \mathbf{Q}\left(\tilde{\phi}\right)\right] + \frac{\mu_0 I}{4\pi}\int d\tilde{\phi}\,\mathbf{Q}\left(\tilde{\phi}\right)$$

Compute **Q** by Taylor expansion of integrand about $\tilde{\phi} = \phi$.

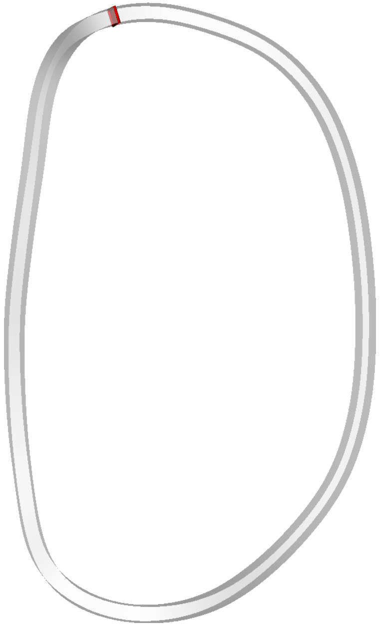
Result:

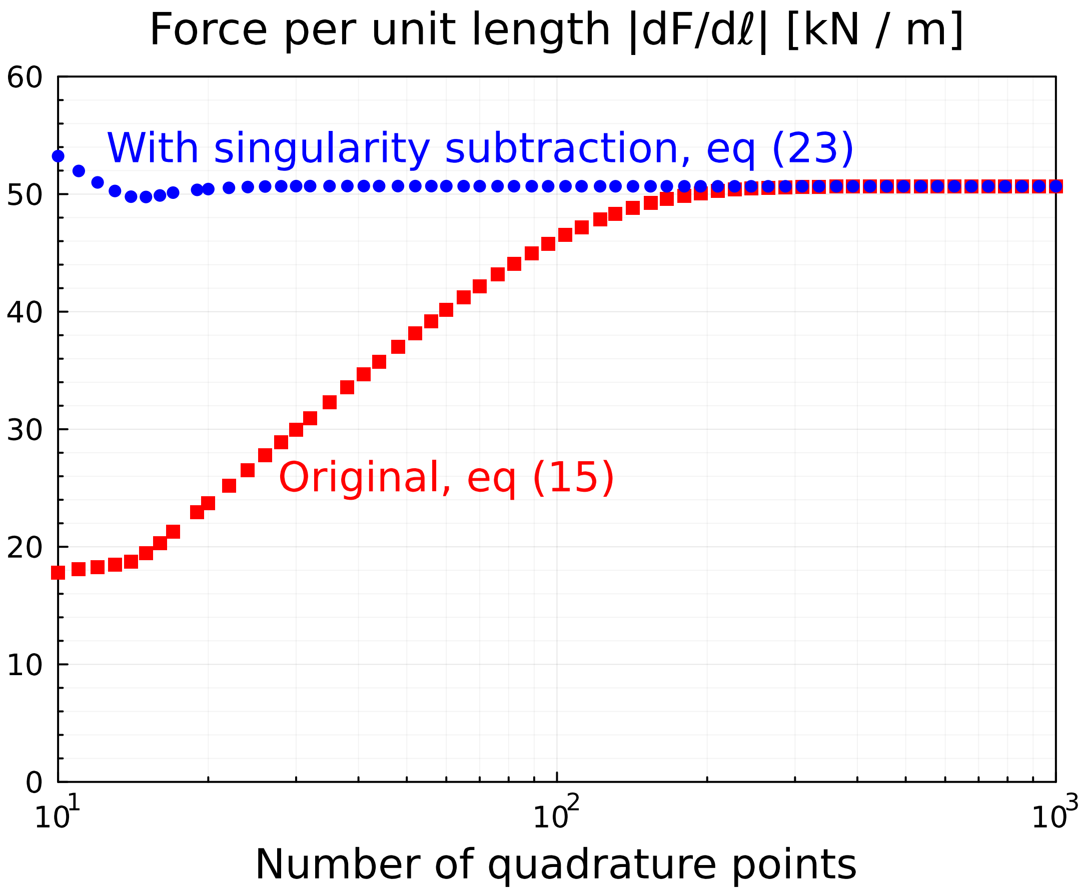
$$\mathbf{B}_{reg} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\tilde{\phi}\left[\frac{1}{\left(|\mathbf{r}\text{-}\tilde{\mathbf{r}}|^2+\Delta\right)^{3/2}}\frac{d\tilde{\mathbf{r}}}{d\tilde{\phi}}\times(\mathbf{r}-\tilde{\mathbf{r}}) + \frac{d^2\mathbf{r}}{d\phi^2}\times\frac{d\mathbf{r}}{d\phi}\frac{1-\cos\left(\tilde{\phi}-\phi\right)}{\left(\left[2-2\cos\left(\tilde{\phi}-\phi\right)\right]\left(\frac{d\mathbf{r}}{d\phi}\right)^2+\Delta\right)^{3/2}}\right]$$

$$+\frac{\mu_0 I}{8\pi}\left[\frac{d^2\mathbf{r}}{d\phi^2}\times\frac{d\mathbf{r}}{d\phi}\right]\left|\frac{d\mathbf{r}}{d\phi}\right|^{-3}\left[-2+\ln\left(\frac{64}{\Delta}\left|\frac{d\mathbf{r}}{d\phi}\right|^2\right)\right].$$

The singularity-subtraction method allows B and the force to be evaluated with very few quadrature points.

HSX coil 1





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A possible reduced model for the critical current?

Given a model for how the local critical current density depends on **B**, e.g.

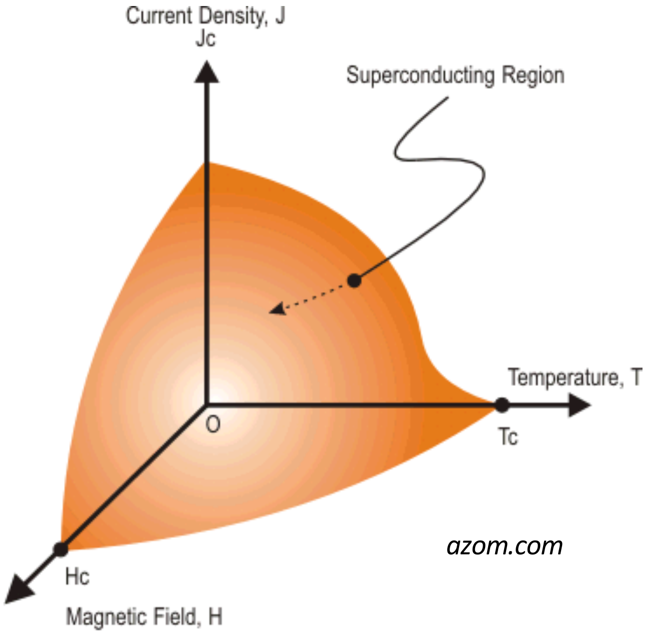
$$j_c(x, y) = \frac{j_{c0}}{\left(1 + \frac{\sqrt{k^2 B_x^2(x,y) + B_y^2(x,y)}}{B_0}\right)^{\beta}}$$

Gömöry and Klinčok (2006)

estimate the global critical current as

$$I_c = \min_{\phi} \int_{x-section} d^2a \; j_c(\boldsymbol{B}(u, v, \phi))$$

Not self-consistent, but is it good enough to be useful?

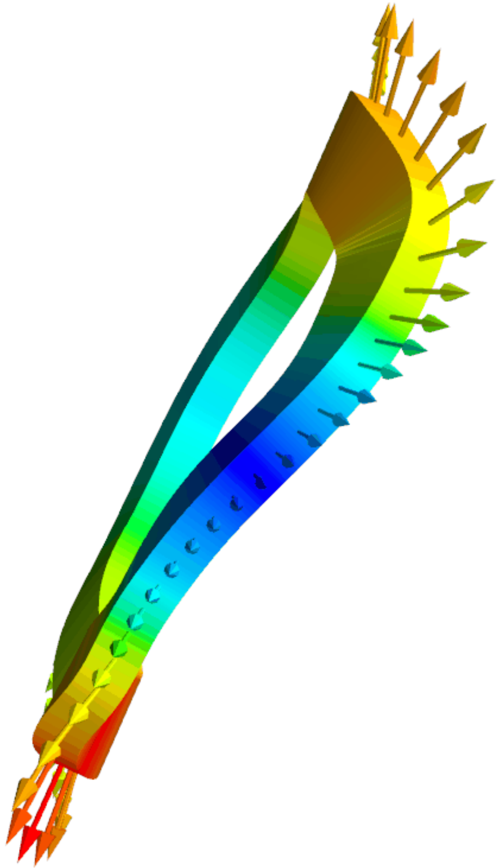


Conclusions & future work

- To compute internal **B** field, self-force, & stored energy of a coil, the finite cross-section matters, & it is not accurate to just drop the singular point.
- These quantities can be computed using just a 1D/2D integral if formulated carefully.
- New method agrees with high-fidelity finite-cross-section calculations & analytic results.

Next steps:

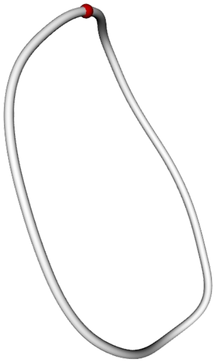
- Apply in coil optimization
- Improve applicability to superconductors
- Would welcome collaboration with this!

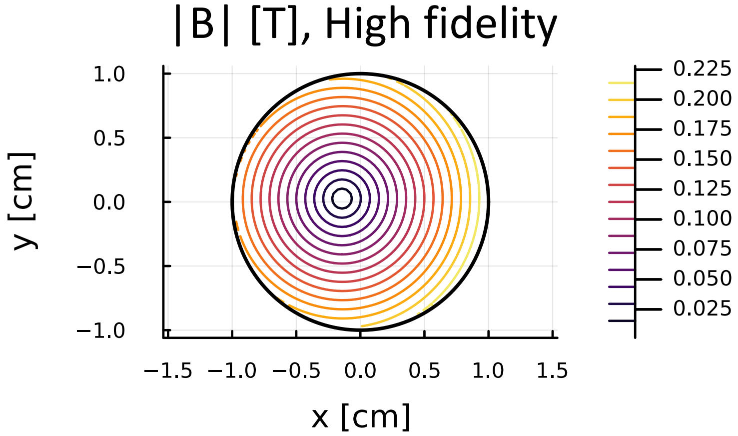


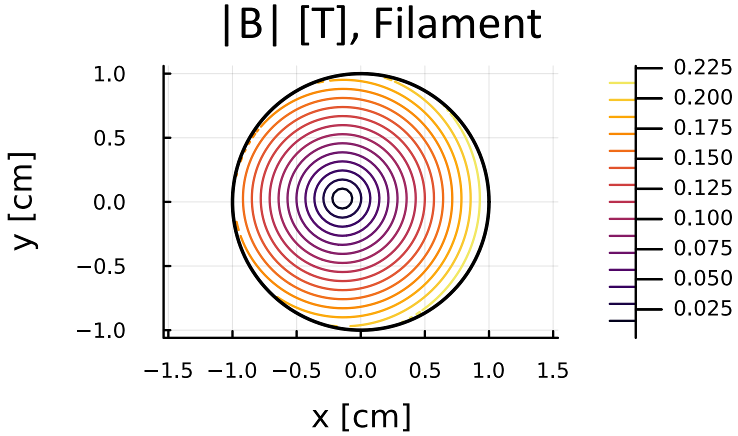
Extra slides

The new filament model agrees with the high-fidelity 3D integral for **B** in stellarator coils

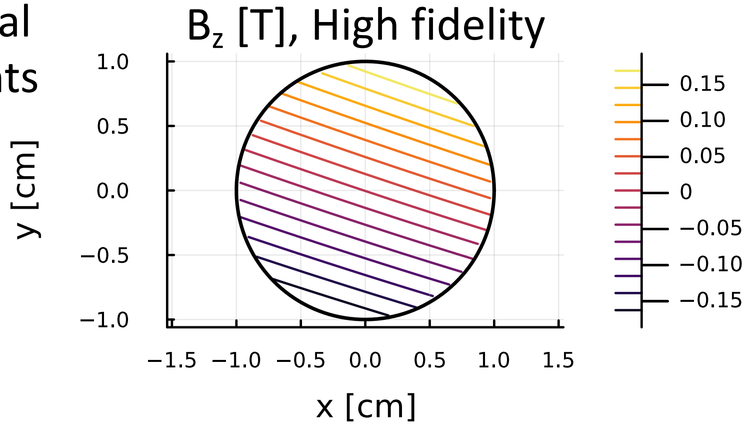
HSX coil 1

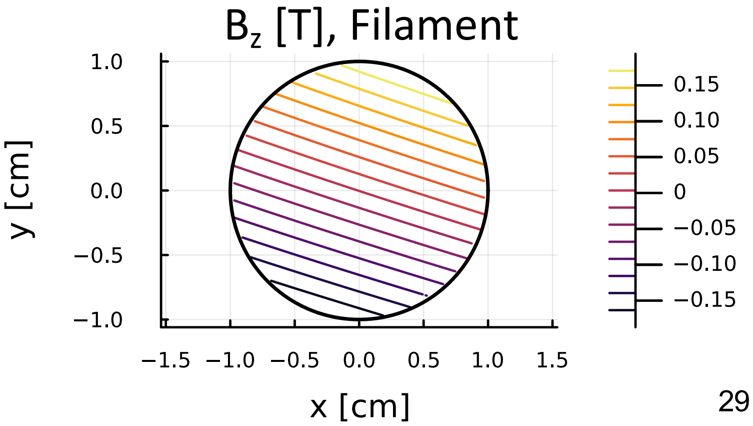






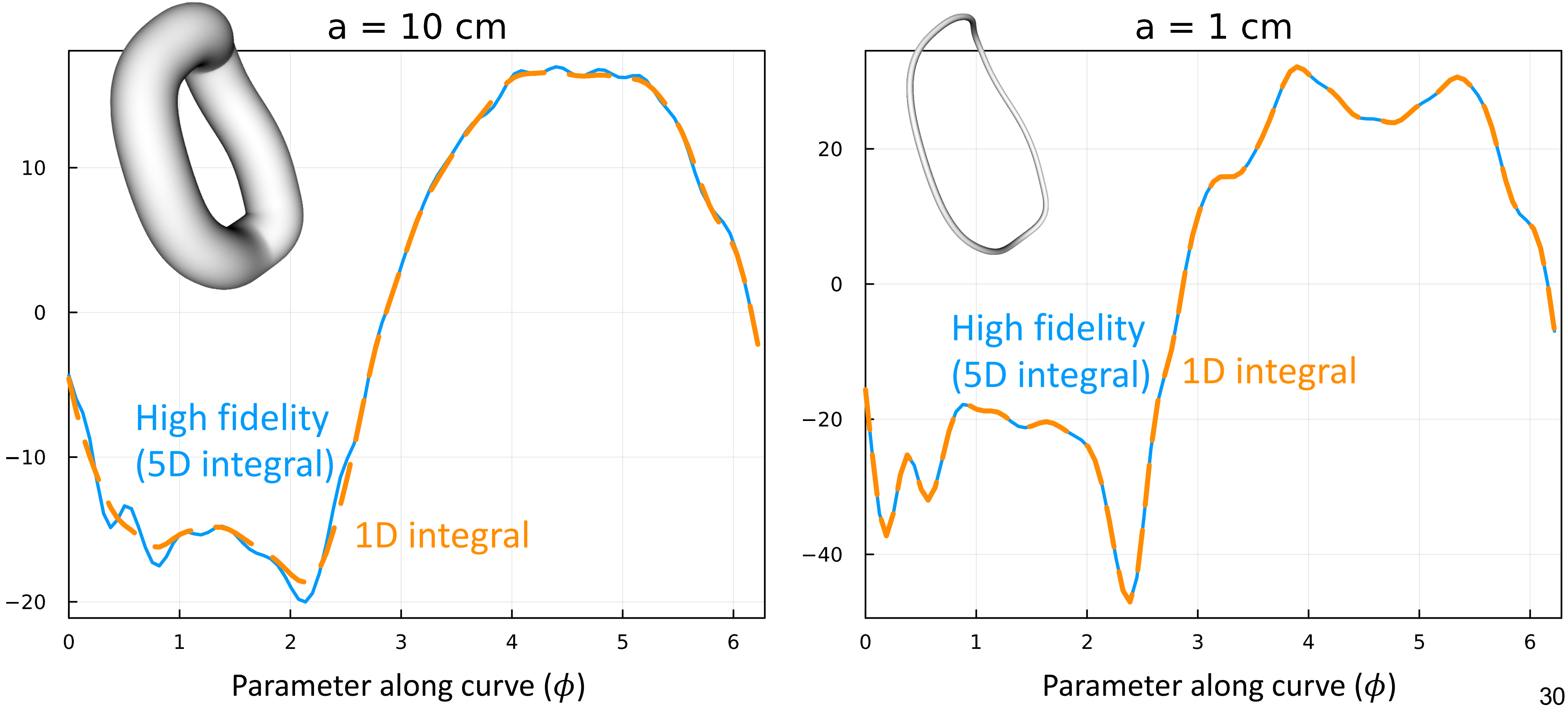
The individual **B** components also agree:





The 1D integral accurately approximates high-fidelity calculations for the self-force in stellarator coils

$dF_x/d\ell$ [kN / m] for HSX coil 1, @ 150 kA



Summary of main results

Internal field:

$$\mathbf{B} = \mathbf{B}_{local} + \mathbf{B}_{reg}$$

$$\mathbf{r}(\rho,\theta,\phi) = \mathbf{r}_c(\phi) + \rho a \cos\theta \mathbf{n}(\phi) + \rho a \sin\theta \mathbf{b}(\phi)$$

$$\mathbf{B}_{local} = \frac{\mu_0 I \rho}{2\pi a}\left[-\mathbf{n}\sin\theta + \mathbf{b}\cos\theta\right] + \frac{\mu_0 I \kappa}{8\pi}\left[-\frac{\rho^2}{2}\sin 2\theta \mathbf{n} + \left(\frac{3}{2} - \rho^2 + \frac{\rho^2}{2}\cos 2\theta\right)\mathbf{b}\right]$$

$$\begin{aligned}\mathbf{B}_{reg} \quad = \quad & \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \left[\frac{1}{\left(|\mathbf{r}\text{-}\mathbf{r}'|^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}} \frac{d\mathbf{r}'}{d\phi'} \times (\mathbf{r} - \mathbf{r}') + \frac{1}{2}\frac{d^2\mathbf{r}}{d\phi^2} \times \frac{d\mathbf{r}}{d\phi} \frac{2 - 2\cos(\phi' - \phi)}{\left([2 - 2\cos(\phi' - \phi)]\left(\frac{d\mathbf{r}}{d\phi}\right)^2 + \frac{a^2}{\sqrt{e}}\right)^{3/2}}\right] \\ & + \frac{\mu_0 I}{4\pi}\left[\frac{d^2\mathbf{r}}{d\phi^2} \times \frac{d\mathbf{r}}{d\phi}\right]\left|\frac{d\mathbf{r}}{d\phi}\right|^{-3}\left[\frac{3}{4} - \ln\left(\frac{8}{a}\left|\frac{d\mathbf{r}}{d\phi}\right|\right)\right].\end{aligned}$$

Self-force:

$$\frac{d\mathbf{F}}{d\ell} = I\mathbf{t} \times \mathbf{B}_{reg}$$